EXPLICIT MERTENS SUMS

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Received: 6/29/15, Accepted: 3/19/17, Published: 4/24/17

Abstract

Making effective use of zero-free regions for the Riemann ζ-function and the computed zeros of $\zeta(s)$, we give explicit bounds for some well-known sums and products over prime numbers.

1. Introduction and Results

Explicit estimates in prime number theory have a long history, starting for the modern part with the two seminal papers [22] and [23]. The main development since then has been directed towards the Chebyshev ψ-function: verifying the Riemann hypothesis up to large heights ([26], [10], [19]), getting estimates for ψ ([24], [7], [9]) or getting better infinite zero-free regions ([24], [13]). Related quantities like $\sum_{p\le x} 1/p$ or $\prod_{p\le x}(1-p^{-1})$ were considered only marginally. It may be surprising, but it is not automatic to derive quantitatively good estimates for such "derived quantities" from the estimates for the ψ-function, as is explained in [5]. Recently [21] dealt efficiently with $\sum_{p\le x} \Lambda(n)/n$ and this work may be seen as continuing this line of work. In passing we correct a mistake therein.

Here is one of our typical results:

Corollary 1. *We have*

$$\sum_{p\le x} \frac{1}{p} = \log\log x + B + \mathcal{O}^*\left(\frac{4}{\log^3 x}\right) \quad (x \ge 2),$$

$$\sum_{p\le x} \frac{1}{p} = \log\log x + B + \mathcal{O}^*\left(\frac{2.3}{\log^3 x}\right) \quad (x \ge 1000)$$

and for $x \ge 24284$,

$$\sum_{p\le x} \frac{1}{p} = \log\log x + B + \mathcal{O}^*\left(\frac{1}{\log^3 x}\right).$$

Here and henceforth, $f(x) = \mathcal{O}^*\big(g(x)\big)$ means $|f(x)| \le g(x)$.

This is to be compared with [23, (3.17),(3.18)] where the authors have the error term ${}^1/_{(2\log^2 x)}$. We heavily rely on Pari/GP (see [25]) computations for small values of the variable x. It is also of interest to get better error terms, even if large values of the variable x are required, and in this direction we prove the following:

Corollary 2. *When* $\log x \ge 4635$*, we have*

$$\sum_{p\le x}\frac{1}{p} = \log\log x + B + \mathcal{O}^*\left(1.1\frac{\exp(-\sqrt{0.175\log x})}{(\log x)^{3/4}}\right). \tag{1}$$

Such results are dependent on the size of the zero-free region for the Riemann zeta function and may change if there are improvements on zero-free regions, so we provide a result to reflect the size of the known zero-free region. Let us assume that $\zeta(s)$ does not vanish in the region

$$\Re s \ge 1 - \frac{1}{R\log(|\Im s|)} \quad (|\Im s| \ge t_0)$$

where $R > 0$. For instance, thanks to [13], we can choose $R = 5.69693$ and $t_0 = 10$. We use the notation

$$\lambda(x) = \sum_{p\le x}\frac{1}{p}. \tag{2}$$

Under such a hypothesis, we have the following:

Corollary 3. *For* $\log x \ge 814R$*, we have*

$$\lambda(x) = \log\log x + B + \mathcal{O}^*\left(\frac{1.6}{R^{\frac14}\log^{\frac34} x}\exp\big(-\sqrt{\frac{\log x}{R}}\big)\right).$$

This is derived from our main theorem, which is the following:

Theorem 4. *For* $x \ge \exp(814R)$*, we have*

$$\lambda(x) = \log\log x + B + \frac{\vartheta(x)-x}{x\log x} + \mathcal{O}^*\left(\frac{7\times10^{-6}}{\log x}\exp\big(-\sqrt{\frac{2\log x}{R}}\big)\right). \tag{3}$$

For $x \ge 2$*, we have*

$$\lambda(x) = \log\log x + B + \frac{\vartheta(x)-x}{x\log x} + \mathcal{O}^*\left(\frac{1+\log x}{\log^2 x}\alpha^*(x)\right)$$

with

$$\alpha^*(x) = \frac{2.1}{\sqrt{x}} + \frac{4.5}{x^{\frac23}} + \frac{2.84}{x} + 1.751\times10^{-12}.$$

Proof of Corollary 3. This comes directly from (3) and a recent result of Dusart (see [8, Theorem 1.1]) which says that, if $\log x \geq 70R$, then

$$\left|\frac{\vartheta(x)-x}{x}\right| < \sqrt{\frac{8}{\pi}}\left(\frac{\log x}{R}\right)^{\frac{1}{4}} \exp\big(-\sqrt{\frac{\log x}{R}}\big). \tag{4}$$

$\square$

Using the value $R = 5.69693$ mentioned earlier then gives (1).

1.1. Eulerian Products

As an application of our estimates on $\lambda(x)$, we give explicit estimates for the (finite Euler) products $\prod_{p\leq x}(1+{}^{\epsilon}/_{p})$.

Theorem 5. *Let ϵ be a complex number with $|\epsilon| < 2$. Then for $x \geq \exp(22)$, we have*

$$\prod_{p\leq x}\left(1+\frac{\epsilon}{p}\right) = e^{\gamma(\epsilon)+\epsilon B}\big(\log x\big)^{\epsilon}\left\{1+\mathcal{O}^*\left(\frac{0.841}{\log^3 x}\right)\right\}$$

where

$$\gamma(\epsilon) = \sum_p \sum_{n=2}^{\infty}(-1)^{n+1}\frac{\epsilon^n}{np^n}.$$

The cases $\epsilon = \pm 1$ are most commonly studied, with Mertens himself treating the case $\epsilon = -1$ in [16], without giving explicit error terms. A preliminary form for this result is found in [15]. One may compare this with the error term ${}^{1}/_{(2\log^2 x)}$ given in [23, Theorem 7] for the case $\epsilon = -1$. In [4] it is proved that the difference

$$\prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-1} - e^{\gamma}\log x$$

changes sign infinitely often. Similar products are studied in [2, 3], the latter dealing with aspects other than explicit bounds.

