REPRESENTATION OF REAL NUMBERS BY THE ALTERNATING CANTOR SERIES

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Abstract

The article is devoted to alternating Cantor series. It is proved that any real number belonging to $[a_0-1, a_0]$, where $a_0 = \sum_{k=1}^{\infty} \frac{d_{2k}-1}{d_1 d_2 \cdots d_{2k}}$, has not more than two representations by such series, and only the numbers from a certain countable subset of real numbers have two representations. The geometry of these representations, properties of cylinder and semicylinder sets, and the simplest metric problems are investigated. Some applications of such series to fractal theory and the relation between positive and alternating Cantor series are described. The shift operator with some its applications, as well as the set of incomplete sums are studied. Necessary and sufficient conditions for a rational number to be representable by an alternating Cantor series are formulated.

Introduction

The investigation of various numeral systems is useful for the development of metric, probability, and fractal theories of real numbers, for the study of fractal and other properties of mathematical objects possessing a complicated local structure such as continuous nowhere differentiable or singular functions, random variables of Jessen-Wintner type, DP-transformations (transformations preserving the fractal Hausdorff-Besicovitch dimension), dynamical systems with chaotic trajectories, etc. [2, 3].

There exist systems of real number representations with a finite or an infinite alphabet, with redundant digits or with zero redundancy. The s-adic and nega-s-adic numeral systems [5] are examples of real number encodings with a finite alphabet, whereas number representations by Lüroth series [6], regular continued fractions, polybasic nega-$\tilde{Q}$-representations [11], etc., are examples of encoding them with an infinite alphabet. A representation

$$x = \frac{\varepsilon_1}{d_1} + \frac{\varepsilon_2}{d_1 d_2} + \cdots + \frac{\varepsilon_n}{d_1 d_2 \cdots d_n} + \ldots, \varepsilon_n \in A_{d_n}, \tag{1}$$

of a real number x by a positive Cantor series [1, 7, 8], is an example of a polybasic numeral system with zero redundancy. Here (d_n) is a fixed sequence of positive integers, $d_n > 1$, and (A_{d_n}) is a sequence of the sets $A_{d_n} = \{0, 1, \dots, d_n - 1\}$. This encoding of real numbers has a finite alphabet when (d_n) is bounded.

The representation of real numbers by positive Cantor series is a generalization of the classical s-adic numeral system. Note that this representation is "similar" to the following series

$$\sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n},$$

where (a_n) is a monotone non-decreasing sequence of positive integers and $a_1 \geq 1$. This series is called an Engel series.

In 1869, Georg Cantor [1] considered series expansions of real numbers (1). There are many papers [1, 4, 7, 10, 8] where properties of real number representations by positive Cantor series are studied, but many problems related to these series are not solved completely. For example, criteria of representation of rational numbers, modeling of functions with a complicated local structure are still open problems.

Since real number expansions by positive Cantor series are useful for studying complicated objects of fractal analysis, the notion of alternating Cantor series is introduced in the present article. Alternating Cantor series, which generalize the nega-s-adic numeral system, were not considered in publications earlier. In this paper the foundations of the metric theory of real number representations by alternating Cantor series are given and some related problems of mathematical analysis are considered.

Consider the main object of this article.

Let (d_n) be a fixed sequence of numbers from $\mathbb{N} \setminus \{1\}$, (A_{d_n}) be a sequence of the sets $A_{d_n} = \{0, 1, 2, \dots, d_n - 1\}$.

Definition 1. A series of the form

$$-\frac{\varepsilon_1}{d_1} + \frac{\varepsilon_2}{d_1 d_2} - \frac{\varepsilon_3}{d_1 d_2 d_3} + \cdots + \frac{(-1)^n \varepsilon_n}{d_1 d_2 d_3 \cdots d_n} + \dots, \tag{2}$$

where $\varepsilon_n \in A_{d_n}$ for any $n \in \mathbb{N}$, is called *an alternating Cantor series.*

The number d_n is called *the nth element* of sum (2), and ε_n is called *the nth digit* of sum (2).

By $\Delta^{-D}_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n \dots}$ denote any number x having expansion (2). This notation is called *the nega-D-representation of* x. Expansion (2) of x is called *the nega-D-expansion of* x.

Remark 1. The term "nega" is used in this article, since the alternating Cantor series expansion is a numeral system with a negative base, i.e.,

$$x = \sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon_n}{d_1 d_2 \cdots d_n} = \frac{\varepsilon_1}{-d_1} + \frac{\varepsilon_2}{(-d_1)(-d_2)} + \cdots + \frac{\varepsilon_n}{(-d_1)(-d_2) \cdots (-d_n)} + \dots.$$

If (d_n) is purely periodic with the simple period (s), where $s > 1$ is a fixed positive integer, then series (2) has the form

$$x = -\frac{\varepsilon_1}{s} + \frac{\varepsilon_2}{s^2} - \frac{\varepsilon_3}{s^3} + \cdots + \frac{(-1)^n \varepsilon_n}{s^n} + \ldots, \ \varepsilon_n \in \{0, 1, \ldots, s-1\}.$$

The last-mentioned series is the nega-s-adic expansion [5, 9] of numbers in $[-\frac{s}{s+1}, \frac{1}{s+1}]$.

