A SIMPLE PROOF OF A CONJECTURE OF DOU ON $(3,7)$-REGULAR BIPARTITIONS MODULO 3

Chandrashekar Adiga
Department of Studies in Mathematics, University of Mysore
Manasagangotri, Mysuru-570006, Karnataka, India
c_adiga@hotmail.com

Ranganatha D.[1]
Department of Studies in Mathematics, University of Mysore
Manasagangotri, Mysuru-570006, Karnataka, India
ddranganatha@gmail.com

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Abstract

We provide a simple proof for the Ramanujan-type congruence for $(3,7)$-regular bipartitions modulo 3 which was conjectured by Donna.Q.J.Dou. Furthermore, we also find some new infinite families of congruences for $(3,7)$-regular bipartitions modulo 3.

1. Introduction

Throughout the paper, we assume that $|q|<1$. For a positive integer n, we use the standard notation

$$(a;q)_0 = 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k) \quad \text{and} \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

A partition of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r}\lambda_i = n$. The λ_i are called the parts of the partition. We shall set $p(0)=1$ and for $n \geq 1$, let $p(n)$ denote the number of partitions of n. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

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In 1919, Ramanujan [8], [9, pp.210–213] found nice congruence properties for $p(n)$ moduli 5, 7 and 11. Namely, for any nonnegative integer n,

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7}$$

and

$$p(11n+6) \equiv \ 0 \pmod{11}.$$

For a positive integer $\ell \geq 2$, a partition is called ℓ-regular if none of its parts is divisible by ℓ. Let $b_\ell(0) = 1$ and for $n \geq 1$, let $b_\ell(n)$ denote the number of ℓ-regular partitions of n. Then the generating function for $b_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty}.$$

A bipartition (λ,μ) of n is a pair of partitions (λ,μ) such that the sum of all the parts equals n. A (k,ℓ)-regular bipartition of n is a bipartition (λ,μ) of n such that λ is a k-regular partition and μ is a ℓ-regular partition. Let $B_{k,\ell}(n)$ denote the number of (k,ℓ)-regular bipartitions of n. Then the generating function of $B_{k,\ell}(n)$ is given by

$$\sum_{n=0}^{\infty} B_{k,\ell}(n)q^n = \frac{(q^k;q^k)_\infty(q^\ell;q^\ell)_\infty}{(q;q)_\infty^2}. \tag{1}$$

Recently, Lin [6, 7] discovered several infinite families of congruences for $B_{7,7}(n)$ and $B_{13,13}(n)$ modulo 3. For example, he proved that for $\alpha \geq 2$ and $n \geq 0$,

$$B_{7,7}\Big(3^\alpha n + \frac{5\cdot 3^{\alpha-1}-1}{2}\Big) \equiv 0 \pmod{3}$$

and

$$B_{13,13}\Big(3^\alpha n + 2\cdot 3^{\alpha-1} - 1\Big) \equiv 0 \pmod{3}.$$

Very recently, by using relations between certain cubic theta functions, Dou [4] proved the following congruence for $(3,11)$-regular bipartitions modulo 11.

Theorem 1. *([4, Theorem 1.1]) For all integers α, n with $\alpha \geq 2$ and $n \geq 0$, we have*

$$B_{3,11}\Big(3^\alpha n + \frac{5\times 3^{\alpha-1}-1}{2}\Big) \equiv 0 \pmod{11}. \tag{2}$$

In the same paper, Dou presented the following two conjectures on $B_{5,7}(n)$ and $B_{3,7}(n)$.

Conjecture 1. ([4, Conjecture 1.1]) For any $n \geq 0$,

$$B_{5,7}(7n+6) \equiv 0 \pmod{7}. \tag{3}$$

Conjecture 2. ([4, Conjecture 1.2, Equations 1.9, 1.10, 1.11]) For any $n \geq 0$,

$$B_{3,7}(An+B) \equiv 0 \pmod{2}, \tag{4}$$

$$B_{3,7}(Cn+D) \equiv 0 \pmod{3}, \tag{5}$$

$$B_{3,7}(En+F) \equiv 0 \pmod{9}, \tag{6}$$

where $(A,B) \in \{(14,4),(14,10),(16,1),(28,6),(32,21)\}$, $(C,D)=(4,3)$, $(E,F) \in \{(7,3),(7,4),(14,13),(21,6),(21,20),(25,3),(25,13),(25,18),(25,23)\}$.