1.2. Mertens Sums

We next study two closely related sums:

$$\Upsilon(x) = \sum_{p\leq x}\frac{\log p}{p}, \quad \tilde{\psi}(x) = \sum_{n\leq x}\frac{\Lambda(n)}{n}.$$

We will content ourselves with giving explicit approximations for very large values of x.

Theorem 6. *The following holds for* $\log x \geq 814R$*:*

$$\Upsilon(x) = \log x + E + \frac{\vartheta(x) - x}{x} + \mathcal{O}^*\left(6.5 \times 10^{-6} \exp\big(- \sqrt{\frac{2\log x}{R}}\big)\right)$$

where $E = -\gamma - \sum_{n=2}^{\infty} \sum_p {}^{\log p}/_{p^n} = -1.332582275733221...$[1]
Also, for all $x \geq 2$*, we have*

$$\Upsilon(x) = \log x + E + \frac{\vartheta(x) - x}{x} + \mathcal{O}^*\left(\frac{2.0494}{x^{\frac{1}{2}}} + \frac{4.5}{x^{\frac{2}{3}}} + \frac{1.838}{x} + 1.75 \times 10^{-12}\right)$$

Corollary 7. *For* $x \geq \exp(814R)$*, we have*

$$\Upsilon(x) = \log x + E + \mathcal{O}^*\left(1.6 \left(\frac{\log x}{R}\right)^{\frac{1}{4}} \exp\big(- \sqrt{\frac{\log x}{R}}\big)\right).$$

Proof. One uses (4) in Theorem 6. □

The value $R = 5.69693$ then gives the following corollary:

Corollary 8. *For* $x \geq \exp(2319)$*, we have*

$$\Upsilon(x) = \log x + E + \mathcal{O}^*\left(1.036(\log x)^{1/4} \exp\big(- \sqrt{0.175 \log x}\big)\right).$$

In [23, Theorem 6], we find an error term of ${}^{1}/_{(2\log x)}$ for this sum. Landau [14, § 55] gives error terms of $\exp\big(- (\log x)^{\frac{1}{14}}\big)$ for both $\Upsilon(x)$ and $\lambda(x)$.

Finally, we rectify the estimate for $\tilde{\psi}(x)$ given in [21, Theorem 1.1]:

Theorem 9. *When* $\log x \geq 407R$*, the following holds:*

$$\tilde{\psi}(x) = \log x - \gamma + \frac{\psi(x) - x}{x} + \mathcal{O}^*\left(\frac{0.05}{\sqrt{x}}\right) + \mathcal{O}^*\left(6.4 \times 10^{-6} \exp\big(- \sqrt{\frac{2\log x}{R}}\big)\right).$$

Here γ *is the usual Euler-Mascheroni constant. Also, for all* $x \geq 2$*, we have*

$$\tilde{\psi}(x) = \log x - \gamma + \frac{\psi(x) - x}{x} + \mathcal{O}^*\left(\frac{0.047}{\sqrt{x}} + \frac{1.884}{x} + 1.75 \times 10^{-12}\right).$$

We find this sum for example in [1, Theorem 4.9] or [11, Theorem 424] in the more rudimentary form $\tilde{\psi}(x) = \log x + \mathcal{O}(1)$.

Corollary 10. *For* $\log x \geq 407R$*, we have*

$$\tilde{\psi}(x) = \log x - \gamma + \mathcal{O}^*\left(1.6 \left(\frac{\log x}{R}\right)^{\frac{1}{4}} \exp\big(- \sqrt{\frac{\log x}{R}}\big)\right).$$

[1] See [23, (2.11)] for the numerical value.

Proof. We use the estimate

$$0 \le \psi(x) - \vartheta(x) \le 1.0012\sqrt{x} + 3x^{\frac{1}{3}} \quad (x > 0) \tag{5}$$

(see [24, Theorem 6]) together with (4) and round off appropriately. □

One may also give an analogue of Corollary 8 for this function.

Notation. We have already introduced the symbols $\lambda, \tilde{\psi}$ and Υ. Following [21], by $f(x) = \mathcal{O}^*(g(x))$ we mean $|f(x)| \le g(x)$. We define the following functions:

$$J(x) := \sum_{\rho} \frac{x^{\rho-1}}{\rho(\rho-1)}, \tag{6}$$

$$S_m(x) := \sum_{\rho} \frac{x^{\beta-1}}{|\gamma|^{m+1}} \tag{7}$$

for $x > 0$ and $m \ge 1$. In both cases, the sum runs through all the nontrivial zeros ρ of the the Riemann zeta function. It has been verified (see [10]) that at least the first 10^{13} zeros of $\zeta(s)$ lie on the critical line $\Re s = \frac{1}{2}$. Hence we may consider the Riemann Hypothesis verified up to height $T_0 = 2.44 \times 10^{12}$. As mentioned earlier, we also suppose that there is no nontrivial zero $\rho = \beta + i\gamma$ of $\zeta(s)$ satisfying

$$\beta \ge 1 - \varphi(\gamma) = 1 - \frac{1}{R \log|\gamma|} \quad (|\gamma| \ge t_0), \tag{8}$$

where R is a positive constant. For explicit computations, we will however assume throughout that $R \ge 1$. We use γ as the imaginary part of a nontrivial zero of $\zeta(s)$ and as the Euler-Mascheroni constant. This is unlikely to cause any confusion. The symbols ψ and ϑ always denote the Chebyshev functions, whereas we have defined φ in (8), abrogating its traditional use as the Euler totient. We follow other usual number-theoretic conventions, such as writing s for a complex variable, etc. Further notations will be introduced as necessary.