The following series are alternating Cantor series:

1. $$\sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon_n}{s^{\alpha_1+\alpha_2+\cdots+\alpha_n}},$$

   where α_n belongs to some finite subset of positive integers, $\varepsilon_n \in \{0, 1, \ldots, s^{\alpha_n} - 1\}$ for each $n \in \mathbb{N}$, and $1 < s \in \mathbb{N}$ is a fixed number;

2. $$\sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon_n}{2 \cdot 3 \cdot \ldots \cdot (n+1)}, \ \varepsilon_n \in \{0, 1, \ldots, n\};$$

3. $$\sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon_n}{p_1 p_2 \cdots p_n},$$

   where (p_n) is the increasing sequence of all prime numbers.

Lemma 1. *Every alternating Cantor series is absolutely convergent and its sum belongs to* $[a_0 - 1, a_0]$, *where*

$$a_0 = \sum_{n=1}^{\infty} \frac{d_{2n} - 1}{d_1 d_2 \cdots d_{2n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{d_1 d_2 \cdots d_n}.$$

Proof. This statement follows from the propositions:

- the series
  $$\sum_{n=1}^{\infty} \frac{\varepsilon_n}{d_1 d_2 \cdots d_n}$$
  is convergent;

- the condition
  $$-1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{d_1 d_2 \ldots d_n} \le S \le \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{d_1 d_2 \cdots d_n},$$
  holds, where S is equal to the sum of series (2).

□

Lemma 2. *Let series* (2) *be a fixed series,*

$$r_n = \frac{(-1)^n}{d_1 d_2 \cdots d_n} \sum_{k=1}^{\infty} \frac{(-1)^k \varepsilon_{n+k}}{d_{n+1} \cdots d_{n+k}}, \text{ and } a_n = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{d_{n+1} d_{n+2} \cdots d_{n+k}}$$

for this series; then the following inequalities hold:

$$\frac{a_n - 1}{d_1 \cdots d_n} \le r_n \le \frac{a_n}{d_1 \cdots d_n} \text{ whenever } n \text{ is even;}$$

$$-\frac{a_n}{d_1 d_2 \cdots d_n} \le r_n \le \frac{1 - a_n}{d_1 d_2 \cdots d_n} \text{ whenever } n \text{ is odd.}$$

1. Representation of Real Numbers by Alternating Cantor Series

Lemma 3. *Each number* $x \in [a_0 - 1, a_0]$ *can be represented by series* (2).

Proof. It is obvious that $a_0 - 1 = \Delta^{-D}_{[d_1-1]0[d_3-1]0\ldots}$ and $a_0 = \Delta^{-D}_{0[d_2-1]0[d_4-1]\ldots}$.

Since x is an arbitrary number from $(a_0 - 1, a_0)$,

$$-\frac{\varepsilon_1}{d_1} - \sum_{k=2}^{\infty} \frac{d_{2k-1} - 1}{d_1 d_2 \cdots d_{2k-1}} < x \le -\frac{\varepsilon_1}{d_1} + \sum_{k=1}^{\infty} \frac{d_{2k} - 1}{d_1 d_2 \cdots d_{2k}}$$

with $0 \le \varepsilon_1 \le d_1 - 1$, and

$$[a_0 - 1, a_0] = I_0 = \bigcup_{i=0}^{d_1 - 1} \left[-\frac{i}{d_1} - \sum_{k=2}^{\infty} \frac{d_{2k-1} - 1}{d_1 d_2 \cdots d_{2k-1}}, -\frac{i}{d_1} + \sum_{k=1}^{\infty} \frac{d_{2k} - 1}{d_1 d_2 \cdots d_{2k}} \right],$$

we have

$$-\sum_{k=2}^{\infty} \frac{d_{2k-1} - 1}{d_1 d_2 \cdots d_{2k-1}} < x + \frac{\varepsilon_1}{d_1} \le \sum_{k=1}^{\infty} \frac{d_{2k} - 1}{d_1 d_2 \cdots d_{2k}}.$$

Let $x + \frac{\varepsilon_1}{d_1} = x_1$. Then we obtain the following cases:

1. If the equality

$$x_1 = \sum_{k=1}^{\infty} \frac{d_{2k} - 1}{d_1 d_2 \cdots d_{2k}}$$

   holds, then

$$x = \Delta^{-D}_{\varepsilon_1 [d_2-1]0[d_4-1]0\ldots} \text{ or } x = \Delta^{-D}_{[\varepsilon_1-1]0[d_3-1]0[d_5-1]0\ldots}.$$

2. If the first case does not hold, then $x = -\frac{\varepsilon_1}{d_1} + x_1$, where

$$\frac{\varepsilon_2}{d_1 d_2} - \sum_{k=2}^{\infty} \frac{d_{2k-1} - 1}{d_1 d_2 \cdots d_{2k-1}} \le x_1 < \frac{\varepsilon_2}{d_1 d_2} + \sum_{k=2}^{\infty} \frac{d_{2k} - 1}{d_1 d_2 \cdots d_{2k}}.$$

In this case, let $x_2 = x_1 - \frac{\varepsilon_2}{d_1 d_2}$. Then:

1. if the equality

$$x_2 = \sum_{k=2}^{\infty} \frac{d_{2k-1} - 1}{d_1 d_2 \cdots d_{2k-1}}$$

holds, then

$$x = \Delta^{-D}_{\varepsilon_1 \varepsilon_2 [d_3 - 1] 0 [d_5 - 1] 0 \ldots} \text{ or } x = \Delta^{-D}_{\varepsilon_1 [\varepsilon_2 - 1] 0 [d_4 - 1] 0 [d_6 - 1] 0 \ldots}.$$

