REPRESENTATION OF REAL NUMBERS BY THE ALTERNATING CANTOR SERIES

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Abstract
The article is devoted to alternating Cantor series. It is proved that any real number belonging to \([a_0 - 1, a_0]\), where \(a_0 = \sum_{k=1}^{\infty} \frac{d_k-1}{d_1 \cdots d_k}\), has not more than two representations by such series, and only the numbers from a certain countable subset of real numbers have two representations. The geometry of these representations, properties of cylinder and semicylinder sets, and the simplest metric problems are investigated. Some applications of such series to fractal theory and the relation between positive and alternating Cantor series are described. The shift operator with some its applications, as well as the set of incomplete sums are studied. Necessary and sufficient conditions for a rational number to be representable by an alternating Cantor series are formulated.

Introduction
The investigation of various numeral systems is useful for the development of metric, probability, and fractal theories of real numbers, for the study of fractal and other properties of mathematical objects possessing a complicated local structure such as continuous nowhere differentiable or singular functions, random variables of Jessen-Wintner type, DP-transformations (transformations preserving the fractal Hausdorff-Besicovitch dimension), dynamical systems with chaotic trajectories, etc. [2, 3].

There exist systems of real number representations with a finite or an infinite alphabet, with redundant digits or with zero redundancy. The s-adic and nega-s-adic numeral systems [5] are examples of real number encodings with a finite alphabet, whereas number representations by Liouth series [6], regular continued fractions, polybasic nega-\(Q\)-representations [11], etc., are examples of encoding them with an infinite alphabet. A representation

\[ x = \frac{\varepsilon_1}{d_1} + \frac{\varepsilon_2}{d_1d_2} + \cdots + \frac{\varepsilon_n}{d_1d_2\cdots d_n} + \cdots, \varepsilon_n \in A_{d_n}, \]

(1)
of a real number \( x \) by a positive Cantor series \([1, 7, 8]\), is an example of a polybasic numeral system with zero redundancy. Here \((d_n)\) is a fixed sequence of positive integers, \(d_n > 1\), and \((A_{d_n})\) is a sequence of the sets \(A_{d_n} = \{0, 1, \ldots, d_n - 1\}\). This encoding of real numbers has a finite alphabet when \((d_n)\) is bounded.

The representation of real numbers by positive Cantor series is a generalization of the classical \(s\)-adic numeral system. Note that this representation is “similar” to the following series

\[
\sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n},
\]

where \((a_n)\) is a monotone non-decreasing sequence of positive integers and \(a_1 \geq 1\). This series is called an Engel series.

In 1869, Georg Cantor \([1]\) considered series expansions of real numbers (1). There are many papers \([1, 4, 7, 10, 8]\) where properties of real number representations by positive Cantor series are studied, but many problems related to these series are not solved completely. For example, criteria of representation of rational numbers, modeling of functions with a complicated local structure are still open problems.

Since real number expansions by positive Cantor series are useful for studying complicated objects of fractal analysis, the notion of alternating Cantor series is introduced in the present article. Alternating Cantor series, which generalize the nega-\(s\)-adic numeral system, were not considered in publications earlier. In this paper the foundations of the metric theory of real number representations by alternating Cantor series are given and some related problems of mathematical analysis are considered.

Consider the main object of this article.

Let \((d_n)\) be a fixed sequence of numbers from \(\mathbb{N} \setminus \{1\}\), \((A_{d_n})\) be a sequence of the sets \(A_{d_n} = \{0, 1, 2, \ldots, d_n - 1\}\).

**Definition 1.** A series of the form

\[
\frac{-\varepsilon_1}{d_1} + \frac{\varepsilon_2}{d_1d_2} - \frac{\varepsilon_3}{d_1d_2d_3} + \cdots + \frac{(-1)^n\varepsilon_n}{d_1d_2d_3 \cdots d_n} + \cdots,
\]

where \(\varepsilon_n \in A_{d_n}\) for any \(n \in \mathbb{N}\), is called an alternating Cantor series.

The number \(d_n\) is called the \(n\)th element of sum (2), and \(\varepsilon_n\) is called the \(n\)th digit of sum (2).

By \(\Delta_{\varepsilon_1 \varepsilon_2 \varepsilon_3 \ldots}^{\frac{1}{d_1} \frac{1}{d_1d_2} \frac{1}{d_1d_2d_3} \ldots}\) denote any number \(x\) having expansion (2). This notation is called the nega-\(D\)-representation of \(x\). Expansion (2) of \(x\) is called the nega-\(D\)-expansion of \(x\).

**Remark 1.** The term “nega” is used in this article, since the alternating Cantor series expansion is a numeral system with a negative base, i.e.,

\[
x = \sum_{n=1}^{\infty} \frac{(-1)^n\varepsilon_n}{d_1d_2 \cdots d_n} = \frac{-\varepsilon_1}{-d_1} + \frac{\varepsilon_2}{(-d_1)(-d_2)} + \cdots + \frac{\varepsilon_n}{(-d_1)(-d_2) \cdots (-d_n)} + \cdots
\]
If \((d_n)\) is purely periodic with the simple period \((s)\), where \(s > 1\) is a fixed positive integer, then series (2) has the form

\[
x = -\frac{\varepsilon_1}{s} + \frac{\varepsilon_2}{s^2} - \frac{\varepsilon_3}{s^3} + \cdots + \frac{(-1)^n \varepsilon_n}{s^n} + \cdots, \quad \varepsilon_n \in \{0, 1, \ldots, s - 1\}.
\]

The last-mentioned series is the nega-\(s\)-adic expansion \([5, 9]\) of numbers in \([-\frac{s}{s+1}, \frac{1}{s+1}]\).

The following series are alternating Cantor series:

1. \[
\sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon_n}{s^{\alpha_1 + \alpha_2 + \cdots + \alpha_n}},
\]
where \(\alpha_n\) belongs to some finite subset of positive integers, \(\varepsilon_n \in \{0, 1, \ldots, s^{\alpha_n} - 1\}\) for each \(n \in \mathbb{N}\), and \(1 < s \in \mathbb{N}\) is a fixed number;

2. \[
\sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon_n}{2 \cdot 3 \cdot \cdots \cdot (n + 1)}, \quad \varepsilon_n \in \{0, 1, \ldots, n\};
\]

3. \[
\sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon_n}{p_1 p_2 \cdots p_n},
\]
where \((p_n)\) is the increasing sequence of all prime numbers.

**Lemma 1.** Every alternating Cantor series is absolutely convergent and its sum belongs to \([a_0 - 1, a_0]\), where

\[
a_0 = \sum_{n=1}^{\infty} \frac{d_{2n} - 1}{d_1 d_2 \cdots d_{2n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{d_1 d_2 \cdots d_n}.
\]

**Proof.** This statement follows from the propositions:

- the series
  \[
  \sum_{n=1}^{\infty} \frac{\varepsilon_n}{d_1 d_2 \cdots d_n}
  \]
  is convergent;

- the condition
  \[
  -1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{d_1 d_2 \cdots d_n} \leq S \leq \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{d_1 d_2 \cdots d_n},
  \]
holds, where \(S\) is equal to the sum of series (2).
Lemma 2. Let series (2) be a fixed series,
\[ r_n = \frac{(-1)^n}{d_1d_2 \cdots d_n} \sum_{k=1}^{\infty} \frac{(-1)^k e_{n+k}}{d_{n+1} \cdots d_{n+k}}, \]
and 
\[ a_n = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{d_{n+1}d_{n+2} \cdots d_{n+k}} \]
for this series; then the following inequalities hold:
\[ \frac{a_n - 1}{d_1 \cdots d_n} \leq r_n \leq \frac{a_n}{d_1 \cdots d_n} \text{ whenever } n \text{ is even}; \]
\[ -\frac{a_n}{d_1d_2 \cdots d_n} \leq r_n \leq \frac{1 - a_n}{d_1d_2 \cdots d_n} \text{ whenever } n \text{ is odd.} \]

1. Representation of Real Numbers by Alternating Cantor Series

Lemma 3. Each number \( x \in [a_0 - 1, a_0] \) can be represented by series (2).

Proof. It is obvious that \( a_0 - 1 = \Delta_{D_{d_1]|d_2-1|0[d_4-1]0} \) and \( a_0 = \Delta_{D_{d_4-1}0[d_4-1]...} \).

Since \( x \) is an arbitrary number from \( (a_0 - 1, a_0) \),
\[ -\frac{\varepsilon_1}{d_1} - \sum_{k=2}^{\infty} \frac{d_{2k-1} - 1}{d_1d_2 \cdots d_{2k-1}} < x \leq \frac{\varepsilon_1}{d_1} + \sum_{k=1}^{\infty} \frac{d_{2k} - 1}{d_1d_2 \cdots d_{2k}} \]
with \( 0 \leq \varepsilon_1 \leq d_1 - 1 \), and
\[ [a_0 - 1, a_0] = I_0 = \bigcup_{i=0}^{d_1-1} \left[ -\frac{i}{d_1} - \sum_{k=2}^{\infty} \frac{d_{2k-1} - 1}{d_1d_2 \cdots d_{2k-1}} - \frac{i}{d_1} + \sum_{k=1}^{\infty} \frac{d_{2k} - 1}{d_1d_2 \cdots d_{2k}} \right], \]
we have
\[ -\sum_{k=2}^{\infty} \frac{d_{2k-1} - 1}{d_1d_2 \cdots d_{2k-1}} < x + \frac{\varepsilon_1}{d_1} \leq \sum_{k=1}^{\infty} \frac{d_{2k} - 1}{d_1d_2 \cdots d_{2k}}. \]

Let \( x + \frac{\varepsilon_1}{d_1} = x_1 \). Then we obtain the following cases:

1. If the equality
\[ x_1 = \sum_{k=1}^{\infty} \frac{d_{2k} - 1}{d_1d_2 \cdots d_{2k}} \]
holds, then
\[ x = \Delta_{d_1|d_2-1|0[d_4-1]0} \text{ or } x = \Delta_{d_4-1}0[d_4-1]0[d_5-1]0... \]

2. If the first case does not hold, then \( x = -\frac{\varepsilon_2}{d_1} + x_1 \), where
\[ \frac{\varepsilon_2}{d_1d_2} - \sum_{k=2}^{\infty} \frac{d_{2k-1} - 1}{d_1d_2 \cdots d_{2k-1}} \leq x_1 < \frac{\varepsilon_2}{d_1d_2} + \sum_{k=2}^{\infty} \frac{d_{2k} - 1}{d_1d_2 \cdots d_{2k}}. \]
In this case, let \( x_2 = x_1 - \frac{\varepsilon_2}{d_1d_2} \). Then:

1. If the equality
   \[
   x_2 = \sum_{k=2}^{\infty} \frac{d_{2k-1} - 1}{d_1d_2 \cdots d_{2k-1}}
   \]
   holds, then
   \[
   x = \Delta_{\varepsilon_1 \varepsilon_2 [d_3-1]0[d_5-1]0 \cdots} \quad \text{or} \quad x = \Delta_{\varepsilon_1 [\varepsilon_2-1]0[d_4-1]0[d_6-1]0 \cdots}.
   \]