Organization of the paper. The results stated in Section 1 are not restated. Section 2 is independent of other sections, and so may be read separately. The theorems stated in Section 1 are proved in Section 3; only the proof of Theorem 5 there depends on Section 4 which comes after it. This last section consists mainly of numerical computations (using Pari/GP) to bridge the gap between extremely big values of x and bounded intervals. It uses the results stated in Section 1.

2. Lemmas on the Zeros of $\zeta(s)$

As is customary, $N(T)$ denotes the number of zeros $\rho = \beta + i\gamma$ of the Riemann zeta function $\zeta(s)$ with $0 < \gamma \le T$ and $0 < \beta < 1$. We have the following explicit form of the von Mangoldt formula (see [20, Lemma 1]):

Lemma 11. *For $T \geq 1\,000$, we have*

$$N(T) = N^*(T) + \mathcal{O}^*(0.67 \log \frac{T}{2\pi}) \tag{9}$$

where

$$N^*(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}.$$

We also quote [20, Lemma 2]:

Lemma 12. *For $m \geq 1$ and $T \geq 1\,000$, we have*

$$\sum_{\substack{\rho \\ |\gamma|>T}} \frac{1}{|\gamma|^{m+1}} = \frac{1}{m\pi T^m} (\log \tfrac{T}{2\pi} + \frac{1}{m}) + \mathcal{O}^*\big(\frac{1.34}{T^{m+1}} (2 \log \tfrac{T}{2\pi} + 1)\big). \tag{10}$$

We quote [21] for an estimate of $J(x)$:

Lemma 13. *We have*

$$|J(x)| \leq \frac{0.047}{\sqrt{x}} + 1.75 \times 10^{-12} \quad (x \geq 2). \tag{11}$$

Further, when $\log x \geq 407R$, we have the inequality

$$|J(x)| \leq \frac{0.047}{\sqrt{x}} + 6.4 \times 10^{-6} \exp\big(-\sqrt{\frac{2 \log x}{R}}\big). \tag{12}$$

To give an estimate for $S_m(x)$, we follow the method of derivation of (12) in [21]. We need this:

Lemma 14. *Let $n \geq 1$ and $T \geq 1$. When $\log x \geq \frac{1}{2} n R \log^2 T$, we have the inequality*

$$I_n(T, x) := \int_T^\infty \frac{x^{-\varphi(t)}}{t^{n+1}} \log t \, \mathrm{d}t \leq \frac{4 + 2n \log T}{n^2 T^{\frac{n}{2}}} \exp\big(-\sqrt{\frac{2n \log x}{R}}\big). \tag{13}$$

When $\log x \leq \frac{1}{2} n R \log^2 T$, we have

$$\int_T^\infty \frac{x^{-\varphi(t)}}{t^{n+1}} \log t \, \mathrm{d}t \leq \big(\frac{4 + 2n \log T}{n^2 T^{\frac{n}{2}}}\big) x^{-\frac{1}{R \log T}}.$$

Proof. We transform the integral by writing $u = \log t$ to

$$I_n(T, x) = \int_{\log T}^\infty \exp\big(-\frac{\log x}{Ru} - nu\big) u \, \mathrm{d}u.$$

Now, this may be rewritten as

$$I_n(T, x) = \int_{\log T}^\infty \exp\big(-\frac{\log x}{Ru} - \frac{nu}{2}\big) e^{-\frac{nu}{2}} u \, \mathrm{d}u.$$

The function
$$\exp\big(-\frac{\log x}{Ru}-\frac{1}{2}nu\big) \tag{14}$$
has a maximum at $u=\sqrt{\frac{2\log x}{nR}}$ (which is greater than or equal to $\log T$ by assumption) so we have

$$\begin{aligned}I_n(T,x) &\le \exp\big(-\sqrt{\frac{2n\log x}{R}}\big)\int_{\log T}^{\infty} ue^{-\frac{nu}{2}}\,\mathrm{d}u\\ &= \frac{4+2n\log T}{n^2T^{\frac{n}{2}}}\exp\big(-\sqrt{\frac{2n\log x}{R}}\big).\end{aligned}$$

The second assertion is obvious since then the function (14) is decreasing in the interval of integration. □

Of course, the factor $\frac{1}{2}$ in (14) may be replaced by any positive $\epsilon<1$; in that case, (13) will become

$$I_n(T,x)\le\frac{1+(1-\epsilon)n\log T}{(1-\epsilon)^2n^2T^{(1-\epsilon)n}}\exp\big(-2\sqrt{\frac{\epsilon n\log x}{R}}\big),$$

valid for $\log x\ge\epsilon nR\log^2 T$. This is interesting if we want to gain in powers of T in the denominator to the detriment of the factor of $\log x$ inside the exponential, and vice versa. For example, by choosing $\epsilon=\frac{1}{4}$, we get that

$$I_n(T,x)\le\frac{16+12n\log T}{9n^2T^{\frac{3}{4}n}}\exp\big(-\sqrt{\frac{n\log x}{R}}\big)$$

when $\log x\ge\frac{1}{4}nR\log^2 T$.