2. In the converse case,

$$x = -\frac{\varepsilon_1}{d_1} + \frac{\varepsilon_2}{d_1 d_2} + x_2, \text{ where}$$

$$-\frac{\varepsilon_3}{d_1 d_2 d_3} - \sum_{k=3}^{\infty} \frac{d_{2k-1} - 1}{d_1 d_2 \cdots d_{2k-1}} < x_2 \le -\frac{\varepsilon_3}{d_1 d_2 d_3} + \sum_{k=2}^{\infty} \frac{d_{2k} - 1}{d_1 d_2 \cdots d_{2k}}, \text{ etc.}$$

Therefore,

$$-\sum_{k > \frac{m+2}{2}} \frac{d_{2k-1} - 1}{d_1 d_2 \cdots d_{2k-1}} < x_m - \frac{(-1)^{m+1} \varepsilon_{m+1}}{d_1 d_2 \cdots d_{m+1}} < \sum_{k > \frac{m+1}{2}} \frac{d_{2k} - 1}{d_1 d_2 \cdots d_{2k}}$$

for some positive integer m. Moreover, the following cases are possible:

1. $$x_{m+1} = \begin{cases} \sum_{k > \frac{m+2}{2}} \frac{d_{2k-1} - 1}{d_1 d_2 \cdots d_{2k-1}} & \text{if } m \text{ is odd} \\ \sum_{k > \frac{m+1}{2}} \frac{d_{2k} - 1}{d_1 d_2 \cdots d_{2k}} & \text{if } m \text{ is even.} \end{cases}$$

   In this case,

$$x = \Delta^{-D}_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_{m+1} [d_{m+2} - 1] 0 [d_{m+4} - 1] 0 \ldots}$$

   or

$$x = \Delta^{-D}_{\varepsilon_1 \ldots \varepsilon_m [\varepsilon_{m+1} - 1] 0 [d_{m+3} - 1] 0 [d_{m+5} - 1] 0 \ldots}.$$

2. If there does not exist a number $m \in \mathbb{N}$ such that the last-mentioned system is satisfied, then

$$x = -\frac{\varepsilon_1}{d_1} + x_1 = \cdots = -\frac{\varepsilon_1}{d_1} + \frac{\varepsilon_2}{d_1 d_2} - \frac{\varepsilon_3}{d_1 d_2 d_3} + \cdots + \frac{(-1)^n \varepsilon_n}{d_1 d_2 \cdots d_n} + x_n = \ldots.$$

Hence,

$$x = \sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon_n}{d_1 d_2 \cdots d_n}.$$

□

Lemma 4. *The numbers*

$$x = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{m-1}\varepsilon_m\varepsilon_{m+1}\ldots} \text{ and } x^{'} = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{m-1}\varepsilon^{'}_m\varepsilon^{'}_{m+1}\ldots},$$

where $\varepsilon_m \neq \varepsilon^{'}_m$*, are equal if and only if one of the following systems*

$$\begin{cases} \varepsilon_{m+2i-1} & = & d_{m+2i-1}-1 \\ \varepsilon_{m+2i} & =0 & =\varepsilon^{'}_{m+2i-1} \\ \varepsilon^{'}_{m+2i} & = & d_{m+2i}-1 \\ \varepsilon^{'}_m & = & \varepsilon_m - 1 \end{cases} \quad or \begin{cases} \varepsilon_{m+2i} & = & d_{m+2i}-1 \\ \varepsilon_{m+2i-1} & =0 & =\varepsilon^{'}_{m+2i} \\ \varepsilon^{'}_{m+2i-1} & = & d_{m+2i-1}-1 \\ \varepsilon^{'}_m - 1 & = & \varepsilon_m \end{cases}$$

is satisfied for all $i \in \mathbb{N}$.

Proof. We prove necessity. Let $\varepsilon_m = \varepsilon^{'}_m + 1$. Then

$$0 = x - x^{'} = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{m-1}\varepsilon_m\varepsilon_{m+1}\ldots} - \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{m-1}\varepsilon^{'}_m\varepsilon^{'}_{m+1}\ldots} = \frac{(-1)^m}{d_1d_2\cdots d_m}$$
$$+ \frac{(-1)^{m+1}(\varepsilon_{m+1}-\varepsilon^{'}_{m+1})}{d_1d_2\cdots d_{m+1}} + \cdots + \frac{\varepsilon_{m+i}-\varepsilon^{'}_{m+i}}{d_1d_2\cdots d_{m+i}}(-1)^{m+i} + \ldots$$
$$= \frac{(-1)^m}{d_1d_2\cdots d_m}\left(1 + \sum_{i=1}^{\infty}\frac{(-1)^i(\varepsilon_{m+i}-\varepsilon^{'}_{m+i})}{d_{m+1}d_{m+2}\cdots d_{m+i}}\right),$$

where

$$\sum_{i=1}^{\infty}\frac{(-1)^i(\varepsilon_{m+i}-\varepsilon^{'}_{m+i})}{d_{m+1}d_{m+2}\cdots d_{m+i}} \geq -\sum_{i=1}^{\infty}\frac{d_{m+i}-1}{d_{m+1}d_{m+2}\cdots d_{m+i}} = -1.$$

The last-mentioned inequality is an equality when

$$\varepsilon_{m+2i} = \varepsilon^{'}_{m+2i-1} = 0, \ \varepsilon_{m+2i-1} = d_{m+2i-1}-1, \ \text{ and } \varepsilon^{'}_{m+2i} = d_{m+2i}-1.$$

In our case, the conditions of the first system follow from $x = x^{'}$. It is easy to see that the conditions of the second system follow from $x = x^{'}$ when $\varepsilon^{'}_m = \varepsilon_m + 1$.