2. In the converse case,
   \[
   x = -\frac{\varepsilon_1}{d_1} + \frac{\varepsilon_2}{d_1d_2} + x_2, \quad \text{where}
   \]
   \[
   -\frac{\varepsilon_3}{d_1d_2d_3} - \sum_{k=3}^{\infty} \frac{d_{2k-1} - 1}{d_1d_2 \cdots d_{2k-1}} < x_2 \leq -\frac{\varepsilon_3}{d_1d_2d_3} + \sum_{k=2}^{\infty} \frac{d_{2k} - 1}{d_1d_2 \cdots d_{2k}}, \quad \text{etc.}
   \]

Therefore,
\[
-\sum_{k=\frac{m+2}{2}}^{\infty} \frac{d_{2k-1} - 1}{d_1d_2 \cdots d_{2k-1}} < x_m - \frac{(-1)^{m+1} \varepsilon_{m+1}}{d_1d_2 \cdots d_{m+1}} < \sum_{k=\frac{m+2}{2}}^{\infty} \frac{d_{2k} - 1}{d_1d_2 \cdots d_{2k}}
\]
for some positive integer \( m \). Moreover, the following cases are possible:

1. \[
x_{m+1} = \begin{cases} 
  \sum_{k=\frac{m+2}{2}}^{\infty} \frac{d_{2k-1} - 1}{d_1d_2 \cdots d_{2k-1}} & \text{if } m \text{ is odd} \\
  \sum_{k=\frac{m+1}{2}}^{\infty} \frac{d_{2k-1} - 1}{d_1d_2 \cdots d_{2k}} & \text{if } m \text{ is even.}
\end{cases}
\]

In this case,
\[
x = \Delta_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{m+1} [d_{m+2}-1]0[d_{m+4}-1]0 \cdots} \quad \text{or}
\]
\[
x = \Delta_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m [\varepsilon_{m+1}-1]0[d_{m+3}-1]0[d_{m+5}-1]0 \cdots}.
\]

2. If there does not exist a number \( m \in \mathbb{N} \) such that the last-mentioned system is satisfied, then
\[
x = -\frac{\varepsilon_1}{d_1} + x_1 = \cdots = -\frac{\varepsilon_1}{d_1} + \frac{\varepsilon_2}{d_1d_2} - \frac{\varepsilon_3}{d_1d_2d_3} + \cdots + \frac{(-1)^n \varepsilon_n}{d_1d_2 \cdots d_n} + x_n = \ldots.
\]

Hence,
\[
x = \sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon_n}{d_1d_2 \cdots d_n}.
\]
Lemma 4. The numbers

\[ x = \Delta_{\varepsilon_2 \ldots \varepsilon_m \varepsilon_{m+1}} D_{\varepsilon_2 \ldots \varepsilon_m \varepsilon_{m+1}} \quad \text{and} \quad x' = \Delta_{\varepsilon_2 \ldots \varepsilon_m \varepsilon_{m+1}} D_{\varepsilon_2 \ldots \varepsilon_m \varepsilon_{m+1}}', \]

where \( \varepsilon_m \neq \varepsilon_m' \), are equal if and only if one of the following systems

\[
\begin{align*}
\varepsilon_m + 2i - 1 &= d_m + 2i - 1 \\
\varepsilon_m + 2i &= 0 = \varepsilon_m' + 2i - 1 \\
\varepsilon_m + 2i &= d_m + 2i - 1 \\
\varepsilon_m &= \varepsilon_m - 1
\end{align*}
\]

or

\[
\begin{align*}
\varepsilon_m + 2i - 1 &= d_m + 2i - 1 \\
\varepsilon_m + 2i &= 0 = \varepsilon_m' + 2i \\
\varepsilon_m + 2i - 1 &= d_m + 2i - 1 \\
\varepsilon_m - 1 &= \varepsilon_m
\end{align*}
\]

is satisfied for all \( i \in \mathbb{N} \).

Proof. We prove necessity. Let \( \varepsilon_m = \varepsilon_m' + 1 \). Then

\[
0 = x - x' = \Delta_{\varepsilon_2 \ldots \varepsilon_m \varepsilon_{m+1}} D_{\varepsilon_2 \ldots \varepsilon_m \varepsilon_{m+1}} - \Delta_{\varepsilon_2 \ldots \varepsilon_m \varepsilon_{m+1}} D_{\varepsilon_2 \ldots \varepsilon_m \varepsilon_{m+1}}' = \frac{(-1)^m}{d_1 d_2 \cdots d_m}
\]

\[
+ \frac{(-1)^{m+1}(\varepsilon_m - \varepsilon_m + 1)}{d_1 d_2 \cdots d_m} + \cdots + \frac{\varepsilon_m + 1 - \varepsilon_m' + 1}{d_1 d_2 \cdots d_m}(-1)^{m+i} + \cdots
\]

\[
= \frac{(-1)^m}{d_1 d_2 \cdots d_m} \left( 1 + \sum_{i=1}^{\infty} \frac{(-1)^i(\varepsilon_m + i - \varepsilon_m' + i)}{d_{m+1} d_{m+2} \cdots d_{m+i}} \right),
\]

where

\[
\sum_{i=1}^{\infty} \frac{(-1)^i(\varepsilon_m + i - \varepsilon_m' + i)}{d_{m+1} d_{m+2} \cdots d_{m+i}} = - \sum_{i=1}^{\infty} \frac{d_{m+i} - 1}{d_{m+1} d_{m+2} \cdots d_{m+i}} = -1.
\]

The last-mentioned inequality is an equality when

\[
\varepsilon_m + 2i = \varepsilon_m' + 2i - 1 = 0, \quad \varepsilon_m + 2i - 1 = d_m + 2i - 1, \quad \text{and} \quad \varepsilon_m' + 2i = d_m + 2i - 1.
\]

In our case, the conditions of the first system follow from \( x = x' \). It is easy to see that the conditions of the second system follow from \( x = x' \) when \( \varepsilon_m = \varepsilon_m' + 1 \).

The proof of sufficiency is trivial. \( \square \)

Definition 2. The nega-D-representation \( \Delta_{\varepsilon_2 \ldots \varepsilon_m} D_{\varepsilon_2 \ldots \varepsilon_m} \) of some number \( x \in [a_0 - 1, a_0] \) is called periodic if there exist numbers \( m \in \mathbb{Z}_0 \) and \( t \in \mathbb{N} \) such that the equality \( \varepsilon_{m+n} = \varepsilon_{m+j} \) holds for each \( n \in \mathbb{N}, j \in \mathbb{N} \).

Denote by \( \Delta_{\varepsilon_2 \ldots \varepsilon_m(\varepsilon_{m+1} \varepsilon_{m+2} \ldots \varepsilon_{m+t})} D_{\varepsilon_2 \ldots \varepsilon_m(\varepsilon_{m+1} \varepsilon_{m+2} \ldots \varepsilon_{m+t})} \) any number whose nega-D-representation is periodic with the period \((\varepsilon_{m+1} \varepsilon_{m+2} \ldots \varepsilon_{m+t})\) of length \( t \).

A periodic representation is:

- purely periodic if \( m = 0 \);
- mixed periodic if \( m > 0 \).
Definition 3. Denote by
\[ x = \Delta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_m}^{-D} \phi_1(d_{m+1})\phi_2(d_{m+2})\ldots\phi_t(d_{m+t+1})\phi_1(d_{m+t+2})\ldots\phi_t(d_{m+2t}) \]
a any quasi-periodic number \( x \). Here \( m \in \mathbb{Z}_0 \), \( t \in \mathbb{N} \), and \( \phi_1, \phi_2, \ldots, \phi_t \) are functions such that \( \phi_i(d_n) \in \mathcal{A}_d \) for any \( n \in \mathbb{N} \) and \( i = 1, t \). That is \( \phi_i(d_n) \) is a regularity that depends on the parameter \( d_n \).

The following numbers are quasi-periodic:
\[ \Delta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n 0[d_{n+2}+1]/[d_{n+3}]}^{-D} \varepsilon_1\varepsilon_2\ldots\varepsilon_n 0[d_{n+1}+1]/[d_{n+2}+1] \ldots \]
etc.

Definition 4. A number \( x \in I_0 = [a_0 - 1, a_0] \) is called negatively \( D \)-rational if
\[ x = \Delta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}^{-D} \varepsilon_1\varepsilon_2\ldots\varepsilon_n [d_{n+1}+1]/[d_{n+2}+1] \ldots \]
or
\[ x = \Delta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}^{-D} \varepsilon_1\varepsilon_2\ldots\varepsilon_n [d_{n+1}+1]/[d_{n+2}+1] \ldots \]
The other numbers in \( I_0 \) are called negatively \( D \)-irrational.

The next proposition follows from Lemma 3 and Lemma 4.

Theorem 1. Every negatively \( D \)-irrational number has the unique negatively \( D \)-representation. Every negatively \( D \)-rational number has two negatively \( D \)-representations, i.e.,
\[ \Delta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}^{-D} \varepsilon_1\varepsilon_2\ldots\varepsilon_n [d_{n+1}+1]/[d_{n+2}+1] \ldots = \Delta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}^{-D} \varepsilon_1\varepsilon_2\ldots\varepsilon_n [d_{n+1}+1]/[d_{n+2}+1] \ldots \]

Remark 2. There exist sequences \( (d_n) \) such that a negatively \( D \)-rational number is an irrational number. For example, the following numbers are negatively \( D \)-rational:

\[ x = \sum_{i=1}^{n} \frac{(-1)^{i} \varepsilon_i}{d_1 d_2 \ldots d_i} + \frac{(-1)^{n}}{d_1 d_2 \ldots d_n} \left( -1 - \sum_{j=1}^{\infty} \frac{(-1)^{j}}{2 \cdot 3 \cdot \ldots \cdot (j+1)} \right) \]
\[ = \sum_{i=1}^{n} \frac{(-1)^{i} \varepsilon_i}{d_1 d_2 \ldots d_i} + \frac{(-1)^{n}}{d_1 d_2 \ldots d_n} \left( -1 + \frac{1}{e} \right), \]
\[ x = \sum_{i=1}^{n} \frac{(-1)^{i} \varepsilon_i}{d_1 d_2 \ldots d_i} + \frac{(-1)^{n}}{d_1 d_2 \ldots d_n} \left( -1 - \sum_{j=1}^{\infty} \frac{(-1)^{j}}{2 \cdot 4 \cdot \ldots \cdot 2j} \right) \]
\[ = \sum_{i=1}^{n} \frac{(-1)^{i} \varepsilon_i}{d_1 d_2 \ldots d_i} + \frac{(-1)^{n+1}}{d_1 d_2 \ldots d_n} \cdot \frac{\sqrt{e}}{e}, \]
since
\[ \Delta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}^{-D} \varepsilon_1\varepsilon_2\ldots\varepsilon_n [d_{n+1}+1]/[d_{n+2}+1] \ldots = g_n + \frac{(-1)^{n}}{d_1 d_2 \ldots d_n} \left( -1 - \sum_{j=1}^{\infty} \frac{(-1)^{j}}{d_{n+1}+1 \ldots d_{n+j}} \right) \]
and

\[ \Delta_{\varepsilon_1 \ldots \varepsilon_{n-1} | \varepsilon_{n-1} | 0 | d_{n+2} - 1 | 0 \ldots} = g_n + \frac{(-1)^n+1}{d_1 d_2 \cdots d_n} \left( 1 + \sum_{j=1}^{\infty} \frac{(-1)^j}{d_{n+1} \cdots d_{n+j}} \right), \]

where

\[ g_n = \sum_{i=1}^{n} \frac{(-1)^i \varepsilon_i}{d_i d_2 \cdots d_i}. \]

To avoid some inconveniences in the future, we can modify expansion (2) of \( x \in [-1 + a_0, a_0] \) to the following form

\[ x = \sum_{n=1}^{\infty} \frac{1 + \varepsilon_n}{d_1 d_2 \cdots d_n} (-1)^{n+1}, \]

where \( x \) represented in form (3) belongs to \([0,1]\), \( \varepsilon_n \in A_{d_n} \), and \( a_0 = -\Delta_{(1)}^{-D} \).