Lemma 15. *For $x\ge 1$, we have*

$$\begin{aligned}S_m(x)\le\frac{S_m(1)}{\sqrt{x}}&+\big(\frac{0.67}{T_0}\log\frac{T_0}{2\pi}-\frac{\log 2\pi}{2m\pi}\big)\frac{x^{-\varphi(T_0)}}{T_0^m}+\frac{1}{2\pi}I_m(T_0,x)\\ &+0.67\big(m+1+\frac{\log x}{R\log^2 T_0}\big)I_{m+1}(T_0,x).\end{aligned} \tag{15}$$

Moreover, if $m\le 3.6\times10^{10}\le\big(T_0\log(2\pi)\big)/\big(1.34\pi\log(\frac{T_0}{2\pi})\big)$, then we can ignore the second term in (15), that is,

$$S_m(x)\le\frac{S_m(1)}{\sqrt{x}}+\frac{1}{2\pi}I_m(T_0,x)+0.67\big(m+1+\frac{\log x}{R\log^2 T_0}\big)I_{m+1}(T_0,x).$$

Proof. Since $1-\rho$ is a nontrivial zero whenever ρ is, we have

$$\begin{aligned} S_m(x) &= \sum_{\gamma>0} \frac{x^{-\beta}+x^{\beta-1}}{\gamma^{m+1}} \\ &= \frac{2}{\sqrt{x}} \sum_{0<\gamma\le T_0} \frac{1}{\gamma^{m+1}} + \sum_{\gamma>T_0} \frac{x^{-\beta}+x^{\beta-1}}{\gamma^{m+1}}. \end{aligned}$$

Using (8), we easily see that $x^{-\beta}+x^{\beta-1} \le x^{-\frac{1}{2}} + x^{-\varphi(\gamma)}$, so that

$$\sum_{\gamma>T_0} \frac{x^{-\beta}+x^{\beta-1}}{\gamma^{m+1}} \le \frac{1}{\sqrt{x}} \sum_{\gamma>T_0} \frac{1}{\gamma^{m+1}} + \sum_{\gamma>T_0} \frac{x^{-\varphi(\gamma)}}{\gamma^{m+1}};$$

we can apply (10) to the first sum, and we evaluate the second sum as follows. Set $\varphi_m(t) = \frac{x^{-\varphi(t)}}{t^{m+1}}$. We write

$$\begin{aligned} \sum_{\gamma>T_0} \frac{x^{-\varphi(\gamma)}}{\gamma^{m+1}} &= -\int_{T_0}^{\infty} N(t)\varphi_m'(t)\,\mathrm{d}t - N(T_0)\varphi_m(T_0) \\ &= \int_{T_0}^{\infty} (N^*(t)-N(t))\varphi_m'(t)\,\mathrm{d}t - \int_{T_0}^{\infty} N^*(t)\varphi_m'(t)\,\mathrm{d}t - N(T_0)\varphi_m(T_0). \end{aligned}$$

Integration by parts (of the middle term) and an appeal to the asymptotic (9) yields

$$\begin{aligned} \sum_{\gamma>T_0} \frac{x^{-\varphi(\gamma)}}{\gamma^{m+1}} &= (N^*(T_0)-N(T_0))\varphi_m(T_0) + \int_{T_0}^{\infty} (N^*(t)-N(t))\varphi_m'(t)\,\mathrm{d}t \\ &\quad + \frac{1}{2\pi} \int_{T_0}^{\infty} \frac{x^{-\varphi(t)}}{t^{m+1}} \log\frac{t}{2\pi}\,\mathrm{d}t \\ &= (N^*(T_0)-N(T_0)) \frac{x^{-\varphi(T_0)}}{T_0^{m+1}} - \frac{\log 2\pi}{2\pi} \int_{T_0}^{\infty} \frac{x^{-\varphi(t)}}{t^{m+1}}\,\mathrm{d}t \\ &\quad + \mathcal{O}^*\big(0.67(m+1+\frac{\log x}{R\log^2 T_0})I_{m+1}(T_0,x)\big) + \mathcal{O}^*\big(\frac{1}{2\pi} I_m(T_0,x)\big). \end{aligned}$$

The first assertion follows readily; the second is obvious in view of the first. □

Corollary 16. *For $\log x \ge R\log^2 T_0$, we have*

$$S_2(x) \le \frac{0.001460}{\sqrt{x}} + 2\times 10^{-12} \exp\big(-2\sqrt{\frac{\log x}{R}}\big).$$

For $\log x \le R\log^2 T_0$, the following inequality holds:

$$\begin{aligned} S_2(x) \le{}& \frac{0.001460}{\sqrt{x}} + 2\times 10^{-12} x^{-\frac{1}{29R}} \\ &+ 3.5\times 10^{-18}\big(3+\frac{\log x}{813R}\big)x^{-\frac{1}{29R}}. \end{aligned}$$

Proof. The number 0.001460 comes from a Pari/GP computation making use of the file of the first 100000 zeros of $\zeta(s)$ provided by [17] (see also [18]); we only need to use the first few thousand zeros to get a 5-digit precision (in fact, we have used the first 20000 zeros and get 0.00145909... Since the 20001$^{\text{st}}$ zero has $\gamma = 18047.13453033...$ and $1/\gamma^3$ is then about 1.7×10^{-13}, we do not run the risk of committing a significant error, given the 28-digit precision in Pari/GP). ☐

Note that the equation (10) and our evaluation of $S_m(x)$ prove that for small x, one may profitably use the bound given in the following:

Lemma 17. *For $x \geq 1$, we have the inequality*

$$S_m(x) \leq \frac{S_m(1)}{\sqrt{x}} + \frac{1}{2\pi m T_0^m}\big(\log\frac{T_0}{2\pi} + \frac{1}{m}\big) + \frac{0.67}{T_0^{m+1}}\big(2\log\frac{T_0}{2\pi} + 1\big).$$

This gives, in particular, that

$$S_2(x) \leq \frac{0.001460}{\sqrt{x}} + 3.7 \times 10^{-25} \quad (x \geq 2). \tag{16}$$

Finally, we will extensively use the following formula ([20, Lemma 4]) which relates important functions of prime numbers with the nontrivial zeros of the Riemann zeta function:

Lemma 18. *Let $g \in \mathcal{C}^1[a,b]$ with $2 \leq a < b < +\infty$. Then we have*

$$\int_a^b (\psi(t) - t) g(t)\,\mathrm{d}t = -\sum_\rho \int_a^b \frac{t^\rho}{\rho} g(t)\,\mathrm{d}t + \int_a^b (\log 2\pi - \tfrac{1}{2}\log(1 - t^{-2})) g(t)\,\mathrm{d}t,$$

where the sum runs through all the nontrivial zeros of the Riemann zeta function.