The proof of *sufficiency* is trivial. □

Definition 2. The nega-D-representation $\Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}$ of some number $x \in [a_0 - 1, a_0]$ is called *periodic* if there exist numbers $m \in \mathbb{Z}_0$ and $t \in \mathbb{N}$ such that the equality $\varepsilon_{m+nt+j} = \varepsilon_{m+j}$ holds for each $n \in \mathbb{N}$, $j \in \mathbb{N}$.

Denote by $\Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_m(\varepsilon_{m+1}\varepsilon_{m+2}\ldots\varepsilon_{m+t})}$ any number whose nega-D-representation is periodic with *the period* $(\varepsilon_{m+1}\varepsilon_{m+2}\ldots\varepsilon_{m+t})$ of *lenght* t.

A periodic representation is:

- *purely periodic* if $m = 0$;
- *mixed periodic* if $m > 0$.

Definition 3. Denote by

$$x = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_m\phi_1(d_{m+1})\phi_2(d_{m+2})\ldots\phi_t(d_{m+t})\phi_1(d_{m+t+1})\phi_2(d_{m+t+2})\ldots\phi_t(d_{m+2t})\ldots}$$

any *quasi-periodic number* x. Here $m \in \mathbb{Z}_0$, $t \in \mathbb{N}$, and $\phi_1, \phi_2, \ldots, \phi_t$ are functions such that $\phi_i(d_n) \in A_{d_n}$ for any $n \in \mathbb{N}$ and $i = \overline{1,t}$. That is $\phi_i(d_n)$ is a regularity that depends on the parameter d_n.

The following numbers are quasi-periodic:

$$\Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n 0[d_{n+2}-1][d_{n+3}-1]\ldots[d_{n+i}-1]\ldots},\ \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n 0[d_{n+1}-1]0[d_{n+3}-1]0[d_{n+5}-1]\ldots},\ \text{etc.}$$

Definition 4. A number $x \in I_0 = [a_0 - 1, a_0]$ is called *nega-D-rational* if

$$x = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{n-1}\varepsilon_n[d_{n+1}-1]0[d_{n+3}-1]0[d_{n+5}-1]\ldots}$$

or

$$x = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{n-1}[\varepsilon_n-1]0[d_{n+2}-1]0[d_{n+4}-1]0[d_{n+6}-1]\ldots}.$$

The other numbers in I_0 are called *nega-D-irrational*.

The next proposition follows from Lemma 3 and Lemma 4.

Theorem 1. *Every nega-D-irrational number has the unique nega-D-representation. Every nega-D-rational number has two nega-D-representations, i.e.,*

$$\Delta^{-D}_{\varepsilon_1\ldots\varepsilon_{n-1}\varepsilon_n[d_{n+1}-1]0[d_{n+3}-1]0[d_{n+5}-1]\ldots} = \Delta^{-D}_{\varepsilon_1\ldots\varepsilon_{n-1}[\varepsilon_n-1]0[d_{n+2}-1]0[d_{n+4}-1]0[d_{n+6}-1]\ldots}.$$

Remark 2. There exist sequences (d_n) such that a nega-D-rational number is an irrational number. For example, the following numbers are nega-D-rational:

$$\begin{aligned}
x &= \sum_{i=1}^{n} \frac{(-1)^i \varepsilon_i}{d_1 d_2 \cdots d_i} + \frac{(-1)^n}{d_1 d_2 \cdots d_n}\left(-1 - \sum_{j=1}^{\infty} \frac{(-1)^j}{2 \cdot 3 \cdot \ldots \cdot (j+1)}\right) \\
&= \sum_{i=1}^{n} \frac{(-1)^i \varepsilon_i}{d_1 d_2 \cdots d_i} + \frac{(-1)^n}{d_1 d_2 \cdots d_n}\left(-1 + \frac{1}{e}\right), \\
x &= \sum_{i=1}^{n} \frac{(-1)^i \varepsilon_i}{d_1 d_2 \cdots d_i} + \frac{(-1)^n}{d_1 d_2 \cdots d_n}\left(-1 - \sum_{j=1}^{\infty} \frac{(-1)^j}{2 \cdot 4 \cdot \ldots \cdot 2j}\right) \\
&= \sum_{i=1}^{n} \frac{(-1)^i \varepsilon_i}{d_1 d_2 \cdots d_i} + \frac{(-1)^{n+1}}{d_1 d_2 \cdots d_n} \cdot \frac{\sqrt{e}}{e},
\end{aligned}$$

since

$$\Delta^{-D}_{\varepsilon_1\ldots\varepsilon_n[d_{n+1}-1]0[d_{n+3}-1]0\ldots} = g_n + \frac{(-1)^n}{d_1 d_2 \cdots d_n}\left(-1 - \sum_{j=1}^{\infty} \frac{(-1)^j}{d_{n+1} \cdots d_{n+j}}\right)$$

and

$$\Delta^{-D}_{\varepsilon_1\ldots\varepsilon_{n-1}[\varepsilon_n-1]0[d_{n+2}-1]0\ldots} = g_n + \frac{(-1)^{n+1}}{d_1d_2\cdots d_n}\left(1+\sum_{j=1}^{\infty}\frac{(-1)^j}{d_{n+1}\cdots d_{n+j}}\right),$$

where

$$g_n = \sum_{i=1}^{n}\frac{(-1)^i\varepsilon_i}{d_1d_2\cdots d_i}.$$

To avoid some inconveniences in the future, we can modify expansion (2) of $x \in [-1+a_0, a_0]$ to the following form

$$x = \sum_{n=1}^{\infty}\frac{1+\varepsilon_n}{d_1d_2\cdots d_n}(-1)^{n+1}, \tag{3}$$

where x represented in form (3) belongs to $[0,1]$, $\varepsilon_n \in A_{d_n}$, and $a_0 = -\Delta^{-D}_{(1)}$.