It is easy to see that

\[ \inf \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{d_1 d_2 \cdots d_n} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \varepsilon_n}{d_1 d_2 \cdots d_n} \right) = g' = \sum_{i=1}^{\infty} \frac{d_{2i-1} - 1}{d_1 d_2 \cdots d_{2i-1}} = 0 \]

and

\[ \sup \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{d_1 d_2 \cdots d_n} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \varepsilon_n}{d_1 d_2 \cdots d_n} \right) = g' + \sum_{i=1}^{\infty} \frac{d_{2i-1} - 1}{d_1 d_2 \cdots d_{2i-1}} = 1, \]

where

\[ g' = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{d_1 d_2 \cdots d_n}. \]

By \( \Delta_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n} \) denote any number \( x \in [0,1] \) having expansion (3). This notation is called the nega-\((d_n)\)-representation of \( x \in [0,1] \). The number \( d_n \) in (3) is called the \( n \)th element and \( \varepsilon_n = \varepsilon_n(x) \) is the \( n \)th digit of expansion (3).

2. Some Properties

Suppose that \( x = \Delta_{\varepsilon_1(x) \varepsilon_2(x) \ldots \varepsilon_n(x)} \) and \( y = \Delta_{\varepsilon_1(y) \varepsilon_2(y) \ldots \varepsilon_n(y)} \).

**Proposition 1.** The inequality \( x < y \) holds for any numbers \( x \) and \( y \) from \([-1 + a_0, a_0]\) if and only if there exists a number \( m \) such that

\[ \varepsilon_n(x) = \varepsilon_n(y) \text{ for } n < 2m \text{ and } \varepsilon_{2m}(x) < \varepsilon_{2m}(y) \]

or

\[ \varepsilon_n(x) = \varepsilon_n(y) \text{ for } n < 2m - 1 \text{ and } \varepsilon_{2m-1}(x) > \varepsilon_{2m-1}(y). \]
Proposition 2. Suppose that $x_1 = \Delta^{-D}_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_{2k-1}(0)}$, $x_2 = \Delta^{-D}_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_{2k}(0)}$, and $x_3 = \Delta^{-D}_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_{2k+1}(0)}$, and $\varepsilon_i \not= 0$ for all $i = 1,2k+1$. Then the following two-sided inequality holds:

$$x_1 < x_3 < x_2.$$ 

Proof. This statement follows from the relationship

$$x_3 = x_1 + \frac{1}{d_1 d_2 \cdots d_{2k}} \left( \varepsilon_{2k} - \frac{\varepsilon_{2k+1}}{d_{2k+1}} \right) = x_2 - \frac{\varepsilon_{2k+1}}{d_1 d_2 \cdots d_{2k+1}}.$$ 

\(\square\)

Proposition 3. Suppose that $z_1 = \Delta^{-D}_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_{2k}(0)}$, $z_2 = \Delta^{-D}_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_{2k+1}(0)}$, and $z_3 = \Delta^{-D}_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_{2k+2}(0)}$, and $\varepsilon_i \not= 0$ for all $i = 1,2k+2$. Then the two-sided inequality

$$z_2 < z_3 < z_1$$

holds.

Proof. Since the relationship

$$z_3 = z_1 - \frac{1}{d_1 d_2 \cdots d_{2k+1}} \left( \varepsilon_{2k+1} - \frac{\varepsilon_{2k+2}}{d_{2k+2}} \right) = z_2 + \frac{\varepsilon_{2k+2}}{d_1 d_2 \cdots d_{2k+2}}$$

holds, we see that our statement is true. \(\square\)

3. Relations Between Positive and Alternating Cantor Series

Let $(d_n)$ be a fixed sequence of positive integers, $d_n > 1$. For any $x \in [0,1]$ there exists a sequence $(\alpha_n)$, where $\alpha_n \in A_{d_n}$, such that

$$\Delta^D_{\alpha_1 \alpha_2 \ldots \alpha_n} = x = \sum_{n=1}^{\infty} \frac{\alpha_n}{d_1 d_2 \cdots d_n}.$$ 

It is obvious that

$$x = \frac{\alpha_1 d_2 + \alpha_2}{d_1 d_2} + \frac{\alpha_3 d_4 + \alpha_4}{d_1 d_2 d_3 d_4} + \cdots + \frac{\alpha_{2n-1} d_{2n} + \alpha_{2n}}{d_1 d_2 \cdots d_{2n}} + \ldots.$$ 

This representation is the representation of $x$ by a positive Cantor series with the sequence of elements $(d'_n)$, where $d'_n = d_{2n-1}d_{2n}$. In fact, $0 \leq \alpha_{2n-1}d_{2n} + \alpha_{2n} \leq d_{2n-1}d_{2n} - 1$ and therefore,

$$\Delta^D_{\beta_1 \beta_2 \ldots \beta_n} = x = \sum_{n=1}^{\infty} \frac{\beta_n}{p_1 p_2 \cdots p_n}.$$ 

(4)
where $\beta_n = \alpha_{2n-1}d_{2n} + \alpha_{2n}$, $p_n = d_{2n-1}d_{2n}$ for any $n \in \mathbb{N}$.

Let us consider representation (2). Using the same technique, we get

$$x = \frac{\varepsilon_2 - \varepsilon_1d_2}{d_1d_2} + \frac{\varepsilon_4 - \varepsilon_3d_4}{d_1d_2d_3d_4} + \ldots + \frac{\varepsilon_2n - \varepsilon_{2n-1}d_{2n}}{d_1d_2 \ldots d_{2n}} + \ldots.$$

But $(\varepsilon_{2n} - \varepsilon_{2n-1}d_{2n})$ belongs to $\{0, 1, \ldots, d_{2n-1}d_{2n} - 1\}$ for not all values of $\varepsilon_{2n-1}$ and $\varepsilon_{2n}$.

Consider expansion (3). Indeed, for

$$\Delta_{\delta_1\delta_2\ldots\delta_n}^{-}(d_n) = x = \sum_{n=1}^{\infty} \frac{1 + \delta_n}{d_1d_2 \ldots d_n} (-1)^{n+1},$$

where

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{d_1d_2 \ldots d_n} \equiv \Delta_{0(d_2-1)0[d_4-1]0\ldots}^{-D},$$

we obtain

$$x = \sum_{n=1}^{\infty} \frac{d_{2n} - 1}{d_1d_2 \ldots d_{2n}} + \frac{\delta_1d_2 - \delta_2}{d_1d_2} + \frac{\delta_3d_4 - \delta_4}{d_1d_2d_3d_4} + \ldots + \frac{\delta_{2n-1}d_{2n} - \delta_{2n}}{d_1d_2 \ldots d_{2n}} + \ldots.$$

Thus the number $(\delta_{2n-1}d_{2n} - \delta_{2n} + d_{2n} - 1)$ always belongs to $\{0, 1, \ldots, d_{2n-1}d_{2n} - 1\}$ for any nega-$(d_n)$-representation and

$$\Delta_{\gamma_1\gamma_2\ldots\gamma_n}^{-}(d_n) = x = \sum_{n=1}^{\infty} \frac{(\delta_{2n-1} + 1)d_{2n} - \delta_{2n} - 1}{d_1d_2 \ldots d_{2n}},$$

where $\gamma_n = (\delta_{2n-1} + 1)d_{2n} - \delta_{2n} - 1 = \delta_{2n-1}d_{2n} + d_{2n} - 1 - \delta_{2n}$.

The next statement follows from (4) and (5).

**Lemma 5.** The following functions are identity transformations:

$$x = \Delta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}^{-}(d_n) \overset{f}{\longrightarrow} \Delta_{\varepsilon_1[d_2-1-\varepsilon_2]\ldots[d_{2n-1}-1-\varepsilon_{2n}]}^{-}(d_n), \quad f(x) = y,$$

$$x = \Delta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}^{-}(d_n) \overset{g}{\longrightarrow} \Delta_{\varepsilon_1[d_2-1-\varepsilon_2]\ldots[d_{2n-1}-1-\varepsilon_{2n}]}^{-}(d_n), \quad g(x) = y.$$

Therefore the following functions are DP-functions (functions preserving the fractal Hausdorff-Besicovitch dimension):

$$x = \Delta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}^{-}(d_n) \overset{f}{\longrightarrow} \Delta_{[d_1-1-\varepsilon_1]\varepsilon_2\ldots[d_{2n-1}-1-\varepsilon_{2n-1}]\varepsilon_{2n}}^{-}(d_n), \quad f(x) = y,$$

$$x = \Delta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}^{-}(d_n) \overset{g}{\longrightarrow} \Delta_{[d_1-1-\varepsilon_1]\varepsilon_2\ldots[d_{2n-1}-1-\varepsilon_{2n-1}]\varepsilon_{2n}}^{-}(d_n), \quad g(x) = y.$$

**Lemma 6.** The following relationships are true:
4. Shift Operators

Let $\mathcal{F}^{-D}_{[-1+a_0,a_0]}$ be the set of all nega-$D$-expansions of real numbers from $[-1+a_0,a_0]$.

Define the shift operator $\hat{\phi}$ of expansion (2) on $\mathcal{F}^{-D}_{[-1+a_0,a_0]}$ by the rule

$$\hat{\phi} \left( \sum_{n=1}^{\infty} \frac{(-1)^n \xi_n}{d_1 d_2 \cdots d_n} \right) = \sum_{n=2}^{\infty} \frac{(-1)^n \xi_{n-1}}{d_2 d_3 \cdots d_n}.$$ 

In other words,

$$\hat{\phi}(\Delta_{\xi_1 \xi_2 \cdots \xi_n}) = \Delta_{\xi_2 \xi_3 \cdots \xi_n} = -d_1 \Delta_{D}^{-1} \Delta_{\xi_2 \xi_3 \cdots \xi_n}.$$ 

This operator generates some function $\hat{\phi}$ such that

$$\hat{\phi} : [-1+a_0,a_0] \rightarrow [-a_0 d_1, 1-a_0 d_1].$$

By definition, put

$$\hat{\phi}^k \left( \sum_{n=1}^{\infty} \frac{(-1)^n \xi_n}{d_1 d_2 \cdots d_n} \right) = \sum_{n=k+1}^{\infty} \frac{(-1)^{n-k} \xi_n}{d_{k+1} \cdots d_n},$$

$$\hat{\phi}^k(\Delta_{\xi_1 \xi_2 \cdots \xi_n}) = \Delta_{-D}^{-k} \Delta_{\xi_{k+1} \xi_{k+2}} = (-1)^k d_1 d_2 \cdots d_k \Delta_{D}^{-1} \Delta_{0}^{-1} \xi_{k+1} \xi_{k+2} \cdots .$$

Define the generalized shift operator $\phi_m$ of expansion (2) by the rule

$$\hat{\phi}_m \left( \sum_{n=1}^{\infty} \frac{(-1)^n \xi_n}{d_1 d_2 \cdots d_n} \right) = -\frac{\xi_1}{d_1} + \cdots + \frac{(-1)^{m-1} \xi_{m-1}}{d_1 d_2 \cdots d_{m-1}} + \frac{(-1)^m \xi_{m-1}}{d_1 d_2 \cdots d_{m-1} d_{m+1}} + \cdots ,$$

i.e.,

$$\hat{\phi}_m(\Delta_{\xi_1 \xi_2 \cdots \xi_{m-1} \xi_{m+1}}) = \Delta_{D}^{-m} \Delta_{\xi_1 \xi_2 \cdots \xi_{m-1} \xi_{m+1}} .$$

Remark 3. Since $(\xi_n)$ and $(d_n)$ are fixed sequences in (2) for a given $x \in [a_0-1,a_0]$, we see that the operator $\hat{\phi}$ or $\phi_m$ takes each number to a number represented in terms of the “other” numeral system.
It is easy to see that the operator \( \hat{\varphi} \) has exactly \( d_1 \) invariant points. These are points of the form

\[
-\frac{i}{d_1 + 1}, \quad i = 0, d_1 - 1.
\]

The operator \( \hat{\varphi} \) is not a bijection because the points \( \Delta_{\varepsilon_2 \varepsilon_3 \ldots}^{-D} \) are preimages of the point \( \Delta_{\varepsilon_2 \varepsilon_3 \ldots}^{-1} \).