3. Proof of the Theorems

Proof of Theorem 4. We easily see, using (Stieltjes) integration by parts, that

$$\begin{aligned}\lambda(x) &= \int_{2^-}^x \frac{\mathrm{d}\vartheta(t)}{t\log t} = \frac{\vartheta(x)}{x\log x} + \int_2^x \frac{1+\log t}{t^2\log^2 t}\vartheta(t)\,\mathrm{d}t \\ &= \log\log x + B + \frac{\vartheta(x) - x}{x\log x} - \int_x^\infty \frac{1+\log t}{t^2\log^2 t}(\vartheta(t) - t)\,\mathrm{d}t \qquad (17)\end{aligned}$$

where

$$\begin{aligned}B &= \frac{1}{\log 2} - \log\log 2 + \int_2^\infty \frac{1+\log t}{t^2\log^2 t}(\vartheta(t) - t)\,\mathrm{d}t \\ &= \gamma + \sum_p \big\{\log(1 - \frac{1}{p}) + \frac{1}{p}\big\} \\ &= 0.26149\,72128\,47643\ldots\end{aligned}$$

is called the Meissel-Mertens constant; see [23, (2.10)] for the numerical value and [12, p. 23] for the second line. Here γ is the Euler-Mascheroni constant. The integral in (17) is

$$\begin{aligned}&\int_x^\infty \frac{1+\log t}{t^2\log^2 t}(\vartheta(t)-t)\,\mathrm{d}t\\&=\int_x^\infty \frac{1+\log t}{t^2\log^2 t}(\vartheta(t)-\psi(t))\,\mathrm{d}t+\int_x^\infty \frac{1+\log t}{t^2\log^2 t}(\psi(t)-t)\,\mathrm{d}t. \qquad (18)\end{aligned}$$

Since we know that (see [24, Theorem 6])

$$0\le \psi(t)-\vartheta(t)\le 1.0012\sqrt{t}+3t^{\frac{1}{3}} \quad (t>0),$$

we have

$$\int_x^\infty \frac{1+\log t}{t^2\log^2 t}(\psi(t)-\vartheta(t))\,\mathrm{d}t\le \big(\frac{2.0024}{\sqrt{x}}+\frac{9}{2x^{\frac{2}{3}}}\big)\frac{1+\log x}{\log^2 x}, \qquad (19)$$

which gives an explicit estimate for the first integral in (18). In order to estimate the second integral, we use Lemma 18 with $g(t)=\frac{1+\log t}{t^2\log^2 t}$ to get

$$\begin{aligned}\int_x^Y \frac{1+\log t}{t^2\log^2 t}(\psi(t)-t)\,\mathrm{d}t&=-\sum_\rho\int_x^Y\frac{t^{\rho-2}}{\rho}\frac{1+\log t}{\log^2 t}\,\mathrm{d}t\\&\quad+\int_x^Y(\log 2\pi-\tfrac{1}{2}\log(1-t^{-2}))\frac{1+\log t}{t^2\log^2 t}\,\mathrm{d}t. \qquad (20)\end{aligned}$$

We easily see that

$$\int_x^Y \frac{1+\log t}{t^2\log^2 t}\,\mathrm{d}t\le\frac{1+\log x}{x\log^2 x}$$

and

$$\begin{aligned}-\int_x^Y \log(1-t^{-2}))\frac{1+\log t}{t^2\log^2 t}\,\mathrm{d}t&\le-\frac{1+\log x}{\log^2 x}\int_x^Y\frac{\log(1-t^{-2})}{t^2}\,\mathrm{d}t\\&=\frac{1+\log x}{\log^2 x}\bigg(\log\frac{Y+1}{Y-1}+\log\frac{x-1}{x+1}+\frac{1}{Y}\log\frac{Y^2-1}{Y^2}\\&\quad-\frac{1}{x}\log\frac{x^2-1}{x^2}+\frac{2}{x}-\frac{2}{Y}\bigg).\end{aligned}$$

We would like to send Y to infinity; for this, it suffices to prove the absolute convergence of all the sums and integrals in (20). First of all, integration by parts gives

$$\begin{aligned}\int_x^Y\frac{t^{\rho-2}}{\rho}\frac{1+\log t}{\log^2 t}\,\mathrm{d}t&=\frac{1+\log Y}{\log^2 Y}\frac{Y^{\rho-1}}{\rho(\rho-1)}-\frac{1+\log x}{\log^2 x}\frac{x^{\rho-1}}{\rho(\rho-1)}\\&\quad+\int_x^Y\frac{t^{\rho-2}}{\rho(\rho-1)}\frac{2+\log t}{\log^3 t}\,\mathrm{d}t.\end{aligned}$$

This last integral is clearly absolutely convergent and since we know that the sum $\sum_\rho \frac{1}{\rho(\rho-1)}$ converges absolutely, we can let Y tend to infinity on the right of (20) and on the left as well. We thus obtain

$$\int_x^\infty \frac{1+\log t}{t^2\log^2 t}(\psi(t)-t)\,\mathrm{d}t = \frac{1+\log x}{\log^2 x}\sum_\rho \frac{x^{\rho-1}}{\rho(\rho-1)} - \sum_\rho \int_x^\infty \frac{t^{\rho-2}}{\rho(\rho-1)}\frac{2+\log t}{\log^3 t}\,\mathrm{d}t$$
$$+ \mathcal{O}^*\big(\frac{1+\log x}{x\log^2 x}\log(2\pi e)\big).$$