It is easy to see that

$$\inf\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{d_1d_2\cdots d_n} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\varepsilon_n}{d_1d_2\cdots d_n}\right) = g^{'} - \sum_{i=1}^{\infty}\frac{d_{2i}-1}{d_1d_2\cdots d_{2i}} = 0$$

and

$$\sup\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{d_1d_2\cdots d_n} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\varepsilon_n}{d_1d_2\cdots d_n}\right) = g^{'} + \sum_{i=1}^{\infty}\frac{d_{2i-1}-1}{d_1d_2\cdots d_{2i-1}} = 1,$$

where

$$g^{'} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{d_1d_2\cdots d_n}.$$

By $\Delta^{-(d_n)}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}$ denote any number $x \in [0,1]$ having expansion (3). This notation is called *the nega-(d_n)-representation of* $x \in [0,1]$. The number d_n in (3) is called *the nth element* and $\varepsilon_n = \varepsilon_n(x)$ is *the nth digit* of expansion (3).

2. Some Properties

Suppose that $x = \Delta^{-D}_{\varepsilon_1(x)\varepsilon_2(x)\ldots\varepsilon_n(x)\ldots}$ and $y = \Delta^{-D}_{\varepsilon_1(y)\varepsilon_2(y)\ldots\varepsilon_n(y)\ldots}$.

Proposition 1. *The inequality $x < y$ holds for any numbers x and y from $[-1+a_0, a_0]$ if and only if there exists a number m such that*

$$\varepsilon_n(x) = \varepsilon_n(y) \text{ for } n < 2m \text{ and } \varepsilon_{2m}(x) < \varepsilon_{2m}(y)$$

or

$$\varepsilon_n(x) = \varepsilon_n(y) \text{ for } n < 2m-1 \text{ and } \varepsilon_{2m-1}(x) > \varepsilon_{2m-1}(y).$$

Proposition 2. *Suppose that* $x_1 = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{2k-1}(0)}$, $x_2 = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{2k}(0)}$, *and* $x_3 = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{2k+1}(0)}$, *and* $\varepsilon_i \neq 0$ *for all* $i = \overline{1, 2k+1}$. *Then the following two-sided inequality holds:*

$$x_1 < x_3 < x_2.$$

Proof. This statement follows from the relationship

$$x_3 = x_1 + \frac{1}{d_1 d_2 \cdots d_{2k}}\left(\varepsilon_{2k} - \frac{\varepsilon_{2k+1}}{d_{2k+1}}\right) = x_2 - \frac{\varepsilon_{2k+1}}{d_1 d_2 \cdots d_{2k+1}}.$$

□

Proposition 3. *Suppose that* $z_1 = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{2k}(0)}$, $z_2 = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{2k+1}(0)}$, *and* $z_3 = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{2k+2}(0)}$, *and* $\varepsilon_i \neq 0$ *for all* $i = \overline{1, 2k+2}$. *Then the two-sided inequality*

$$z_2 < z_3 < z_1$$

holds.

Proof. Since the relationship

$$z_3 = z_1 - \frac{1}{d_1 d_2 \cdots d_{2k+1}}\left(\varepsilon_{2k+1} - \frac{\varepsilon_{2k+2}}{d_{2k+2}}\right) = z_2 + \frac{\varepsilon_{2k+2}}{d_1 d_2 \cdots d_{2k+2}}$$

holds, we see that our statement is true. □

3. Relations Between Positive and Alternating Cantor Series

Let (d_n) be a fixed sequence of positive integers, $d_n > 1$. For any $x \in [0, 1]$ there exists a sequence (α_n), where $\alpha_n \in A_{d_n}$, such that

$$\Delta^{D}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} = x = \sum_{n=1}^{\infty} \frac{\alpha_n}{d_1 d_2 \cdots d_n}.$$

It is obvious that

$$x = \frac{\alpha_1 d_2 + \alpha_2}{d_1 d_2} + \frac{\alpha_3 d_4 + \alpha_4}{d_1 d_2 d_3 d_4} + \cdots + \frac{\alpha_{2n-1} d_{2n} + \alpha_{2n}}{d_1 d_2 \cdots d_{2n}} + \ldots.$$

This representation is the representation of x by a positive Cantor series with the sequence of elements $(d^{'}_n)$, where $d^{'}_n = d_{2n-1} d_{2n}$. In fact, $0 \leq \alpha_{2n-1} d_{2n} + \alpha_{2n} \leq d_{2n-1} d_{2n} - 1$ and therefore,

$$\Delta^{D^{'}_1}_{\beta_1\beta_2\ldots\beta_n\ldots} = x = \sum_{n=1}^{\infty} \frac{\beta_n}{p_1 p_2 \cdots p_n}, \tag{4}$$

where $\beta_n = \alpha_{2n-1}d_{2n} + \alpha_{2n}$, $p_n = d_{2n-1}d_{2n}$ for any $n \in \mathbb{N}$.