If a sequence \( (d_n) \) is purely periodic with a period of length \( k \), then the mapping \( \hat{\varphi} \) has periodic points with a period of length \( k, k \in \mathbb{N} \), i.e.,

\[
\hat{\varphi}^{k+j}(\Delta_{\varepsilon_1 \varepsilon_2 \ldots}^{-D}) = \hat{\varphi}^{kt+j}(\Delta_{\varepsilon_1 \varepsilon_2 \ldots}^{-D}), \quad t = 0, 1, 2, \ldots.
\]

**Lemma 7.** If a sequence \( (d_n) \) is purely periodic with a simple period, then the following set

\[
C[-D, V] = \{ x : x = \Delta_{\varepsilon_2 \varepsilon_3 \ldots}^{-D}, \varepsilon_n \in \{ v_1, v_2, \ldots, v_m \} \subset \{ 0, 1, \ldots, d_n - 1 \} \}
\]

is an invariant set under the mapping \( \hat{\varphi} \). Here \( v_1, v_2, \ldots, v_m \) are fixed positive integers, \( 1 < m \leq d_n - 1 \), and \( d_n > 2 \).

**Lemma 8.** If sequences \( (d_n) \) and \( (V_n) \) are purely periodic with a period of length \( k \), then the set

\[
C[-D, V_n] = \{ x : x = \Delta_{\varepsilon_2 \varepsilon_3 \ldots}^{-D}, \varepsilon_n \in V_n = \{ v_1^{(n)}, v_2^{(n)}, \ldots, v_m^{(n)} \} \supset A_{d_n} \}
\]

is an invariant set under the mapping \( \hat{\varphi}^k \).

Define the shift operator \( \varphi \) of the nega-D-representation of \( x \) by the rule

\[
\varphi(x) = \varphi(\Delta_{\varepsilon_2 \varepsilon_3 \ldots}^{-D}) = \Delta_{\varepsilon_2 \varepsilon_3 \ldots}^{-d_2 \ldots d_n} = \sum_{n=1}^{\infty} (-1)^n \varepsilon_{n+1} / d_1 d_2 \cdots d_n.
\]

It is obvious that the mapping \( \varphi \) is not always well-defined. In fact, the inequality \( \varepsilon_{n+1} \leq d_n - 1 \) holds for not all alternating Cantor series. The next statement follows from the last-mentioned inequality.

**Lemma 9.** The operator \( \varphi \) is well-defined if and only if the inequality

\[
d_{n+1} \leq d_n
\]

holds for any \( n \in \mathbb{N} \).

**Remark 4.** In Lemma 9, we understand that \( \varphi \) is well-defined in the wide sense, i.e., for each \( x \in [a_0 - 1, a_0] \). In fact, for any sequence \( (d_n) \) there exist points from \( [a_0 - 1, a_0] \) such that the function \( \varphi \) is well-defined at these points.
Lemma 10. If there exists a number \( m \in \mathbb{N} \) such that \( \hat{\varphi}^m(x) = x \), then

\[
x = \left( 1 + \frac{1}{(-1)^m d_1 d_2 \cdots d_m - 1} \right) \Delta_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m}^D(0).
\]

Proof. Let \( x = \Delta_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m}^D \). Then

\[
x = \Delta_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m}^D(0) - \sum_{i=m+1}^{\infty} \varepsilon_i \Delta_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m}^D(0).
\]

This concludes the proof.

Lemma 11. Let \( x \) be a fixed number. If there exist \( m \in \mathbb{Z}_0 \) and \( c \in \mathbb{N} \) such that \( \hat{\varphi}^m(x) = \hat{\varphi}^{m+c}(x) \), then

\[
x = \frac{\Delta_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m}^D(0) + (-1)^{c+1} d_m d_{m+1} \cdots d_{m+c} \Delta_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{m+c}}^D(0)}{1 + (-1)^{c+1} d_m d_{m+1} \cdots d_{m+c}}.
\]

Proof. The statement follows from the next equality:

\[
x - \Delta_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m}^D(0) = (-1)^{c+1} d_m d_{m+1} \cdots d_{m+c}(x - \Delta_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{m+c}}^D(0)).
\]

Lemma 12. The equalities

\[
\hat{\varphi}^k(x) = (-1)^k d_1 d_2 \cdots d_k x + (-1)^{k+1} d_1 d_2 \cdots d_k \Delta_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k}^D(0)
\]

and

\[
x = (-1)^k \hat{\varphi}^k(x) + (-1)^k d_1 d_2 \cdots d_k \Delta_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k}^D(0)
\]

hold for an arbitrary \( k \in \mathbb{N} \).

The next proposition follows from the last-mentioned lemma.

Lemma 13. The equality

\[
(-1)^c d_m d_{m+1} \cdots d_{m+c} \cdot \hat{\varphi}^m(x) - \hat{\varphi}^{m+c}(x)
\]

\[
= (-1)^{m+c} d_1 d_2 \cdots d_{m+c} \cdot \Delta_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{m+c}}^D(0)
\]

holds for arbitrary numbers \( m \in \mathbb{N} \) and \( c \in \mathbb{N} \).

Theorem 2. The mapping \( \hat{\varphi} \) is decreasing on each first rank interval \( \nabla_{\varepsilon}^D = (\inf \Delta_{\varepsilon}^D, \sup \Delta_{\varepsilon}^D) \).
Proof. Let points \( x_1 = \Delta_{\varepsilon_2 \varepsilon_3 \ldots \varepsilon_n}^{-D} \) and \( x_2 = \Delta_{\varepsilon_2 \varepsilon_3 \ldots \varepsilon_n}^{-D} \) (\( x_1 < x_2 \)) be arbitrary points from the interval \( \nabla_c^{-D} \).

Since the equality \( \varepsilon_n(\hat{\varphi}(x)) = \varepsilon_{n+1}(x) \) holds and Proposition 1 is true for \( x_1 \) and \( x_2 \), we see that the inequality \( \hat{\varphi}(x_1) > \hat{\varphi}(x_2) \) holds.

The next statement follows from this theorem.

**Corollary 1.** The mapping \( \hat{\varphi} \) has a derivative almost everywhere (with respect to the Lebesgue measure).

**Theorem 3.** The mapping \( \hat{\varphi} \) is continuous at each point of the first rank interval \( \nabla_c^{-D} \) and the endpoints of this interval are points of discontinuity of the mapping.

**Proof.** Let \( x = \Delta_{\varepsilon_2 \varepsilon_3 \ldots \varepsilon_n}^{-D} \) be an arbitrary nega-D-irrational point from \( \nabla_c^{-D} \). Let \( (x_m) \) be an arbitrary sequence of points from \( \nabla_c^{-D} \) such that \( \lim_{m \to \infty} x_m = x \). Then

\[
\lim_{m \to \infty} x_m = x \Leftrightarrow \lim_{n \to \infty} n_m = \infty,
\]

where \( n_m = \min\{n : \varepsilon_n(x_m) \neq \varepsilon_n(x)\} \). The last-mentioned equivalence follows from the definition and basic properties of the nega-D-representation.

Since \( \varepsilon_n(\hat{\varphi}(x)) = \varepsilon_{n+1}(x) \) holds, we have \( \lim_{m \to \infty} \hat{\varphi}(x_m) = \hat{\varphi}(x) \). Therefore the mapping \( \hat{\varphi} \) is continuous at the point \( x \).

Let \( x_0 \) be a certain nega-D-rational point from \( \nabla_c^{-D} \), i.e.,

\[
x_0 = \Delta_{\varepsilon_2 \varepsilon_3 \ldots \varepsilon_n [d_{n+1}-1]0[d_{n+3}-1]0}^{-D} = \Delta_{\varepsilon_2 \varepsilon_3 \ldots \varepsilon_n [\varepsilon_{n-1}]0[d_{n+2}-1]0[d_{n+4}-1]0}^{-D}.
\]

At the same time

\[
\lim_{x \to x_0} \hat{\varphi}(x) = \Delta_{\varepsilon_2 \ldots \varepsilon_n [d_{n+1}-1]0[d_{n+3}-1]0}^{-D} = \Delta_{\varepsilon_2 \ldots \varepsilon_n [\varepsilon_{n-1}]0[d_{n+2}-1]0[d_{n+4}-1]0}^{-D}.
\]

Indeed, the existence of the left-hand and right-hand finite limits at each point follows from monotonicity and boundedness of the mapping.

Consider the problem of continuity of \( \hat{\varphi} \) at the point

\[
x_1 = \Delta_{c[d_2-1]0[d_4-1]0}^{-D} = \Delta_{[c-1]0[d_3-1]0[d_5-1]0}^{-D} = x_2, \quad c \neq 0.
\]

The endpoints of the interval \( \nabla_c^{-D} \) are the jump points of \( \hat{\varphi} \) because

\[
\hat{\varphi}(x_1) = \Delta_{[d_2-1]0[d_4-1]0}^{-D} \neq \Delta_{[d_3-1]0[d_5-1]0}^{-D} = \hat{\varphi}(x_2).
\]

**Theorem 4.** If the mapping \( \hat{\varphi} \) has a derivative at the point \( x = \Delta_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n}^{-D} \), then

\[
(\hat{\varphi}(x))' = -d_1.
\]
Proof. Suppose that \( \hat{\phi} \) has a derivative at the point \( x_0 \). Let \( (x_n) \) be a sequence of \( \Delta^{-D}_{\varepsilon_1(x_0)\varepsilon_2(x_0)\ldots\varepsilon_n(x_0)\varepsilon_{n+1}(x_\ldots)\ldots} \). Here \( \varepsilon_k(x) \neq \varepsilon_k(x_0) \) for all \( k > n \). Then

\[
(\hat{\phi}(x))' = \lim_{\Delta x \to 0} \frac{\hat{\phi}(x) - \hat{\phi}(x_0)}{x - x_0} = \lim_{\Delta x \to 0} \frac{\Delta^{-D}_{\varepsilon_2(x)\varepsilon_3(x)\ldots} - \Delta^{-D}_{\varepsilon_2(x_0)\varepsilon_3(x_0)\ldots}}{\Delta x} \\
= - \lim_{\varepsilon_n(x) \to \varepsilon_n(x_0)} \sum_{i=2}^{\infty} \frac{(-1)^i \varepsilon_i(x)}{d_2 \ldots d_i} - \sum_{i=2}^{\infty} \frac{(-1)^i \varepsilon_i(x_0)}{d_2 \ldots d_i} = - \sum_{i=1}^{\infty} \frac{(-1)^i \varepsilon_i(x_0)}{d_1 d_2 \ldots d_i} - \sum_{j=1}^{\infty} \frac{(-1)^j \varepsilon_j(x_0)}{d_1 d_2 \ldots d_j} = \lim_{\varepsilon_n(x) \to \varepsilon_n(x_0)} \sum_{j=1}^{\infty} \frac{(-1)^j \varepsilon_j(x_0)}{d_1 d_2 \ldots d_j} \\
= - \lim_{\varepsilon_n(x) \to \varepsilon_n(x_0)} \sum_{j=1}^{\infty} \frac{(-1)^j \varepsilon_j(x_0)}{d_1 d_2 \ldots d_j} + \varepsilon_1(x) - \varepsilon_1(x_0) = - \lim_{\varepsilon_n(x) \to \varepsilon_n(x_0)} \sum_{j=1}^{\infty} \frac{(-1)^j \varepsilon_j(x_0)}{d_1 d_2 \ldots d_j} \cdot d_1 = -d_1.
\]

Corollary 2. The derivative of \( \hat{\phi}^k \) does not exist at an arbitrary nega-D-rational point \( \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{n-1}\varepsilon_n[d_{n+1}-1]d_{n+3}-10\ldots\ldots} \) when \( k > n - 1 \).