Now,

$$\begin{aligned}\int_x^\infty \frac{t^{\rho-2}}{\rho(\rho-1)}\frac{2+\log t}{\log^3 t}\,\mathrm{d}t &= -2\int_x^\infty \frac{t^{\rho-2}}{\rho(\rho-1)}\big(\int_t^\infty \frac{3+\log u}{u\log^4 u}\,\mathrm{d}u\big)\,\mathrm{d}t\\ &= -2\int_x^\infty \frac{3+\log u}{u\log^4 u}\big(\int_x^u \frac{t^{\rho-2}}{\rho(\rho-1)}\,\mathrm{d}t\big)\,\mathrm{d}u\\ &= -2\int_x^\infty \frac{3+\log u}{u\log^4 u}\big(\frac{u^{\rho-1}-x^{\rho-1}}{\rho(\rho-1)^2}\big)\,\mathrm{d}u.\end{aligned}$$

The absolute value of the left member is therefore

$$\begin{aligned}&\le \frac{4x^{\beta-1}}{|\rho(\rho-1)^2|}\int_x^\infty \frac{3+\log u}{u\log^4 u}\,\mathrm{d}u = \frac{2x^{\beta-1}}{|\rho(\rho-1)^2|}\frac{2+\log x}{\log^3 x}\\ &\le \frac{4+2\log x}{\log^3 x}\frac{x^{\beta-1}}{|\gamma|^3}.\end{aligned}$$

Using this and (18) in (17) we get

$$\begin{aligned}\lambda(x) &= \log\log x + B + \frac{\vartheta(x)-x}{x\log x}\\ &\quad+ \frac{1+\log x}{\log^2 x}J(x) + \mathcal{O}^*\left(\frac{4+2\log x}{\log^3 x}S_2(x)\right)\\ &\quad+ \mathcal{O}^*\left(\big(\frac{2.0024}{\sqrt{x}} + \frac{9}{2x^{\frac{2}{3}}} + \frac{\log(2\pi e)}{x}\big)\frac{1+\log x}{\log^2 x}\right).\end{aligned}$$

This is valid for any $x \ge 2$.

Lemma 13 and the estimate (16) give

$$\begin{aligned}\lambda(x) &= \log\log x + B + \frac{\vartheta(x)-x}{x\log x} + \mathcal{O}^*\left(\frac{1+\log x}{\log^2 x}\alpha(x)\right)\\ &\quad+ \mathcal{O}^*\left(\frac{4+2\log x}{\log^3 x}\big(\frac{0.001460}{\sqrt{x}} + 3.7\times 10^{-25}\big)\right)\end{aligned} \tag{21}$$

for all $x \ge 2$, where

$$\alpha(x) = \frac{2.0494}{\sqrt{x}} + \frac{4.5}{x^{\frac{2}{3}}} + \frac{2.84}{x} + 1.75\times 10^{-12}.$$

Retaining only the biggest terms and rounding off appropriately gives the first statement of the theorem. For bigger values of x, we use Lemmas 13–15 and the fact that

$$\Big(\frac{2.0494}{\sqrt{x}}+\frac{9}{2x^{\frac{2}{3}}}+\frac{\log(2\pi e)}{x}\Big)\frac{1+\log x}{\log^2 x}\leq\frac{3}{\sqrt{x}\log x},\ \frac{1+\log x}{\log^2 x}\leq\frac{1.09}{\log x}$$
$$\frac{4+2\log x}{\log^3 x}\leq\frac{2.36}{\log^2 x}$$

as soon as $x\geq 74000$. The proof is complete. □

Proof of Theorem 5. Let us put $\psi_\epsilon(x)=\sum_{p\leq x}\log(1+\frac{\epsilon}{p})$ for $0<|\epsilon|<2$. We have

$$\log(1+\frac{\epsilon}{p})-\frac{\epsilon}{p}=-\sum_{n=2}^{\infty}(-1)^n\frac{\epsilon^n}{np^n}$$

so that

$$\psi_\epsilon(x)-\epsilon\lambda(x)=\gamma(\epsilon)+\sum_{p>x}\sum_{n=2}^{\infty}(-1)^n\frac{\epsilon^n}{np^n} \tag{22}$$

with

$$\gamma(\epsilon)=\sum_{p}\sum_{n=2}^{\infty}(-1)^{n+1}\frac{\epsilon^n}{np^n}.$$

The sum in (22) is easily seen to be less than $\frac{|\epsilon|}{2}\log\frac{x-1}{x-1-|\epsilon|}$. Thus

$$|\psi_\epsilon(x)-\epsilon\lambda(x)-\gamma(\epsilon)|<\frac{|\epsilon|}{2}\log\frac{x-1}{x-1-|\epsilon|}.$$

Taking exponentials and using our estimates on $\lambda(x)$ in Corollary 20 (below), we obtain the result. If necessary, one may employ elementary inequalities, such as

$$(1+t)^\varepsilon<1+\varepsilon t\quad(0<t,\varepsilon<1),$$

and so on, in order to obtain the form we have given of the error term. The theorem is stated with an error term of $\mathcal{O}(1/\log^3 x)$ for convenience, although one may state it for error terms based on other functions, such as the one in Corollary 3. Also, for smaller values of x one may apply the results in Corollary 20. □

