ON COMBINATORIAL IDENTITIES OF ENGBERS AND STOCKER

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Abstract
We extend two combinatorial identities published by Engbers and Stocker in 2016. Among others, we prove that if \( b, n \) and \( r \) are integers such that \( b \geq 1 \) and \( n-1 \geq r \geq 0 \), then
\[
\sum_{k=0}^{r} \binom{r}{k}^2 \binom{k+n}{2r+b} = \sum_{k=0}^{n-1} \binom{k}{r}^2 \binom{n-k-1}{b-1}.
\]
The special case \( b = 1 \) is due to Engbers and Stocker.

1. Introduction and Statement of Results

The work on this note is inspired by an interesting research paper published by Engbers and Stocker [1] in 2016. The authors use combinatorial techniques to show that the identities
\[
\sum_{k=0}^{r} \binom{r}{k}^2 \binom{k+n}{2r+1} = \sum_{k=r}^{n-1} \binom{k}{r}^2
\]
and
\[
\sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \binom{n}{k+1} = \sum_{k=r}^{n-1} \binom{k}{r}^2
\]
are valid for all integers \( n \) and \( r \) with \( n-1 \geq r \geq 0 \). Actually, they prove a bit more. They present summation formulas involving \( \sum_{k=r}^{n-1} \binom{k}{s}^3 \), where \( s \) is a natural number. The identities (1) and (2) turn out to be the most attractive special cases.
Here, we provide a different kind of extension. We study the sums

\[ S_{r,n}(b) = \sum_{k=0}^{r} \binom{r}{k}^2 \binom{k+n}{2r+b} \]

and

\[ T_{r,n}(b) = \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \binom{n}{k+b}, \]

where \( b, n \) and \( r \) are integers with \( n-1 \geq r \geq 0 \). In the next sections, we use the concept of generating functions to prove new extensions of (1) and (2). Our extension of (1) reads as follows.

**Theorem 1.** Let \( b, n \) and \( r \) be integers with \( n-1 \geq r \geq 0 \).

(i) If \( b \geq 1 \), then

\[ S_{r,n}(b) = \sum_{k=r}^{n-1} \binom{k}{r}^2 \binom{n-k-1}{b-1}. \]

(ii) If \( b \leq 0 \), then

\[ S_{r,n}(b) = \sum_{k=0}^{-b} (-1)^{-b-k} \binom{k+n}{r}^2 \binom{-b}{k}. \]

The case \( b = 1 \) gives (1) whereas the special cases \( b = 0 \) and \( b = -1 \) lead to the elegant identities

\[ \sum_{k=0}^{r} \binom{r}{k}^2 \binom{k+n}{2r} = \binom{n}{r}^2 \]

and

\[ \sum_{k=0}^{r} \binom{r}{k}^2 \binom{k+n}{2r-1} = \binom{n+1}{r}^2 - \binom{n}{r}^2. \]

The sum

\[ U_{r,n}(b) = \sum_{k=1}^{r} \binom{r}{k-1} \binom{r}{k} \binom{k+n}{2r+b} \]

is closely related to \( S_{r,n}(b) \). We apply

\[ 2 \binom{r}{k-1} \binom{r}{k} = \binom{r+1}{k}^2 - \binom{r}{k-1}^2 - \binom{r}{k}^2 \]

to the previous equation and obtain the representation

\[ U_{r,n}(b) = \frac{1}{2} \left( S_{r+1,n}(b-2) - S_{r,n+1}(b) - S_{r,n}(b) \right). \]
Using (5) with \( b = 1, 0, -1 \), respectively, we conclude from Theorem 1 that the following counterparts of (1), (3) and (4) are valid:

\[
\sum_{k=1}^{r} \binom{r}{k-1} \binom{r}{k} \binom{k+n}{2r+1} = \binom{n}{r} \binom{n}{r+1} - \sum_{k=r}^{n-1} \binom{k}{r}^2,
\]
\[
\sum_{k=1}^{r} \binom{r}{k-1} \binom{r}{k} \binom{k+n}{2r} = \binom{n}{r-1} \binom{n+1}{r+1},
\]
\[
\sum_{k=1}^{r} \binom{r}{k-1} \binom{r}{k} \binom{k+n}{2r-1} = \binom{n+1}{r-1} \binom{n+2}{r+1} - \binom{n}{r-1} \binom{n+1}{r+1}.
\]

In Section 3, we prove a generalization of (2).