Let us consider representation (2). Using the same technique, we get

$$x = \frac{\varepsilon_2 - \varepsilon_1 d_2}{d_1 d_2} + \frac{\varepsilon_4 - \varepsilon_3 d_4}{d_1 d_2 d_3 d_4} + \cdots + \frac{\varepsilon_{2n} - \varepsilon_{2n-1} d_{2n}}{d_1 d_2 \cdots d_{2n}} + \ldots.$$

But $(\varepsilon_{2n} - \varepsilon_{2n-1}d_{2n})$ belongs to $\{0, 1, \ldots, d_{2n-1}d_{2n} - 1\}$ for not all values of ε_{2n-1} and ε_{2n}.

Consider expansion (3). Indeed, for

$$\Delta^{-(d_n)}_{\delta_1\delta_2\ldots\delta_n\ldots} = x = \sum_{n=1}^{\infty} \frac{1 + \delta_n}{d_1 d_2 \cdots d_n}(-1)^{n+1},$$

where

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{d_1 d_2 \cdots d_n} \equiv \Delta^{-D}_{0[d_2-1]0[d_4-1]0\ldots},$$

we obtain

$$x = \sum_{n=1}^{\infty} \frac{d_{2n} - 1}{d_1 d_2 \cdots d_{2n}} + \frac{\delta_1 d_2 - \delta_2}{d_1 d_2} + \frac{\delta_3 d_4 - \delta_4}{d_1 d_2 d_3 d_4} + \cdots + \frac{\delta_{2n-1} d_{2n} - \delta_{2n}}{d_1 d_2 \cdots d_{2n}} + \ldots.$$

Thus the number $(\delta_{2n-1}d_{2n} - \delta_{2n} + d_{2n} - 1)$ always belongs to $\{0, 1, \ldots, d_{2n-1}d_{2n} - 1\}$ for any nega-(d_n)-representation and

$$\Delta^{D'_1}_{\gamma_1\gamma_2\ldots\gamma_n\ldots} = x = \sum_{n=1}^{\infty} \frac{(\delta_{2n-1} + 1)d_{2n} - \delta_{2n} - 1}{d_1 d_2 \cdots d_{2n}}, \tag{5}$$

where $\gamma_n = (\delta_{2n-1} + 1)d_{2n} - \delta_{2n} - 1 = \delta_{2n-1}d_{2n} + d_{2n} - 1 - \delta_{2n}$.

The next statement follows from (4) and (5).

Lemma 5. *The following functions are identity transformations:*

$$x = \Delta^D_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} \xrightarrow{f} \Delta^{-(d_n)}_{\varepsilon_1[d_2-1-\varepsilon_2]\ldots\varepsilon_{2n-1}[d_{2n}-1-\varepsilon_{2n}]\ldots} = f(x) = y,$$

$$x = \Delta^{-(d_n)}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} \xrightarrow{g} \Delta^D_{\varepsilon_1[d_2-1-\varepsilon_2]\ldots\varepsilon_{2n-1}[d_{2n}-1-\varepsilon_{2n}]\ldots} = g(x) = y.$$

Therefore the following functions are DP-functions (functions preserving the fractal Hausdorff-Besicovitch dimension):

$$x = \Delta^D_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} \xrightarrow{f} \Delta^{-(d_n)}_{[d_1-1-\varepsilon_1]\varepsilon_2\ldots[d_{2n-1}-1-\varepsilon_{2n-1}]\varepsilon_{2n}\ldots} = f(x) = y,$$

$$x = \Delta^{-(d_n)}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} \xrightarrow{g} \Delta^D_{[d_1-1-\varepsilon_1]\varepsilon_2\ldots[d_{2n-1}-1-\varepsilon_{2n-1}]\varepsilon_{2n}\ldots} = g(x) = y.$$

Lemma 6. *The following relationships are true:*

1. $\Delta^D_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} - \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} = 2\Delta^D_{\varepsilon_1 0\varepsilon_3 0\ldots} = -2\Delta^{-D}_{\varepsilon_1 0\varepsilon_3 0\ldots}$;
2. $\Delta^D_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} + \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} = 2\Delta^D_{0\varepsilon_2 0\varepsilon_4\ldots} = 2\Delta^{-D}_{0\varepsilon_2 0\varepsilon_4\ldots}$;
3. $\Delta^D_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} - \Delta^{-(d_n)}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} = 2\Delta^{D'}_{\varepsilon_2\varepsilon_4\varepsilon_6\ldots} - \Delta^{D'}_{[d_2-1][d_4-1]\ldots}$, $\varepsilon_n \in A_{d_n}$;
4. $\Delta^{D'}_{\gamma_1\gamma_2\ldots\gamma_n\ldots} = \Delta^{D'}_{\beta_1\beta_2\ldots\beta_n\ldots} + \Delta^{D'}_{[d_2-1][d_4-1]\ldots[d_{2n}-1]\ldots} - 2\Delta^{D'}_{\varepsilon_2\varepsilon_4\varepsilon_6\ldots}$.

4. Shift Operators

Let $\mathcal{F}^{-D}_{[-1+a_0,a_0]}$ be the set of all nega-D-expansions of real numbers from $[-1+a_0, a_0]$.