In the final version of Dou's paper [4], she made the following note on page 537. *"We are informed that Conjecture 1.1 and Congruence (1.10) are both true and have been verified by the referee using the theory of modular forms. It would be interesting to give an elementary proof of Conjectures 1.1 and 1.2 by using other identities in q-series"*.

The main purpose of this paper is to highlight the importance of a q-series approach in settling part of Dou's Conjecture 1.2, namely Equation 1.10 [4]. We also establish some new infinite families of congruences modulo 3 for $B_{3,7}(n)$ which are analogous to (2).

2. Basic Results

For $|ab|<1$, Ramanujan's general theta function $f(a,b)$ is defined by [1]

$$f(a,b)=\sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

We need the following lemmas to prove our results:

Lemma 1. *([3, Theorem 2.2]). For any prime $p \geq 5$, we have*

$$(q;q)_\infty = \sum_{\substack{m=-\frac{p-1}{2} \\ m \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^m q^{\frac{3m^2+m}{2}} f\left(-q^{\frac{3p^2+(6m+1)p}{2}}, -q^{\frac{3p^2-(6m+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} (q^{p^2};q^{p^2})_\infty, \tag{7}$$

where the choice of the $\pm$ sign is made so that $(\pm p-1)/6$ is an integer. Note that $(3m^2+m)/2 \not\equiv (p^2-1)/24 \pmod{p}$ as m runs through the range of the summation.

Lemma 2. *([5]). We have*

$$(-q;q^2)_\infty(-q^7;q^{14})_\infty - (q;q^2)_\infty(q^7;q^{14})_\infty = 2q(-q^2;q^2)_\infty(-q^{14};q^{14})_\infty. \tag{8}$$

3. Main Results

In this section, we give a simple proof of (5) using q-series identities. We also prove some infinite families of congruences for $(3,7)$-regular bipartitions modulo 3. The following lemma is the crux of Theorem 2 settling part of Dou's Conjecture 1.2 [4].

Lemma 3. *Let* $\sum_{n=0}^{\infty} D(n)q^n = (q^7;q^7)_\infty(q;q)_\infty$. *Then*

$$(q^7;q^7)_\infty(q;q)_\infty = \sum_{n=0}^{\infty} D(2n)q^{2n} - q(q^4;q^4)_\infty(q^{28};q^{28})_\infty. \tag{9}$$

Proof. In view of (8), we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^n D(n)q^n - \sum_{n=0}^{\infty} D(n)q^n \\
&= (q^{14};q^{14})_\infty(q^2;q^2)_\infty\Big((-q^7;q^{14})_\infty(-q;q^2)_\infty - (q^7;q^{14})_\infty(q;q^2)_\infty\Big) \\
&= 2q(q^{14};q^{14})_\infty(q^2;q^2)_\infty(-q^{14};q^{14})_\infty(-q^2;q^2)_\infty \\
&= 2q(q^{28};q^{28})_\infty(q^4;q^4)_\infty,
\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} D(2n+1)q^{2n+1} = -q(q^{28};q^{28})_\infty(q^4;q^4)_\infty. \tag{10}$$

This completes the proof of Lemma 3. □

Theorem 2. *([4, Conjecture 1.2, Equation 1.10]) For any* $n \geq 0$,

$$B_{3,7}(4n+3) \equiv 0 \pmod 3. \tag{11}$$

Proof. Taking $k=3$ and $\ell=7$ in (1), we have

$$\sum_{n=0}^{\infty} B_{3,7}(n)q^n = \frac{(q^3;q^3)_\infty(q^7;q^7)_\infty}{(q;q)_\infty^2}. \tag{12}$$

By the binomial theorem, it is easy to see that for any positive integers k and m,

$$(q^m;q^m)_\infty^{3k} \equiv (q^{3m};q^{3m})_\infty^k \pmod 3. \tag{13}$$

Using (13) in (12), we have

$$\sum_{n=0}^{\infty} B_{3,7}(n)q^n = \frac{(q^3;q^3)_\infty(q^7;q^7)_\infty(q;q)_\infty}{(q;q)_\infty^3} \equiv (q^7;q^7)_\infty(q;q)_\infty \pmod 3. \tag{14}$$

Using (9) in the above congruence, we obtain

$$\sum_{n=0}^{\infty} B_{3,7}(n)q^n \equiv \sum_{n=0}^{\infty} D(2n)q^{2n} - q(q^4;q^4)_\infty(q^{28};q^{28})_\infty \pmod{3}. \tag{15}$$