5. Representations of Rational and Irrational Numbers

The main statements of this section are analogous to the main results of the paper [8].

Theorem 5. A rational number \( x = \frac{p}{q} \) from \([-1 + a_0, a_0] \) has a finite expansion by series (2) if and only if there exists a number \( n_0 \) such that \( d_1 d_2 \ldots d_{n_0} \equiv 0 \pmod{q} \).

Corollary 3. There exist sequences \( (d_n) \) such that every rational number has the finite nega-D-expansion. Consider the following examples:

\[
\sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon_n}{2 \cdot 3 \cdot \ldots \cdot (n+1)}, \sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon_n}{2 \cdot 4 \cdot \ldots \cdot 2n}.
\]

Lemma 14. The equality

\[
\Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n} + \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n} + \Delta^{-D}_{(1)} = 0
\]

holds for each \( n \in \mathbb{N} \).

Corollary 4. There exist alternating Cantor series (3) such that a number of the form \( \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n(0)} \) is an irrational number.

The following propositions are equivalent.

Theorem 6. A number \( x_0 \in [-1 + a_0, a_0] \) is a rational number if and only if there exist \( k \in \mathbb{Z}_0 \) and \( t \in \mathbb{N} \) such that

\[
\hat{\phi}^k(x) = \hat{\phi}^t(x).
\]
**Theorem 7.** A number $x_0 \in [-1 + a_0, a_0]$ is a rational number if and only if there exist $k \in \mathbb{Z}_0$ and $t \in \mathbb{N}$ ($k < t$) such that

$$\Delta_{0}^{D} \frac{-\varepsilon_{k+1} \varepsilon_{k+2} \varepsilon_{k+3} \ldots}{k} = (-1)^{t-k} d_{k+1} d_{k+2} \cdots d_{t} \Delta_{0}^{D} \frac{-\varepsilon_{k+1} \varepsilon_{k+2} \varepsilon_{k+3} \ldots}{t}.$$ 

**Theorem 8.** A number $x$ is a rational number if and only if the sequence $(\hat{\varphi}^k(x))$, where $k = 0, 1, 2, \ldots$, contains at least two identical terms.

6. Foundations of the Metric Theory

Let $(d_n)$ be a fixed sequence of positive integers, $d_n > 1$. Let $c_1, c_2, \ldots, c_m$ be an ordered tuple of integers such that $c_i \in \{0, 1, \ldots, d_i - 1\}$ for $i = 1, m$.

**Definition 5.** A nega-$D$-cylinder of rank $m$ with base $c_1 c_2 \ldots c_m$ is a set $\Delta_{c_1 c_2 \ldots c_m}^{-D}$ formed by all numbers of the segment $[-1 + a_0, a_0]$ with nega-$D$-representations in which the first $m$ digits coincide with $c_1, c_2, \ldots, c_m$, respectively, i.e.,

$$\Delta_{c_1 c_2 \ldots c_m}^{-D} = \{ x : x = \Delta_{c_1 c_2 \ldots c_m}^{-D}, \varepsilon_j = c_j, j = 1, m \}.$$

**Lemma 15.** A nega-$D$-cylinder is a closed interval, i.e.,

$$\Delta_{c_1 c_2 \ldots c_m}^{-D} = \begin{cases} \left[ g_m + \frac{(-1)^m}{d_{1} d_{2} \cdots d_{m}} (a_m - 1), g_m + \frac{(-1)^m}{d_{1} d_{2} \cdots d_{m}} a_m \right] & \text{if } m \text{ is even} \\ \left[ g_m + \frac{(-1)^m}{d_{1} d_{2} \cdots d_{m}} a_m g_m + \frac{(-1)^m}{d_{1} d_{2} \cdots d_{m}} (a_m - 1) \right] & \text{if } m \text{ is odd} \end{cases},$$

where

$$a_m = \sup_{j=1}^{\infty} \frac{(-1)^{j+1}}{d_{m+1} d_{m+2} \cdots d_{m+j}}, \quad g_m = \sum_{i=1}^{m} \frac{(-1)^i c_i}{d_{1} d_{2} \cdots d_{i}}.$$

**Proof.** Let $m$ be even and $x \in \Delta_{c_1 c_2 \ldots c_m}^{-D}$, i.e.,

$$x = \sum_{i=1}^{m} \frac{(-1)^i c_i}{d_{1} d_{2} \cdots d_{i}} + \sum_{j=m+1}^{\infty} \frac{(-1)^j \varepsilon_j}{d_{1} d_{2} \cdots d_{j}},$$

where $\varepsilon_j \in \{0, 1, \ldots, d_j - 1\}$; then

$$x' = g_m - \sum_{k=1}^{\infty} \frac{d_{m+2k-1} - 1}{d_{1} d_{2} \cdots d_{m+2k-1}} \leq x \leq g_m + \sum_{k=1}^{\infty} \frac{d_{m+2k - 1}}{d_{1} d_{2} \cdots d_{m+2k}} = x''.$$

Hence $x \in [x', x'']$ and $\Delta_{c_1 c_2 \ldots c_m}^{-D} \subseteq [x', x'']$.
Since the equalities
\[
\sum_{j=1}^{\infty} \frac{d_{m+2j} - 1}{d_1d_2 \cdots d_{m+2j}} = \frac{1}{d_1d_2 \cdots d_m} \sup_{j=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^j e_m}{d_{m+1}d_{m+2} \cdots d_{m+j}}
\]
and
\[
-\sum_{j=1}^{\infty} \frac{d_{m+2j-1} - 1}{d_1d_2 \cdots d_{m+2j-1}} = \frac{1}{d_1d_2 \cdots d_m} \inf_{j=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^j e_m}{d_{m+1}d_{m+2} \cdots d_{m+j}}
\]
hold, we have \( x \in \Delta_{c_1 c_2 \cdots c_m}^{-D} \) and \( x', x'' \in \Delta_{c_1 c_2 \cdots c_m}^{-D} \).

Lemma 16. Nega-D-cylinders have the following properties:

1. \[
\inf \Delta_{c_1 c_2 \cdots c_m}^{-D} = \begin{cases} 
  g_m - \frac{1}{d_1d_2 \cdots d_m} \sum_{j=1}^{\infty} \frac{d_{m+2j-1} - 1}{d_{m+1}d_{m+2} \cdots d_{m+j-1}} & \text{if } m \text{ is even} \\
  g_m - \frac{1}{d_1d_2 \cdots d_m} \sum_{j=1}^{\infty} \frac{d_{m+2j-1}}{d_{m+1}d_{m+2} \cdots d_{m+j}} & \text{if } m \text{ is odd}
\end{cases}
\]

2. \[
\sup \Delta_{c_1 c_2 \cdots c_m}^{-D} = \begin{cases} 
  g_m + \frac{1}{d_1d_2 \cdots d_m} \sum_{j=1}^{\infty} \frac{d_{m+2j-1}}{d_{m+1}d_{m+2} \cdots d_{m+j}} & \text{if } m \text{ is even} \\
  g_m + \frac{1}{d_1d_2 \cdots d_m} \sum_{j=1}^{\infty} \frac{d_{m+2j-1} - 1}{d_{m+1}d_{m+2} \cdots d_{m+j-1}} & \text{if } m \text{ is odd}
\end{cases}
\]

3. \[
|\Delta_{c_1 c_2 \cdots c_m}^{-D}| = \frac{1}{d_1d_2 \cdots d_m}.
\]

4. \[
\Delta_{c_1 c_2 \cdots c_m}^{-D} \subset \Delta_{c_1 c_2 \cdots c_m}^D.
\]

5. \[
\Delta_{c_1 c_2 \cdots c_m}^{-D} = \bigcup_{c=0}^{d_{m+1}-1} \Delta_{c_1 c_2 \cdots c_m}^D.
\]

6. \[
\lim_{m \to \infty} |\Delta_{c_1 c_2 \cdots c_m}^{-D}| = 0.
\]

7. \[
\frac{|\Delta_{c_1 c_2 \cdots c_m}^{-D}|}{|\Delta_{c_1 c_2 \cdots c_m}|} = \frac{1}{d_{m+1}}.
\]
8. \[
\begin{aligned}
\sup \Delta^{-D}_{c_1c_2\ldots c_m} &= \inf \Delta^{-D}_{c_1c_2\ldots c_m} & \text{if } m \text{ is odd} \\
\sup \Delta^{-D}_{c_1c_2\ldots c_m} &= \inf \Delta^{-D}_{c_1c_2\ldots c_m} & \text{if } m \text{ is even},
\end{aligned}
\]
where \( c \neq d_{m+1} - 1 \).

9. \[
\Delta^{-D}_{c_1c_2\ldots c_m} \cap \Delta^{-D}_{c_1c_2\ldots c_m} = \begin{cases} 
\Delta^{-D}_{c_1c_2\ldots c_m} & \text{if } c_i = c_i \text{ for } i = 1, \ldots, m \\
\emptyset & \text{if } \exists i (i < m) \text{ such that } c_i \neq c_i \\
\emptyset & \text{if } \exists i \text{ such that } c_i \neq c_i, c_m \neq c_m - 1.
\end{cases}
\]
Here \( c_m \neq 0 \) in the last case.

10. \[
\bigcap_{m=1}^{\infty} \Delta^{-D}_{c_1c_2\ldots c_m} = x = \Delta^{-D}_{c_1c_2\ldots c_m}.
\]

Proof. Properties 1 and 2 follow immediately from the definition of \( \Delta^{-D}_{c_1c_2\ldots c_m} \). Property 3 is a corollary of these properties. Properties 6 and 7 follow from Property 3.