Define *the shift operator* $\hat{\varphi}$ *of expansion* (2) on $\mathcal{F}^{-D}_{[-1+a_0,a_0]}$ by the rule

$$\hat{\varphi}\left(\sum_{n=1}^{\infty}\frac{(-1)^n\varepsilon_n}{d_1d_2\cdots d_n}\right) = \sum_{n=2}^{\infty}\frac{(-1)^{n-1}\varepsilon_n}{d_2d_3\cdots d_n}.$$

In other words,

$$\hat{\varphi}(\Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}) = \Delta^{-D_1}_{\varepsilon_2\varepsilon_3\ldots\varepsilon_n\ldots} = -d_1\Delta^{-D}_{0\varepsilon_2\ldots\varepsilon_n\ldots}.$$

This operator generates some function $\hat{\varphi}$ such that

$$\hat{\varphi} : [-1+a_0, a_0] \to [-a_0d_1, 1-a_0d_1].$$

By definition, put

$$\hat{\varphi}^k\left(\sum_{n=1}^{\infty}\frac{(-1)^n\varepsilon_n}{d_1d_2\cdots d_n}\right) = \sum_{n=k+1}^{\infty}\frac{(-1)^{n-k}\varepsilon_n}{d_{k+1}\cdots d_n},$$

$$\hat{\varphi}^k(\Delta^{-D}_{\varepsilon_1\varepsilon_2\varepsilon_3\ldots\varepsilon_n\ldots}) = \Delta^{-D_k}_{\varepsilon_{k+1}\varepsilon_{k+2}\ldots} = (-1)^k d_1d_2\cdots d_k\Delta^{-D}_{\underbrace{0\ldots0}_{k}\varepsilon_{k+1}\varepsilon_{k+2}\ldots}.$$

Define *the generalized shift operator* $\hat{\varphi}_m$ *of expansion* (2) by the rule

$$\hat{\varphi}_m\left(\sum_{n=1}^{\infty}\frac{(-1)^n\varepsilon_n}{d_1d_2\cdots d_n}\right) = -\frac{\varepsilon_1}{d_1} + \cdots + \frac{(-1)^{m-1}\varepsilon_{m-1}}{d_1d_2\cdots d_{m-1}} + \frac{(-1)^m\varepsilon_{m+1}}{d_1d_2\cdots d_{m-1}d_{m+1}} + \ldots,$$

i.e,

$$\hat{\varphi}_m(\Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{m-1}\varepsilon_m\varepsilon_{m+1}\ldots}) = \Delta^{-D_{\bar{m}}}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{m-1}\varepsilon_{m+1}\ldots}.$$

Remark 3. Since (ε_n) and (d_n) are fixed sequences in (2) for a given $x \in [a_0-1, a_0]$, we see that the operator $\hat{\varphi}$ or $\hat{\varphi}_m$ takes each number to a number represented in terms of the "other" numeral system.

It is easy to see that the operator $\hat{\varphi}$ has exactly d_1 invariant points. These are points of the form

$$-\frac{i}{d_1+1},\ i=\overline{0,d_1-1}.$$

The operator $\hat{\varphi}$ is not a bijection because the points $\Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}$ $(\varepsilon_1=\overline{0,d_1-1})$ are preimages of the point $\Delta^{-D_1}_{\varepsilon_2\varepsilon_3\ldots\varepsilon_n\ldots}$.

If a sequence (d_n) is purely periodic with a period of length k, then the mapping $\hat{\varphi}$ has periodic points with a period of length $k, k\in\mathbb{N}$, i.e.,

$$\hat{\varphi}^{k+j}(\Delta^{-D}_{(\varepsilon_1\varepsilon_2\ldots\varepsilon_k)})=\hat{\varphi}^{kt+j}(\Delta^{-D}_{(\varepsilon_1\varepsilon_2\ldots\varepsilon_k)}),\ t=0,1,2,\ldots.$$

Lemma 7. *If a sequence (d_n) is purely periodic with a simple period, then the following set*

$$C[-D,V]=\{x: x=\Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}, \varepsilon_n\in\{v_1,v_2,\ldots,v_m\}\subset\{0,1,\ldots,d_n-1\}\}$$

is an invariant set under the mapping $\hat{\varphi}$. Here $v_1,v_2,\ldots,v_m$ are fixed positive integers, $1<m\le d_n-1$, and $d_n>2$.

Lemma 8. *If sequences (d_n) and (V_n) are purely periodic with a period of length k, then the set*

$$C[-D,V_n]=\{x: x=\Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}, \varepsilon_n\in V_n=\{v_1^{(n)},v_2^{(n)},\ldots,v_m^{(n)}\}\subset A_{d_n}\}$$

is an invariant set under the mapping $\hat{\varphi}^k$.

Define *the shift operator $\tilde{\varphi}$ of the nega-D-representation of x* by the rule

$$\tilde{\varphi}(x)=\tilde{\varphi}(\Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots})=\Delta^{-D}_{\varepsilon_2\ldots\varepsilon_n\ldots}=\sum_{n=1}^{\infty}\frac{(-1)^n\varepsilon_{n+1}}{d_1d_2\cdots d_n}.$$

It is obvious that the mapping $\tilde{\varphi}$ is not always well-defined. In fact, the inequality $\varepsilon_{n+1}\le d_n-1$ holds for not all alternating Cantor series. The next statement follows from the last-mentioned inequality.

Lemma 9. *The operator $\tilde{\varphi}$ is well-defined if and only if the inequality*

$$d_{n+1}\le d_n$$

holds for any $n\in\mathbb{N}$.

Remark 4. In Lemma 9, we understand that $\tilde{\varphi}$ is well-defined in the wide sense, i.e., for each $x\in[a_0-1,a_0]$. In fact, for any sequence (d_n) there exist points from $[a_0-1,a_0]$ such that the function $\tilde{\varphi}$ is well-defined at these points.