Since there are no terms q^{4n+3} in (15), we find that for any $n \geq 0$

$$B_{3,7}(4n+3) \equiv 0 \pmod{3}.$$

□

Lemma 4. *For any integers $t, n \geq 0$, we have*

$$B_{3,7}\Big(4^t n + \frac{4^t-1}{3}\Big) \equiv (-1)^t B_{3,7}(n) \pmod{3}. \tag{16}$$

Proof. We will prove this lemma by induction on t. Congruence (16) is trivially true for $t = 0$. Extracting the terms of the form q^{4n+1} on both sides of (15) and then using (14), we obtain

$$\sum_{n=0}^{\infty} B_{3,7}(4n+1)q^n \equiv -(q^7;q^7)_\infty(q;q)_\infty \equiv -\sum_{n=0}^{\infty} B_{3,7}(n)q^n \pmod{3},$$

which implies that (16) holds for $t = 1$. Assume that (16) is true for $t = m$ ($m \geq 2$). Using (16) with $t = m$ and (9), we have

$$\sum_{n=0}^{\infty} B_{3,7}\Big(4^m n + \frac{4^m-1}{3}\Big)q^n$$
$$\equiv (-1)^m\Big(\sum_{n=0}^{\infty} B(2n)q^{2n} - q(q^4;q^4)_\infty(q^{28};q^{28})_\infty\Big) \pmod{3}. \tag{17}$$

Extracting the terms involving q^{4n+1} in the above congruence, we obtain

$$\sum_{n=0}^{\infty} B_{3,7}\Big(4^m(4n+1) + \frac{4^m-1}{3}\Big)q^n \equiv (-1)^{m+1}(q^7;q^7)_\infty(q;q)_\infty \pmod{3}.$$

Using (14) in the above congruence and then equating the coefficients of q^n, we see that (16) is holds for $t = m+1$. By induction, we see that (16) is true for any integer $t \geq 0$. □

Theorem 3. *For $t \geq 0$ and $n \geq 0$, we have*

$$B_{3,7}\Big(4^{t+1}n + \frac{5\times 2^{2t+1}-1}{3}\Big) \equiv 0 \pmod{3}, \tag{18}$$

$$B_{3,7}\Big(7\cdot 4^t n + \frac{5\times 2^{2t+1}-1}{3}\Big) \equiv 0 \pmod{3}, \tag{19}$$

$$B_{3,7}\Big(7\cdot 4^t n + \frac{13\cdot 4^t-1}{3}\Big) \equiv 0 \pmod{3} \tag{20}$$

and

$$B_{3,7}\Big(7\cdot 4^t n+\frac{19\cdot 4^t-1}{3}\Big)\equiv 0 \pmod 3. \tag{21}$$

Proof. From [2, Entry 17(v), p. 303], we recall that

$$(q;q)_\infty=(q^{49};q^{49})_\infty\left(\frac{E(q^7)}{C(q^7)}-q\frac{A(q^7)}{E(q^7)}-q^2+q^5\frac{C(q^7)}{A(q^7)}\right),$$

where

$$A(q):=\frac{f(-q^3,-q^4)}{(q^2;q^2)_\infty},\quad E(q):=\frac{f(-q^2,-q^5)}{(q^2;q^2)_\infty}\quad \textit{and}\quad C(q):=\frac{f(-q,-q^6)}{(q^2;q^2)_\infty}.$$

Employing above identity in (14) and then extracting the coefficients of q^{7n+r} for $r\in\{3,4,6\}$, we find that

$$B_{3,7}(7n+3)\equiv\ 0 \pmod 3, \tag{22}$$
$$B_{3,7}(7n+4)\equiv\ 0 \pmod 3, \tag{23}$$

and

$$B_{3,7}(7n+6)\equiv\ 0 \pmod 3. \tag{24}$$

Congruences (18)-(21) easily follows from (22)-(24), (16) and (5). □

In the statement of the following theorem, we use the Legendre symbol $\left(\frac{a}{p}\right)$. Let $p\geq 3$ be a prime. The Legendre symbol $\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right):=\begin{cases}1, & \text{if } a \text{ is a quadratic residue modulo p and } a\not\equiv\ 0 \pmod p,\\ -1, & \text{if } a \text{ is a quadratic nonresidue modulo p},\\ 0, & \text{if } a\equiv\ 0 \pmod p.\end{cases}$$