**Theorem 2.** Let \( b, n \) and \( r \) be integers with \( n - 1 \geq r \geq 0 \).

(i) If \( b \geq 1 \), then

\[ T_{r,n}(b) = \sum_{k=r}^{n-1} \binom{k}{r}^2 \binom{n-k-1}{b-1}. \]

(ii) If \( b \leq 0 \), then

\[ T_{r,n}(b) = \sum_{k=0}^{-b} (-1)^{b-k} \binom{k+n}{r}^2 \binom{-b}{k}. \]

In particular, the special cases \( b = 1 \) and \( b = 0, -1 \) lead to (2) and

\[
\sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \binom{n}{k} = \binom{n}{r}^2,
\]
\[
\sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \binom{n}{k-1} = \binom{n+1}{r}^2 - \binom{n}{r}^2,
\]
respectively.

**2. Proof of Theorem 1**

We define

\[ F_b(x, u) = \sum_{n,r \geq 0} S_{r,n}(b)x^n u^r \]

and

\[ q_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k. \]
Then, see [3, pp. 78, 81],
\[ \sum_{n \geq 0} u^n q_n(x) = \frac{1}{\sqrt{1 - 2(1 + x)u + (1 - x)^2u^2}}. \]

First, we consider the case \( b = 0 \). We obtain
\[
\sum_{n \geq 0} x^n \sum_{k=0}^{r} \binom{r}{k} \left( \frac{2r}{k + n} \right) = \sum_{k=0}^{r} \binom{r}{k} \frac{x^{2r-k}}{(1-x)^{2r+1}} = \frac{x^r}{(1-x)^{2r+1}} q_r(x)
\]
and furthermore
\[
F_0(x, u) = \sum_{r \geq 0} \sum_{n \geq 0} x^n \sum_{k=0}^{r} \binom{r}{k} \left( \frac{2r}{k + n} \right) = \frac{1}{1-x} \sum_{r \geq 0} \left( \frac{ux}{(1-x)^2} \right)^r q_r(x)
\]
\[
= \frac{1}{\sqrt{1 - 2(1 + u)x + (1 - u)^2x^2}}
\]
\[
= \sum_{n, r \geq 0} \binom{n}{r} x^n u^r.
\]

Comparing the coefficients of \( x^n u^r \), we find the identity
\[ S_{r,n}(0) = \sum_{k=0}^{r} \binom{r}{k} \left( \frac{2r}{k + n} \right) = \binom{n}{r}^2. \]

Now, let \( b \geq 1 \). Applying the following variant of the Vandermonde formula, see [2, p. 169],
\[ \frac{k + n}{2r + b} = \sum_{j=0}^{n-1} \binom{k + j}{2r} \binom{n - j - 1}{b - 1} \quad (0 \leq k \leq r) \]
we obtain
\[ S_{r,n}(b) = \sum_{k=0}^{r} \binom{r}{k} \sum_{j=0}^{n-1} \binom{k + j}{2r} \binom{n - j - 1}{b - 1} \]
\[ = \sum_{j=0}^{n-1} \binom{n - j - 1}{b - 1} S_{r,j}(0) = \sum_{j=0}^{n-1} \binom{n - j - 1}{b - 1} \binom{j}{r}^2. \]