Property 4. Let \( m \) be even. Let us prove that the conditions

\[
\begin{aligned}
\inf \Delta^{-D}_{c_1c_2\ldots c_m} &\geq \inf \Delta^{-D}_{c_1c_2\ldots c_m} \\
\sup \Delta^{-D}_{c_1c_2\ldots c_m} &\leq \sup \Delta^{-D}_{c_1c_2\ldots c_m}
\end{aligned}
\]

hold. In fact,

\[
\begin{aligned}
\inf \Delta^{-D}_{c_1c_2\ldots c_m} - \inf \Delta^{-D}_{c_1c_2\ldots c_m} &= g_m + \frac{(-1)^{m+1}c}{d_1d_2\ldots d_{m+1}} \\
+ \frac{(-1)^{m+1}}{d_1d_2\ldots d_{m+1}} \left( \frac{d_{m+3} - 1}{d_{m+2}d_{m+3}} + \frac{d_{m+5} - 1}{d_{m+4}d_{m+5}} + \ldots \right) - g_m \\
- \frac{(-1)^m}{d_1d_2\ldots d_m} \left( - \frac{d_{m+1} - 1}{d_{m+1}} + \frac{d_{m+3} - 1}{d_{m+2}d_{m+3}} + \frac{d_{m+5} - 1}{d_{m+4}} + \ldots \right) \\
= \frac{d_{m+1} - 1 - c}{d_1d_2\ldots d_{m+1}} \geq 0.
\end{aligned}
\]

If the condition \( c = d_{m+1} - 1 \) holds, then the last inequality is an equality. As above,

\[
\begin{aligned}
\sup \Delta^{-D}_{c_1\ldots c_m} - \sup \Delta^{-D}_{c_1\ldots c_m} &= \frac{(-1)^m}{d_1d_2\ldots d_m} \left( \frac{d_{m+2} - 1}{d_{m+1}d_{m+2}} + \frac{d_{m+4} - 1}{d_{m+3}d_{m+4}} + \ldots \right) \\
+ g_m - g_m - \frac{(-1)^{m+1}c}{d_1\ldots d_{m+1}} - \frac{(-1)^{m+1}}{d_1\ldots d_{m+1}} \left( - \frac{d_{m+2} - 1}{d_{m+2}} - \frac{d_{m+4} - 1}{d_{m+3}d_{m+4}} + \ldots \right) \\
= \frac{c}{d_1d_2\ldots d_{m+1}} \geq 0.
\end{aligned}
\]
Here the last inequality is an equality whenever the condition \( c = 0 \) holds.

Similarly, the last-mentioned system of inequalities is true in the case of odd \( m \).

**Property 5** follows from **Property 4** and the definition of \( \Delta_{c_1c_2...c_m}^{-D} \).

**Property 8.** Let \( m \) be odd. Then
\[
\sup \Delta_{c_1c_2...c_m}^{-D} - \inf \Delta_{c_1c_2...c_m}^{-D}[c+1] = \frac{(-1)^{m+1}c}{d_1d_2 \cdots d_{m+1}} + \frac{(-1)^{m+1}}{d_1d_2 \cdots d_{m+1}}a_{m+1}
\]
\[
- \frac{c + 1}{d_1d_2 \cdots d_{m+1}}(-1)^{m+1} - \frac{(-1)^{m+1}}{d_1d_2 \cdots d_{m+1}}(a_{m+1} - 1) = 0.
\]

Let \( m \) be even. Then
\[
\sup \Delta_{c_1c_2...c_m}^{-D} - \inf \Delta_{c_1c_2...c_m}^{-D}[c+1] = \frac{(-1)^{m+1}}{d_1d_2 \cdots d_{m+1}}(c + 1)
\]
\[
+ \frac{(-1)^{m+1}c}{d_1d_2 \cdots d_{m+1}}(a_{m+1} - 1) - \frac{(-1)^{m+1}}{d_1d_2 \cdots d_{m+1}}a_{m+1} = 0.
\]

**Property 9** follows from properties 1, 2, and 8.

**Property 10.** From **Property 4** it follows that
\[
\Delta_{c_1}^{-D} \subset \Delta_{c_1c_2}^{-D} \subset \Delta_{c_1c_2c_3}^{-D} \subset \ldots \subset \Delta_{c_1c_2...c_n}^{-D} \subset \ldots.
\]

Since the last lemma and Cantor’s intersection theorem are true, we obtain
\[
\bigcap_{n=1}^{\infty} \Delta_{c_1c_2...c_n}^{-D} = x = \Delta_{c_1c_2...c_n}^{-D}.
\]

\[\square\]

7. Simplest Metric Problems

Let \( k \) be a fixed positive integer, \( c \) be a fixed digit from \( A_{d_k} \). Consider the following set
\[
\Delta_c^k = \{ x : x = \Delta_{c_1c_2...c_k-1c_{k+1}}^{-D} \}.
\]

**Lemma 17.** The set \( \Delta_c^k (k > 1) \) is the union of nega-\( D \)-cylinders of rank \( k \).

**Proof.** Let \( k = 1 \). Then it is easy to see that \( \Delta_c^1 = \Delta_c^{-D} \).

Let \( k = 2 \). Then
\[
\Delta_c^2 = \Delta_{0c}^{-D} \cup \Delta_{1c}^{-D} \cup \Delta_{2c}^{-D} \cup \ldots \cup \Delta_{[d_1-1]c}^{-D}.
\]

Let \( k = n \). Then
\[
\Delta_c^n = \Delta_{00...0c}^{-D} \cup \Delta_{00...01c}^{-D} \cup \ldots \cup \Delta_{[d_1-1][d_2-1]...[d_{n-1}-1]c}^{-D}.
\]

\[\square\]
Lemma 18. The Lebesgue measure of $\Delta^k_c$ is equal to $\frac{1}{d_k}$.

Proof. It is easy to see that
\[
\lambda(\Delta^k_c) = \sum_{c_1=0}^{d_1-1} \cdots \sum_{c_{k-1}=0}^{d_{k-1}-1} |\Delta^D_{c_1\ldots c_{k-1}}| = \frac{1}{d_k} \sum_{c_1=0}^{d_1-1} \cdots \sum_{c_{k-1}=0}^{d_{k-1}-1} |\Delta^D_{c_1\ldots c_{k-1}}| = \frac{1}{d_k}.
\]

Corollary 5. The Lebesgue measure of $\Delta^k_c = \{x : x = \Delta^D_{c_1\ldots c_k} \neq c\}$ is equal to $1 - \frac{1}{d_k}$.

Lemma 19. The diameter of the set $\Delta^k_c$ is calculated by the following formula
\[
d(\Delta^k_c) = \frac{d_1 d_2 \cdots d_k - d_k + 1}{d_1 d_2 \cdots d_k}.
\]

Proof. Let $k$ be even,
\[
a_k = \sup \sum_{j=1}^{\infty} \frac{(-1)^j c_{k+j}}{d_{k+1} d_{k+2} \cdots d_{k+j}}.
\]

Then
\[
d(\Delta^k_c) = \max \sum_{i=1}^{k-1} \frac{(-1)^i c_i}{d_1 d_2 \cdots d_i} + (\frac{(-1)^k c - a_k}{d_1 d_2 \cdots d_k}) - \min \sum_{i=1}^{k-1} \frac{(-1)^i c_i}{d_1 d_2 \cdots d_i} - \frac{(\frac{(-1)^k c - a_k}{d_1 d_2 \cdots d_k})}{d_1 d_2 \cdots d_k} (a_k - 1)
\]
\[
= \left(\frac{d_2 - 1}{d_1 d_2} + \frac{d_3 - 1}{d_1 d_2 d_3} + \cdots + \frac{d_{k-2} - 1}{d_1 \cdots d_{k-2}}\right)
\]
\[
+ \left(\frac{d_1 - 1}{d_1} + \frac{d_3 - 1}{d_1 d_2 d_3} + \cdots + \frac{d_{k-1} - 1}{d_1 \cdots d_{k-1}}\right) + \frac{(\frac{(-1)^k c - a_k}{d_1 d_2 \cdots d_k})}{d_1 d_2 \cdots d_k} (a_k - 1)
\]
\[
= 1 - \frac{1}{d_1 \cdots d_{k-1}} + \frac{(\frac{(-1)^k c - a_k}{d_1 d_2 \cdots d_k})}{d_1 d_2 \cdots d_k} = \frac{d_1 d_2 \cdots d_k - d_k + 1}{d_1 d_2 \cdots d_k}.
\]

Let $k$ be odd. Then
\[
d(\Delta^k_c) = \left(\frac{d_2 - 1}{d_1 d_2} + \frac{d_3 - 1}{d_1 d_2 d_3} + \cdots + \frac{d_{k-1} - 1}{d_1 \cdots d_{k-1}}\right)
\]
\[
- \left(\frac{d_1 - 1}{d_1} - \frac{d_3 - 1}{d_1 d_2 d_3} - \cdots - \frac{d_{k-2} - 1}{d_1 \cdots d_{k-2}}\right) + \frac{(-1)^{k+1}}{d_1 d_2 \cdots d_k}
\]
\[
= 1 - \frac{1}{d_1 d_2 \cdots d_{k-1}} + \frac{1}{d_1 d_2 \cdots d_k} = \frac{d_1 d_2 \cdots d_k - d_k + 1}{d_1 d_2 \cdots d_k}.
\]
Let \((c_1, c_2, \ldots, c_m)\) and \((k_1, k_2, \ldots, k_m)\) be fixed tuples of positive integers such that \(c_i \in A_{d_{k_i}}, i = 1, m, 0 < k_1 < k_2 < \ldots < k_m\). Consider the following set
\[
\Delta_{c_1c_2\ldots c_m}^{k_1k_2\ldots k_m} = \{x : x = \Delta_{x_1\ldots x_k}^{-D} \in \mathbb{C}_{x_{k_1}+1\ldots x_{k_2}+1\ldots x_{k_m}+1} \in \mathbb{C}_{x_{k_m}+1\ldots x_{k_m}+2}\}.
\]

**Lemma 20.** The Lebesgue measure of the set \(\Delta_{c_1c_2\ldots c_m}^{k_1k_2\ldots k_m}\) is calculated by the formula
\[
\lambda (\Delta_{c_1c_2\ldots c_m}^{k_1k_2\ldots k_m}) = \prod_{i=1}^{m} \frac{1}{d_{k_i}}.
\]

**Proof.** It is easy to see that
\[
\lambda (\Delta_{c_1c_2}^{k_1k_2}) = \frac{1}{d_{k_2}} \frac{d_{k_2-1}}{d_{k_2-1}} \ldots \frac{d_{k_1+1}}{d_{k_1+1}} = \frac{1}{d_{k_1}} = \lambda (\Delta_{c_1}^{k_1}) \cdot \lambda (\Delta_{c_2}^{k_2}).
\]
Clearly,
\[
\lambda (\Delta_{c_1c_2\ldots c_m}^{k_1k_2\ldots k_m}) = \frac{1}{d_{k_m}} \frac{1}{d_{k_{m-1}}} \ldots \frac{1}{d_{k_1}} = \frac{1}{d_{k_1}d_{k_2} \ldots d_{k_m}}.
\]

**Corollary 6.** Sets of the form \(\Delta_{c_1c_2\ldots c_m}^{k_1k_2\ldots k_m}\) are metrically independent, i.e.,
\[
\lambda (\bigcap_{i=1}^{m} \Delta_{c_i}^{k_i}) = \prod_{i=1}^{m} \lambda (\Delta_{c_i}^{k_i}).
\]