Proof of Theorem 6. We use the same procedure as in the proof of Theorem 4; here the functions involved are even simpler. In effect,

$$\begin{aligned}\Upsilon(x)&=\int_{2^-}^{x}\frac{\mathrm{d}\vartheta(t)}{t}=\frac{\vartheta(x)}{x}+\int_{2}^{x}\frac{\vartheta(t)}{t^2}\,\mathrm{d}t\\&=\log x+E+\frac{\vartheta(x)-x}{x}+\int_{x}^{\infty}\frac{\psi(t)-\vartheta(t)}{t^2}\,\mathrm{d}t-\int_{x}^{\infty}\frac{\psi(t)-t}{t^2}\,\mathrm{d}t,\end{aligned}$$

and this last integral is the same as the one occurring in [21, Proof of Lemma 2.2]; the result follows immediately. See also the proof of Theorem 9 which we have given below. □

Proof of Theorem 9. We proceed again by integration by parts:

$$\tilde{\psi}(x) = \int_{2^-}^{x} \frac{\mathrm{d}\psi(t)}{t} = \frac{\psi(x)}{x} + \int_{2}^{x} \frac{\psi(t)}{t^2}\,\mathrm{d}t$$
$$= \log x - \gamma + \frac{\psi(x)-x}{x} - \int_{x}^{\infty} \frac{\psi(t)-t}{t^2}\,\mathrm{d}t$$

where

$$\gamma = \log 2 - 1 - \int_{2}^{\infty} \frac{\psi(t)-t}{t^2}\,\mathrm{d}t$$

is Euler's constant (see [14, § 55]). Proceeding as in [21, Proof of Lemma 2.2], we get

$$\int_{x}^{\infty} \frac{\psi(t)-t}{t^2}\,\mathrm{d}t = J(x) - \frac{B(x)}{x}$$

with

$$B(x) = \frac{x}{2}\log\left(\frac{x+1}{x-1}\right) + \log\left(1-\frac{1}{x^2}\right) - \log(2\pi) - 1. \tag{23}$$

Thus

$$\tilde{\psi}(x) = \log x - \gamma + \frac{\psi(x)-x}{x} - J(x) + \frac{B(x)}{x}.$$

Using our estimates (11) and (12) for $J(x)$ and the fact that

$$|B(x)| \leq \log(2\pi) + 1 + \log 2 - 1.5\log 3 = 1.88310581... \quad (x \geq 2)$$
$$< 1.884,$$

we obtain the result. □

Remark. There is a minor error in [21] in the evaluation of the integral I in the proof of Theorem 1.1. The estimates for $J(x)$ given in [21, Proof of Theorem 1.1] should be replaced by our Lemma 13. The expression for $B(x)$ in [21, Lemma 2.2] should also be replaced by our expression (23).

4. Results for Bounded Intervals

Corollary 3 gives good results for very large values of x. For example, when $x = \exp(20000)$, Corollary 3 says that the error in approximating $\sum_{p\leq x} 1/p$ by $\log\log x + B$ is less than 1.33×10^{-29}, which is very interesting to know, since we cannot easily compute all primes $\leq \exp(20000)$. In this section, we give bounds for moderately big values of x. We first state a corollary of Theorem 4 for big x:

Corollary 19. *For $x \geq \exp(4635)$, we have*

$$\lambda(x) = \log\log x + B + \mathcal{O}^*\Big(\frac{0.21}{\log^3 x}\Big).$$

Proof. This comes immediately from (3) and the estimate on $|\vartheta(x) - x|$ given in [6, Théorème 1.4]. $\square$

n	b_n	ϵ_n	η_n
1	18.42	$1.186414000 \times 10^{-3}$	0.522463178
2	19	$9.416472060 \times 10^{-4}$	0.438928475
3	20	$6.302000000 \times 10^{-4}$	0.314349592
4	21	$4.197685060 \times 10^{-4}$	0.230174445
5	22	$2.786520000 \times 10^{-4}$	0.165125235
6	23	$1.843645000 \times 10^{-4}$	0.117791272
7	24	$1.216119620 \times 10^{-4}$	0.083581394
8	25	$7.998895869 \times 10^{-5}$	0.072862959
9	30	$9.778040657 \times 10^{-6}$	0.024445214
10	50	$9.049928595 \times 10^{-8}$	0.000905011
11	100	$8.842626429 \times 10^{-8}$	0.003537121
12	200	$8.561316979 \times 10^{-8}$	0.013698388
13	400	$8.000089705 \times 10^{-8}$	0.028800954
14	600	$7.442047763 \times 10^{-8}$	0.074422230
15	1000	$6.337118668 \times 10^{-8}$	0.107100266
16	1300	$5.518819789 \times 10^{-8}$	0.124177386
17	1500	$4.980115883 \times 10^{-8}$	0.161361428
18	1800	$4.191337100 \times 10^{-8}$	0.167660488
19	2000	$3.674711889 \times 10^{-8}$	0.194401521
20	2300	$2.917036000 \times 10^{-8}$	0.182325692
21	2500	$2.439460000 \times 10^{-8}$	0.184497402
22	2750	$1.876943507 \times 10^{-8}$	0.168940671
23	3000	$1.376020000 \times 10^{-8}$	0.168583894
24	3500	$6.165300000 \times 10^{-9}$	0.098672807
25	4000	$2.405714403 \times 10^{-9}$	0.053180897
26	4700	$1.734200000 \times 10^{-12}$	0.000087114
27	10000	$6.228800000 \times 10^{-18}$	0.000338143
28	20000	$2.229400000 \times 10^{-25}$	–
n	b_n	ϵ_n	η_n

Table 1: $|\lambda(x) - \log\log x - B| \leq \frac{\eta_n}{\log^3 x}$ for $e^{b_n} \leq x \leq e^{b_{n+1}}$ and $|\psi(x) - x| \leq \epsilon_n x$ for $x \geq e^{b_n}$.