Next, let \( b \leq 0 \). Using the Vandermonde type identity, see [2, p. 169],
\[ \frac{k + n}{2r + b} = \sum_{j=0}^{b} \binom{2r}{j} \binom{k + j + n}{2r} (-1)^{-b-j} \]
we get

\[ S_{r,n}(b) = \sum_{k=0}^{r} \binom{r}{k} \sum_{j=0}^{\frac{b}{2}} \binom{-b}{j} \binom{k+j+n}{2r} (-1)^{-b-j} \]

\[ = \sum_{j=0}^{\frac{b}{2}} \binom{-b}{j} (-1)^{-b-j} S_{r,j+n}(0) = \sum_{j=0}^{\frac{b}{2}} \binom{-b}{j} (-1)^{-b-j} \left( \frac{j+n}{r} \right)^2. \]

This completes the proof of Theorem 1.

3. Proof of Theorem 2

As before, we consider the bivariate generating function

\[ G_0(x,u) = \sum_{n,r \geq 0} T_{r,n}(b) x^n u^r. \]

We have, see [3, page 73]:

\[ \sum_{n \geq 0} \sum_{0 \leq k \leq n/2} \binom{2k}{k} \binom{n}{2k} x^{2k}(1-2x)^{n-2k} = \frac{1}{\sqrt{[1-(1-2x)u]^2 - 4x^2 u^2}}. \]

Next, we replace \( x \) by \( \sqrt{x}/(1 + 2\sqrt{x}) \) and \( u \) by \((1 + 2\sqrt{x})u\). This leads to

\[ \sum_{n \geq 0} \sum_{0 \leq k \leq n/2} \binom{2k}{k} \binom{n}{2k} x^k = \frac{1}{\sqrt{(1-u)^2 - 4ux^2}}. \]

Applying

\[ \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \sum_{n \geq 0} \binom{n}{k} z^n = \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \frac{z^k}{(1-z)^{k+1}} \]

we obtain with \( t = z/(1-z) \):

\[ G_0(z,u) = \sum_{r \geq 0} u^r \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \frac{z^k}{(1-z)^{k+1}} \]

\[ = \frac{1}{1-z} \sum_{r \geq 0} u^r \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} t^k \]

\[ = \frac{1}{1-z} \sum_{k \geq 0} \sum_{r \geq 0} u^{k-r} \binom{2r}{r} \binom{k}{2r} t^k \]
\[
\begin{align*}
&= \frac{1}{1-z} \sum_{k \geq 0} (ut)^k \sum_{r \geq 0} u^{-r} \binom{2r}{r} \binom{k}{2r} \\
&= \frac{1}{1-z} \frac{1}{\sqrt{(1-ut)^2 - 4ut^2}} \\
&= \frac{1}{\sqrt{1 - 2(1 + u)z + (1 - u)^2 z^2}} \\
&= \sum_{n, r \geq 0} \binom{n}{r}^2 z^n u^r.
\end{align*}
\]

We compare the coefficients of \(z^n u^r\) and find

\[
T_{r,n}(0) = \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \binom{n}{k} \binom{u^r}{2r} = \binom{n}{r}^2.
\]

Now, let \(b \geq 1\). Using

\[
\binom{n}{k+b} = \sum_{j=0}^{n-1} \binom{j}{k} \binom{n-j}{b-1}
\]

leads to

\[
T_{r,n}(b) = \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \sum_{j=0}^{n-1} \binom{j}{k} \binom{n-j}{b-1} \binom{n}{r}^2.
\]

Next, let \(b \leq 0\). Since

\[
\binom{n}{k+b} = \sum_{j=0}^{b} \binom{-b}{j} \binom{j+n}{k} (-1)^{-b-j},
\]

we obtain

\[
T_{r,n}(b) = \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \sum_{j=0}^{b} \binom{-b}{j} \binom{j+n}{k} (-1)^{-b-j}
\]

\[
= \sum_{j=0}^{b} \binom{-b}{j} (-1)^{-b-j} T_{r,j+n}(0) = \sum_{j=0}^{b} \binom{-b}{j} (-1)^{-b-j} \binom{j+n}{r}^2.
\]

The proof of Theorem 2 is complete.

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