**Lemma 21.** The diameter \(d (\Delta_{c_1c_2\ldots c_m}^{k_1k_2\ldots k_m})\) of the set \(\Delta_{c_1c_2\ldots c_m}^{k_1k_2\ldots k_m}\) is calculated by the formula
\[
d (\Delta_{c_1c_2\ldots c_m}^{k_1k_2\ldots k_m}) = 1 - \sum_{i=1}^{m} \frac{d_{k_i} - 1}{d_{1}d_{2} \ldots d_{k_i}}.
\]

**Proof.** Let \(K = \{k_1, k_2, \ldots, k_m\}\) and \(l = 1, 2, \ldots\). Then
\[
\sup_{\Delta_{c_1c_2\ldots c_m}^{k_1k_2\ldots k_m}} - \inf_{\Delta_{c_1c_2\ldots c_m}^{k_1k_2\ldots k_m}} = \sum_{2l+j \not\in K, j < k_m} \frac{d_{j-1}}{d_{1}d_{2} \ldots d_{j}} + \frac{(-1)^{k_m}}{d_{1}d_{2} \ldots d_{k_m}} \sum_{p=1}^{\infty} \frac{d_{k_m+2p} - 1}{d_{k_m+1} \ldots d_{k_m+2p}}
\]
\[
+ \frac{(-1)^{k_m}}{d_{1}d_{2} \ldots d_{k_m}} \sum_{p=1}^{\infty} \frac{d_{k_m+2p} - 1}{d_{k_m+1} \ldots d_{k_m+2p}} - \sum_{2l+1+j \not\in K, j < k_m} \frac{1 - d_{j}}{d_{1}d_{2} \ldots d_{j}} - \frac{(-1)^{k_m}}{d_{1}d_{2} \ldots d_{k_m}} \sum_{p=1}^{\infty} \frac{1 - d_{k_m+2p+1}}{d_{k_m+1} \ldots d_{k_m+2p+1}}
\]
\[
= \sum_{0 < j < k_m, j \not\in K} \frac{d_{j} - 1}{d_{1}d_{2} \ldots d_{j}} + \frac{1}{d_{1}d_{2} \ldots d_{k_m}} = 1 - \sum_{i=1}^{m} \frac{d_{k_i} - 1}{d_{1}d_{2} \ldots d_{k_i}}.
\]
8. Alternating Cantor Series and the Hausdorff-Besicovitch Dimension Faithfulness

Let \( \Phi_1 \) be the family of all closed intervals, \( \Phi_2 \) be the family of all possible rank cylinders \( \Delta_{c_1c_2...c_n}^{-D} \), and \( E \) be an arbitrary subset of \([a_0 - 1, a_0]\).

**Theorem 9.** If a sequence \((d_n)\) is bounded, then the family \( \Phi_2 \) of coverings of \([a_0 - 1, a_0]\) is faithful for the Hausdorff-Besicovitch dimension calculation.

**Proof.** Let us find conditions for \((d_n)\) such that the inequality
\[
m^\alpha_\varepsilon(E, \Phi_1) \leq m^\alpha_\varepsilon(E, \Phi_2)
\]
holds for \( \Phi_2 \subseteq \Phi_1 \).

Let \( u \) be an arbitrary closed interval of covering of \( E \), \( k \) be the minimal positive integer such that \( u \) does not contain nega-D-cylinders \( \Delta_{c_1c_2...c_{k-1}}^{-D} \) of rank \( k - 1 \). Then \( u \) belongs to not more than \( d_k \) cylinders of rank \( k \) but \( u \) contains a cylinder of rank \( k + 1 \). Hence,
\[
m^\alpha_\varepsilon(E, \Phi_1) = m^\alpha_\varepsilon(E, \Phi_2) \leq d_k d_{k+1} m^\alpha_\varepsilon(E, \Phi_1),
\]
where
\[
m^\alpha_\varepsilon(E, \Phi) = \inf_{d(E_j) \leq \varepsilon} \sum_j d^\alpha(E_j)
\]
for a fixed \( \varepsilon > 0 \), a fixed \( \alpha > 0 \), and covering of \( E \) by sets \( E_j \) with diameters \( d(E_j) \leq \varepsilon \).

Note that \( d_k d_{k+1} \leq (\max_n \{d_n\})^2 < \infty \) whenever \((d_n)\) is bounded. Indeed,
\[
0 < \lambda_1 = \frac{1}{\max_n \{d_n\}} \leq \frac{|\Delta_{c_1c_2...c_n}^{-D}|}{|\Delta_{c_1c_2...c_n}|} = \frac{1}{d_{n+1}} \leq \frac{1}{2} = \lambda_2 < 1,
\]
where \( \lambda_1 \) and \( \lambda_2 \) are fixed numbers for an arbitrary \( n \in \mathbb{N} \). It is true if and only if a sequence \((d_n)\) is bounded. \( \square \)

9. Sets of Incomplete Sums

Let \((d_n)\) be a fixed sequence of positive integers, \( d_n > 1 \), and let \((\varepsilon_n)\) be a fixed sequence of digits. Then we have a fixed number of the form
\[
\Delta_{\varepsilon_1\varepsilon_2...\varepsilon_n,...}^{-D} s_0 = \sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon_n}{d_1d_2...d_n}.
\]

Let \((A'_n)\) be a sequence of \( A'_n = \{0, \varepsilon_n\} \) and
\[
L'_{s_0} = A'_1 \times A'_2 \times A'_n \times ... = \{\delta : \delta = (\delta_1, \delta_2, ..., \delta_n, ...), \delta_n \in A'_n\}.
\]
Definition 6. A number of the form
\[ s = s(\delta) = \sum_{n=1}^{\infty} \frac{(-1)^n \delta_n}{d_1d_2 \cdots d_n}, \]  \hspace{1cm} (7)  
where \( \delta = (\delta_n) \in L_{s_0}', \) is called an **incomplete sum of alternating Cantor series** (6).

By \( M_{s_0} \) denote the set of all incomplete sums of alternating Cantor series (6), i.e.,
\[ M_{s_0} = \{ x : x = \Delta_{\delta_1 \delta_2 \cdots \delta_n}^{-D} (\delta_n) \in L_{s_0}' \}. \]

It is obvious that
\[ M_{s_0} \subset \left[ -\sum_{k=1}^{\infty} \frac{\varepsilon_{2k-1}}{d_1d_2 \cdots d_{2k-1}}, \sum_{k=1}^{\infty} \frac{\varepsilon_{2k}}{d_1d_2 \cdots d_{2k}} \right] = I_{s_0}^{M_{s_0}} \text{ for } s_0 = \Delta_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n}^{-D} \]
and \( M_{s_0} = \{ 0 \} \) for \( s_0 = 0. \) Moreover,
\[ \bigcup_{s_0} M_{s_0} = \left[ -\sum_{k=1}^{\infty} \frac{d_{2k-1}}{d_1d_2 \cdots d_{2k-1}}, \sum_{k=1}^{\infty} \frac{d_{2k}}{d_1d_2 \cdots d_{2k}} \right]. \]

We begin with definitions.

**Definition 7.** A cylinder of rank \( m \) with base \( c_1c_2 \cdots c_m \) is a set of the following form
\[ \Delta_{c_1c_2 \cdots c_m}^{M_{s_0}} = \left\{ x : x = \sum_{i=1}^{n} \frac{(-1)^i c_i}{d_1d_2 \cdots d_i} + \sum_{j=n+1}^{\infty} \frac{(-1)^j \delta_j}{d_1d_2 \cdots d_j} \right\}, \]

where \( c_1, c_2, \ldots, c_n \) are fixed numbers from \( A'_1, A'_2, \ldots, A'_n, \) respectively, and \( \delta_j \in A'_j. \)

**Definition 8.** A cylindrical closed interval (interval) \( I_{c_1c_2 \cdots c_n}^{M_{s_0}} (\nabla_{c_1c_2 \cdots c_n}^{M_{s_0}}) \) of rank \( n \) with base \( c_1c_2 \cdots c_n \) is a closed interval (interval) whose endpoints coincide with endpoints of \( \Delta_{c_1c_2 \cdots c_n}^{M_{s_0}}. \)

The following properties of cylindrical sets follow from Definition 7:

1. \[ \inf \Delta_{c_1c_2 \cdots c_n}^{M_{s_0}} = \begin{cases} \Delta_{c_1c_2 \cdots c_n 0 \varepsilon_{n+1} 0 \varepsilon_{n+2} \cdots}^{-D} & \text{if } n \text{ is odd} \\ \Delta_{c_1c_2 \cdots c_n 0 \varepsilon_{n+1} 0 \varepsilon_{n+2} \cdots}^{-D} & \text{if } n \text{ is even.} \end{cases} \]

2. \[ \sup \Delta_{c_1c_2 \cdots c_n}^{M_{s_0}} = \begin{cases} \Delta_{c_1c_2 \cdots c_n 0 \varepsilon_{n+1} 0 \varepsilon_{n+2} \cdots}^{-D} & \text{if } n \text{ is odd} \\ \Delta_{c_1c_2 \cdots c_n 0 \varepsilon_{n+1} 0 \varepsilon_{n+2} \cdots}^{-D} & \text{if } n \text{ is even.} \end{cases} \]
3. \[
\lim_{n \to \infty} d(\Delta_{c_1c_2 \ldots c_n}^{M_{n_0}}) = \lim_{n \to \infty} \Delta_{c_1c_2 \ldots c_n}^{D} = 0.\]

4. If \(\varepsilon_{n+1} \neq 0\), then \(\Delta_{c_1c_2 \ldots c_n}^{M_{n_0}} = \Delta_{c_1c_2 \ldots c_n}^{M_{n_0}0} \cup \Delta_{c_1c_2 \ldots c_n}^{M_{n_0} \varepsilon_{n+1}}\).

5. \[
\begin{align*}
\Delta_{c_1c_2 \ldots c_n}^{M_{n_0}} &\subset I_{c_1c_2 \ldots c_n}^{M_{n_0}} \subset \Delta_{c_1c_2 \ldots c_n}^{-D}, \\
M_{n_0} &\subset \bigcup_{c_i \in A_i, i=1,n} \Delta_{c_1c_2 \ldots c_n}^{M_{n_0}} \subset \bigcup_{c_i \in A_i, i=1,n} I_{c_1c_2 \ldots c_n}^{M_{n_0}}. 
\end{align*}
\]

6. \[
\left|\Delta_{c_1c_2 \ldots c_n}^{-D} \setminus \left(\Delta_{c_1c_2 \ldots c_n}^{-D} \cup \Delta_{c_1c_2 \ldots c_n}^{-D} \varepsilon_{n+1}\right)\right| = \begin{cases} \frac{d_{n+1}-2}{d_{n+1}^{D_{n+1}}} & \text{if } \varepsilon_{n+1} > 0 \\
\frac{d_{n+1}-1}{d_{n+1}^{D_{n+1}}} & \text{if } \varepsilon_{n+1} = 0. \end{cases}
\]

Here by \(|\cdot|\) denote the length of an interval and by \(\setminus\) denote the difference of sets.