We now determine the constants required for smaller values of x in order to get an error term of $\mathcal{O}\left({}^{1}/_{\log^3 x}\right)$. For this, we use the second assertion of Theorem 4 together with [6, Table 1.1] and [8, Table 2]. Using the inequality (5) and (21), we obtain

$$|\lambda(x) - \log\log x - B| \le \frac{1}{\log x}\big(\frac{1.0012}{\sqrt{x}} + \frac{3}{x^{\frac{2}{3}}} + \epsilon\big) + \frac{1+\log x}{\log^2 x}\alpha(x) \qquad (24)$$
$$+ \frac{4+2\log x}{\log^3 x}\big(\frac{0.001460}{\sqrt{x}} + 3.7\times 10^{-25}\big),$$

valid for $x \ge e^b$, where the ϵ are as in the aforementioned tables. A Pari/GP computation then gives the inequalities

$$|\lambda(x) - \log\log x - B| \le \frac{\eta_n}{\log^3 x} \quad \big(\exp(b_n) \le x \le \exp(b_{n+1})\big)$$

where b_n, ϵ_n and the corresponding η_n are tabulated in Table 1. Note that b_n and ϵ_n are correlated by the inequality

$$|\psi(x) - x| \le \epsilon_n x \quad \big(x \ge \exp(b_n)\big).$$

Also, we observe that η_n need not decrease with increasing b_n, as is clear from (24).

We also give the following short-interval result as a curiosity and to complement Table 1 (this table starts from $x = 10^8$):

Corollary 20. *We have the following bounds in the indicated intervals:*

$$\lambda(x) = \log\log x + B + \mathcal{O}^*\left(\frac{1.835}{\log^3 x}\right) \quad (2 \le x \le 10),$$
$$\lambda(x) = \log\log x + B + \mathcal{O}^*\left(\frac{3.690}{\log^3 x}\right) \quad (x \ge 10),$$
$$\lambda(x) = \log\log x + B + \mathcal{O}^*\left(\frac{0.820}{\log^3 x}\right) \quad (x \ge 50000),$$
$$\lambda(x) = \log\log x + B + \mathcal{O}^*\left(\frac{0.210}{\log^3 x}\right) \quad (x \ge 2\times 10^6, x \notin [10^8, \exp(22)]).$$

Proof. Write $f(x) = (\lambda(x) - \log\log x - B)\log^3 x$ for $x \ge 2$. We make a Pari/GP computation of all $f(k)$ for integers k in the range $2 \le k \le 10^8$. Table 2 gives the minima m_n and maxima M_n attained by $f(k)$ for k in the interval $x_n \le k \le x_{n+1}$. The columns y_n, Y_n are the unique integers $x_n \le y_n, Y_n \le x_{n+1}$ for which the quantities $m_n = f(y_n)$ and $M_n = f(Y_n)$ are the smallest and biggest, respectively. The quantities m_n and M_n given are truncated after the sixth decimal digit without rounding off. The value given for the last row corresponds to the number $f(10^8)$. Also, our calculations show that f does not change sign in $[2, 10^{18}]$.

We remark that to find the maxima of $f(x)$ for $4 \le x \le 10^8$, it is enough to evaluate $f(x)$ at integral and prime x, since $f(x)$ decreases between two consecutive primes, attaining its local maxima at primes (because the derivative $f'(x)$ of $f(x)$ is negative as soon as $x \ge 4$). $\square$

n	x_n	β_n	y_n	m_n	Y_n	M_n
1	2	1.835	2	0.201485	7	1.834441
2	10	3.055	58	1.186615	73	3.054472
3	100	3.690	556	0.715234	113	3.689944
4	1000	2.247	1422	0.312136	1327	2.246529
5	5000	1.425	7450	0.356194	5881	1.424019
6	10000	1.270	19372	0.159575	10343	1.269310
7	20000	1.107	32050	0.187937	24137	1.106448
8	50000	0.820	69990	0.165231	59797	0.819324
9	100000	0.596	302830	0.067158	102679	0.595960
10	500000	0.343	643846	0.103429	617819	0.342335
11	700000	0.288	993820	0.085181	910229	0.287257
12	1000000	0.275	1090696	0.053584	1195247	0.274719
13	2000000	0.209	4409886	0.036799	2275771	0.208742
14	5000000	0.151	9993078	0.036926	5001779	0.150128
15	10000000	0.120	10219590	0.026636	12871811	0.119603
16	30000000	0.089	36917098	0.009107	30909673	0.088092
17	50000000	0.057	65404318	0.016282	51841303	0.056192
18	70000000	0.055	89823540	0.015339	76020569	0.054421
19	90000000	0.041	93798766	0.015401	97931143	0.040071
20	100000000	–	–	0.025190	–	–

Table 2: $|\lambda(x) - \log\log x - B| \le \frac{\beta_n}{\log^3 x}$ for $x_n \le x \le x_{n+1}$.

We also read from the table that

$$|\lambda(x) - \log\log x - B| \ge \frac{0.009}{\log^3 x}$$

for $2 \le x \le 10^8$, although such a lower bound cannot hold for all x, in view of Corollary 3.

Finally, in view of our computations and theoretical results, the following result is clear:

Theorem 21. *For $x \ge 24284$, we have*

$$\sum_{p \le x} \frac{1}{p} = \log\log x + B + \mathcal{O}^*\left(\frac{1}{\log^3 x}\right). \tag{25}$$

Indeed, our computation shows that (25) does not hold for $x = 24283$ but holds for $24284 \le x \le 10^8$, hence for all $x \ge 24284$ in view of our theoretical results. Corollary 1 can be read off immediately from our tables and other results of this section.

Acknowledgement. The author thanks Olivier Ramaré for his guidance during his thesis work of which this paper forms a part, the anonymous referee for suggesting some corrections, and Prof. Bruce Landman for pointing out several typographical and stylistic inconsistencies in the original manuscript.

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