Theorem 4. *Let $p\geq 5$ be a prime such that $(\frac{-7}{p})=-1$. For $n,t\geq 0$, we have*

$$\sum_{n=0}^{\infty}B_{3,7}\Big(p^{2t}n+\frac{p^{2t}-1}{3}\Big)q^n\equiv (q;q)_\infty(q^7;q^7)_\infty \pmod 3. \tag{25}$$

Proof. Form (14), it follows that (25) is true for $t=0$. Suppose that (25) is true for $t=m$ $(m\geq 1)$. Now, we consider the congruence

$$7\cdot\frac{3m^2+m}{2}+\frac{3k^2+k}{2}\equiv\ \frac{p^2-1}{3} \pmod p, \tag{26}$$

where $-\frac{p-1}{2} \leq k, m \leq \frac{p^2-1}{2}$ and $\left(\frac{-7}{p}\right) = -1$. The congruence (26) can be rewritten as

$$(6k+1)^2 + 7(6m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-7}{p}\right) = -1$, we find that

$$6k+1 \equiv 6m+1 \equiv 0 \pmod{p},$$

which implies that $k = m = \frac{\pm p-1}{6}$ is the only solution of (26). Employing (7) in (25) with $t = m$ and then extracting the terms involving $q^{pn+\frac{p^2-1}{3}}$, we obtain

$$\sum_{n=0}^{\infty} B_{3,7}\Big(p^{2m}\Big(pn + \frac{p^2-1}{3}\Big) + \frac{p^{2m}-1}{3}\Big)q^{pn} \equiv (q^p;q^p)_\infty (q^{7p};q^{7p})_\infty \pmod{3}. \tag{27}$$

Replacing q^p by q in (27), we see that (27) is true for $t = m+1$. □

Corollary 1. *Let $p \geq 5$ be a prime such that $(\frac{-7}{p}) = -1$. For $n, t \geq 0$, we have*

$$B_{3,7}\Big(p^{2t+2}n + \frac{(3k+p)\cdot p^{2t+1}-1}{3}\Big) \equiv 0 \pmod{3},$$

for $1 \leq k \leq p-1$.

Proof. Equating the coefficients of q^{pn+k} for $1 \leq k \leq p-1$ in (27), we obtain the required congruence. □

During revision of our paper, we became aware of related work addressing all of Dou's conjectures submitted soon after the current article. The interested reader may compare our results and techniques with those of Xia and Yao [11] and Wang [10].

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References

[1] C. Adiga, B. C. Berndt, S. Bhargava and G. N. Watson, Chapter 16 of Ramanujan's second notebook: Theta functions and q-series, *Mem. Amer. Math. Soc.* **315** (1985), 1–91.

[2] B. C. Berndt, *Ramanujans Notebooks, Part III,* Springer-Verlag, New York, 1991.

[3] S. -P. Cui and N. S. S. Gu, Arithmetic properties of ℓ-regular partitions, *Adv. Appl. Math.* **51** (2013), 507–523.

[4] D. Q. J. Dou, Congruences for $(3, 11)$-regular bipartitions modulo 11, *Ramanujan J.* **40** (2016), 535–540.

[5] H. M. Farkas and I. Kra, Partitions and theta constant identities, *in: Analysis, Geometry, Number Theory: the Mathematics of Leon Ehrenpreis, Contemp. Math. 251* (2000), 197–203.

[6] B. L. S. Lin, Arithmetic of the 7-regular bipartition function modulo 3, *Ramanujan J.* **37** (2015), 469–478.

[7] B. L. S. Lin, An infinite family of congruences modulo 3 for 13-regular bipartitions, *Ramanujan J.* **39** (2016), 169–178.

[8] S. Ramanujan, Some properties of $p(n)$, the number of partitions of n, *Proc. Cambridge Philos. Soc.* **19** (1919), 210–213.

[9] S. Ramanujan, *Collected Papers*, Cambridge University Press, Cambridge, 1927.

[10] L. Wang, Arithmetic properties of (k, ℓ)-regular bipartitions, *Bull. Aust. Math. Soc.* doi:10.1017/S0004972716000964, to appear.

[11] E. X. W. Xia and O. X. M. Yao, Arithmetic properties for (s, t)-regular bipartition functions, *J. Number Theory* **171** (2017), 1–17.