**Lemma 22.** Let \(s_0 = \Delta_{e_1e_2 \ldots e_n}^{-D}\) be a fixed number, \((c_n)\) be an arbitrary fixed sequence from \(L'_{s_0}\); then the following are true:

1. \[
\bigcap_{i=1}^{\infty} \Delta_{c_1c_2 \ldots c_n}^{-D} = \bigcap_{i=1}^{\infty} \Delta_{c_1c_2 \ldots c_n}^{M_{n_0}} = \Delta_{c_1c_2 \ldots c_n}^{-D},
\]

2. \[
\Delta_{c_1c_2 \ldots c_n}^{M_{n_0}} = \Delta_{c_1c_2 \ldots c_n}^{-D} \cap M_{n_0},
\]

where

\[
M_{n_0} = \bigcap_{n=1}^{\infty} \left( \bigcup_{c_i \in A_i', i=1,n} \Delta_{c_1c_2 \ldots c_n}^{M_{n_0}} \right)
\]

and \(A_i' = \{0, \varepsilon_i\}\) for all positive integers \(i\).

**Proof.** 1. Let \(x = \Delta_{c_1c_2 \ldots c_n}^{-D}\). From the definition of \(\Delta_{c_1c_2 \ldots c_n}^{M_{n_0}}\), it follows that \(\Delta_{c_1c_2 \ldots c_n}^{-D} = x \in \Delta_{c_1c_2 \ldots c_n}^{M_{n_0}}\). Therefore \(x \in \bigcap_{i=1}^{\infty} I_{c_1c_2 \ldots c_n}^{M_{n_0}}\). The first statement follows from Property 5 of \(\Delta_{c_1c_2 \ldots c_n}^{M_{n_0}}\).

2. Let \(x \in M_{n_0}\). Then \(x\) belongs to a certain cylinder \(\Delta_{c_1c_2 \ldots c_n}^{M_{n_0}} \subset \Delta_{c_1c_2 \ldots c_n}^{-D}\). Let us consider the set \(\Delta_{c_1c_2 \ldots c_n}^{-D} \cap M_{n_0}\). Numbers of the form \(\Delta_{c_1c_2 \ldots c_n}^{-D} \delta_{n+1} \delta_{n+2} \delta_{n+k} \ldots\) where \(\delta_{n+k} \in A_{n+k}\) are elements of this set. Consequently, \(\Delta_{c_1c_2 \ldots c_n}^{-D} \cap M_{n_0}\).

Also, if \(x \in (\Delta_{c_1c_2 \ldots c_n}^{-D} \cap M_{n_0})\), then \(x \in \Delta_{c_1c_2 \ldots c_n}^{M_{n_0}}\). \(\square\)
Theorem 10. The set $M_{s_0}$ of incomplete sums of alternating Cantor series (6) is:

1. the one-element set $\{0\}$ whenever $s_0 = 0$;
2. a finite set whenever the condition $\varepsilon_n \neq 0$ holds for a finite number of $n$ in $s_0 = \Delta^{-D}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n}$;
3. the segment $[-\frac{2}{3}; \frac{1}{3}]$ whenever $d_n = \text{const} = 2$ for all $n \in \mathbb{N}$ and $s_0 = -\frac{1}{3}$;
4. a union of finite number of segments whenever there exists a finite number of $m_i$ $(i = 1, k_0$, $k_0$ is a fixed number) that $d_{m_i} \neq 2$ and $s_0 = \Delta^{-D}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m_0}}(1)$;
5. an uncountable, perfect, and nowhere dense set of zero Lebesgue measure whenever $s_0 \neq 0$ and $d_n > 2$ hold for an infinite number of $n$.

Proof. Since an alternating Cantor series is the nega-binary sum

$$\frac{-\varepsilon_1}{2} + \frac{\varepsilon_2}{2^2} - \frac{\varepsilon_3}{2^3} + \cdots + \frac{(-1)^n\varepsilon_n}{2^n} + \ldots,$$

where $\varepsilon_n \in \{0, 1\}$, whenever $d_n = \text{const} = 2$ and the set $M_{s_0}$ is $C[-D, A_n']$, we see that statements 1-4 are true.

We now prove statement 5 is true. Let the mapping $f : M_{s_0} \to C[E, V_n]$, where $V_n = A_n'$, be given by

$$x = \Delta^{-D}_{\delta_1, \delta_2, \ldots, \delta_n} \frac{1}{\sum_{n=1}^{\infty} \prod_{i=1}^{n} (2 + \delta_i)(2 + \delta_1 + \delta_2) \cdots (2 + \delta_1 + \cdots + \delta_n)} = \Delta^{-D}_{\delta_1, \delta_2, \ldots, \delta_n} = f(x) = y.$$

Here $\Delta^{-D}_{\delta_1, \delta_2, \ldots, \delta_n}$ is the representation by an Engel series. This mapping is not a bijection at the nega-D-rational points

$$\Delta^{-D}_{\delta_1, \delta_2, \ldots, \delta_k - 1, \delta_k} = \Delta^{-D}_{\delta_1, \delta_2, \ldots, \delta_k - 1, \delta_k} = \Delta^{-D}_{\delta_1, \delta_2, \ldots, \delta_k - 1, \delta_k}.$$

It is true when $s_0 = \Delta^{-D}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{k-1}}[d_k + 1][d_k + 1][d_k + 1] \cdots$.

Since the set of nega-D-rational numbers is not more than countable in $M_{s_0}$, we can use one of the representations of a nega-D-rational number (e.g., the first representation) when the argument is a nega-D-rational number. Hence $M_{s_0}$ is uncountable, since $C[E, V_n]$ is uncountable.

Let us prove that the set $M_{s_0}$ is nowhere dense. Choose a cylinder $\Delta^{-D}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}}$ such that the condition $\varepsilon_n \neq 0$ holds for $s_0 = \Delta^{-D}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}}$. Consider the mutual
placement of $\Delta_{c_1\cdots c_{n-1}}^{M_0}$ and $\Delta_{c_1\cdots c_{n-1}0}^{M_0}$. Let $n$ be even. Then

$$\inf \Delta_{c_1\cdots c_{n-1}0}^{M_0} - \sup \Delta_{c_1\cdots c_{n-1}}^{M_0} = \sum_{i=1}^{n-1} \frac{(-1)^i c_i}{d_1 d_2 \cdots d_i} + \frac{\varepsilon_n}{d_1 d_2 \cdots d_n}$$

$$- \sum_{k=1}^{\infty} \frac{\varepsilon_{n+2k-1}}{d_1 d_2 \cdots d_{n+2k-1}} - \sum_{i=1}^{n-1} (-1)^i c_i - \sum_{k=1}^{\infty} \frac{\varepsilon_{n+2k}}{d_1 d_2 \cdots d_{n+2k}}$$

$$= \frac{\varepsilon_n}{d_1 d_2 \cdots d_n} - \sum_{k=1}^{\infty} \frac{\varepsilon_{n+k}}{d_1 d_2 \cdots d_{n+k}} = 1 - \sum_{k=1}^{\infty} \frac{\varepsilon_{n+k}}{d_{n+1} \cdots d_{n+k}} \geq 0.$$  

That is the cylinders are left-to-right situated and the last-mentioned difference equals zero when $s_0 = \Delta_{c_1\cdots c_{n-1}1}^{D_{\varepsilon_1\cdots\varepsilon_{n-1}1}[d_{n+1}-1][d_{n+2}-1][d_{n+3}-1]}$. Similarly, the inequality

$$\inf \Delta_{c_1\cdots c_{n-1}0}^{M_0} - \sup \Delta_{c_1\cdots c_{n-1}0}^{M_0} \geq 0$$

holds when $n$ is odd. That is cylinders $\Delta_{c_1\cdots c_{n-1}0}$ are right-to-left situated. Thus for any interval belonging to $[\inf M_0, \sup M_0]$ there exists a subinterval that does not contain points from $M_0$, since $\Delta_{c_1\cdots c_{n-1}0}^{M_0} \cap \Delta_{c_1\cdots c_{n-1}0}^{M_0} \neq \emptyset$ if and only if

$$\varepsilon_n = 0 \text{ or } s_0 = \Delta_{c_1\cdots c_{n-1}0}^{D_{\varepsilon_1\cdots\varepsilon_{n-1}1}[d_{n+1}-1][d_{n+2}-1][d_{n+3}-1]}.$$  

Let us prove that $M_0$ is a closed set without isolated points. Choose an arbitrary limit point $x_0$ from $M_0$. From the definition of a limit point it follows that for all $\varepsilon > 0$ an interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ contains at least one point (that does not coincide with $x_0$) from $M_0$. If there does not exist a unique closed interval $I_{\Delta_{c_1(x_0)\cdots c_n(x_0)}}^{M_0}$ such that $x_0 \in I_{\Delta_{c_1(x_0)\cdots c_n(x_0)}}^{M_0}$, then $x_0$ belongs to one of the adjacent to $M_0$ intervals. Therefore there exists $\varepsilon_0 > 0$ such that $(x_0 - \varepsilon_0, x_0 + \varepsilon_0) \cap M_0 = \emptyset$. In this case, $x_0$ is not a limit point. If there exists a closed interval $I_{\Delta_{c_1(x_0)\cdots c_n(x_0)}}^{M_0}$, then

$$x_0 = \bigcap_{n=1}^{\infty} I_{\Delta_{c_1(x_0)\cdots c_n(x_0)}}^{M_0} \text{ and } x_0 \in M_0.$$  

Hence $M_0$ is a closed set.

Suppose there exists a certain isolated point $x' = \Delta_{\delta_1\cdots\delta_n}^{M_0}$. Then there exists $\varepsilon_0 > 0$ such that

$$(x' - \varepsilon_0, x' + \varepsilon_0) \cap (M_0 \setminus \{x'\}) = \emptyset. \quad (8)$$

Take a number $m$ such that $d(\Delta_{\delta_1\cdots\delta_m}^{M_0}) < \varepsilon_0$ and $\varepsilon_{m+1}(s_0) \neq 0$. Then $\Delta_{\delta_1\cdots\delta_m}^{M_0} \subset (x' - \varepsilon_0, x' + \varepsilon_0)$ and

$$x' \neq x = \Delta_{\delta_1\cdots\delta_m\sigma_{m+2}}^{D_{\delta_1\cdots\delta_m\sigma_{m+2}}} \in (x' - \varepsilon_0, x' + \varepsilon_0) \cap M_0,$$
where
\[ \sigma = \begin{cases} 
\varepsilon_{m+1} & \text{if } \delta_{m+1} = 0 \\
0 & \text{if } \delta_{m+1} \neq 0.
\end{cases} \]

The last condition contradicts (8). The set \( M_{s_0} \) does not contain isolated points.

Let us calculate the Lebesgue measure of \( M_{s_0} \). Let \( F_k \) be the union of closed intervals \( I_{c_1\ldots c_k}^{M_{s_0}} \) of rank \( k \) (\( c_k \in A_k \)). Then \( M_{s_0} \subseteq F_k \subseteq F_{k+1} \) for all \( k \in \mathbb{N} \) and \( \lambda(M_{s_0}) \leq \lim_{k \to \infty} F_k \). Since the condition \( d(\Delta_{c_1\ldots c_n}^{M_{s_0}}) = |I_{c_1\ldots c_n}^{M_{s_0}}| \) and the properties of \( \Delta_{c_1\ldots c_n}^{M_{s_0}} \) hold, it follows that
\[
\lambda(M_{s_0}) \leq \lim_{k \to \infty} \left( \frac{2^k \cdot \Delta_{c_1\ldots c_k}^{I_{c_1\ldots c_n}^{M_{s_0}}}}{d_1 d_2 \cdots d_k} \sum_{i=k+1}^{\infty} \frac{\varepsilon_i(s_0)}{d_{k+1} \cdots d_i} \right) = 0. \]

References