Lemma 10. *If there exists a number $m \in \mathbb{N}$ such that $\hat{\varphi}^m(x) = x$, then*

$$x = \left(1 + \frac{1}{(-1)^m d_1 d_2 \cdots d_m - 1}\right) \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_m(0)}.$$

Proof. Let $x = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}$. Then

$$\begin{aligned} x = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_m\varepsilon_{m+1}\ldots} &= \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_m(0)} + \Delta^{-D}_{\underbrace{0\ldots0}_{m}\varepsilon_{m+1}\varepsilon_{m+2}\ldots} \\ &= \varphi^m(x) = (-1)^m d_1 d_2 \cdots d_m \Delta^{D}_{\underbrace{0\ldots0}_{m}\varepsilon_{m+1}\varepsilon_{m+2}\ldots}. \end{aligned}$$

This concludes the proof. □

Lemma 11. *Let x be a fixed number. If there exist $m \in \mathbb{Z}_0$ and $c \in \mathbb{N}$ such that $\hat{\varphi}^m(x) = \hat{\varphi}^{m+c}(x)$, then*

$$x = \frac{\Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_m(0)} + (-1)^{c+1} d_{m+1} d_{m+2} \cdots d_{m+c} \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{m+c}(0)}}{1 + (-1)^{c+1} d_{m+1} d_{m+2} \cdots d_{m+c}}.$$

Proof. The statement follows from the next equality:

$$x - \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_m(0)} = (-1)^c d_{m+1} d_{m+2} \cdots d_{m+c} (x - \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{m+c}(0)}).$$

□

Lemma 12. *The equalities*

$$\hat{\varphi}^k(x) = (-1)^k d_1 d_2 \cdots d_k x + (-1)^{k+1} d_1 d_2 \cdots d_k \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k(0)}$$

and

$$x = (-1)^k \frac{\hat{\varphi}^k(x) + (-1)^k d_1 d_2 \cdots d_k \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k(0)}}{d_1 d_2 \cdots d_k}$$

hold for an arbitrary $k \in \mathbb{N}$.

The next proposition follows from the last-mentioned lemma.

Lemma 13. *The equality*

$$\begin{aligned} &(-1)^c d_{m+1} d_{m+2} \cdots d_{m+c} \cdot \hat{\varphi}^m(x) - \hat{\varphi}^{m+c}(x) \\ &= (-1)^{m+c} d_1 d_2 \cdots d_{m+c} \cdot \Delta^{-D}_{\underbrace{0\ldots0}_{m}\varepsilon_{m+1}\varepsilon_{m+2}\ldots\varepsilon_{m+c}(0)} \end{aligned}$$

holds for arbitrary numbers $m \in \mathbb{N}$ and $c \in \mathbb{N}$.

Theorem 2. *The mapping $\hat{\varphi}$ is decreasing on each first rank interval $\nabla^{-D}_c = (\inf \Delta^{-D}_c, \sup \Delta^{-D}_c)$.*

Proof. Let points $x_1 = \Delta^{-D}_{c\varepsilon_2(x_1)\varepsilon_3(x_1)\ldots\varepsilon_n(x_1)\ldots}$ and $x_2 = \Delta^{-D}_{c\varepsilon_2(x_2)\varepsilon_3(x_2)\ldots\varepsilon_n(x_2)\ldots}$ ($x_1 < x_2$) be arbitrary points from the interval ∇^{-D}_c.

Since the equality $\varepsilon_n(\hat{\varphi}(x)) = \varepsilon_{n+1}(x)$ holds and Proposition 1 is true for x_1 and x_2, we see that the inequality $\hat{\varphi}(x_1) > \hat{\varphi}(x_2)$ holds. □

The next statement follows from this theorem.

Corollary 1. *The mapping $\hat{\varphi}$ has a derivative almost everywhere (with respect to the Lebesgue measure).*

Theorem 3. *The mapping $\hat{\varphi}$ is continuous at each point of the first rank interval ∇^{-D}_c and the endpoints of this interval are points of discontinuity of the mapping.*

Proof. Let $x = \Delta^{-D}_{c\varepsilon_2\varepsilon_3\ldots\varepsilon_n\ldots}$ be an arbitrary nega-D-irrational point from ∇^{-D}_c. Let (x_m) be an arbitrary sequence of points from ∇^{-D}_c such that $\lim_{m\to\infty} x_m = x$. Then

$$\lim_{m\to\infty} x_m = x \Leftrightarrow \lim_{n\to\infty} n_m = \infty,$$

where $n_m = \min\{n : \varepsilon_n(x_m) \neq \varepsilon_n(x)\}$. The last-mentioned equivalence follows from the definition and basic properties of the nega-D-representation.

Since $\varepsilon_n(\hat{\varphi}(x)) = \varepsilon_{n+1}(x)$ holds, we have $\lim_{m\to\infty} \hat{\varphi}(x_m) = \hat{\varphi}(x)$. Therefore the mapping $\hat{\varphi}$ is continuous at the point x.

Let x_0 be a certain nega-D-rational point from ∇^{-D}_c, i.e.,

$$x_0 = \Delta^{-D}_{c\varepsilon_2\varepsilon_3\ldots\varepsilon_n[d_{n+1}-1]0[d_{n+3}-1]0\ldots} = \Delta^{-D}_{c\varepsilon_2\varepsilon_3\ldots\varepsilon_{n-1}[\varepsilon_n-1]0[d_{n+2}-1]0[d_{n+4}-1]0\ldots}.$$

At the same time

$$\lim_{x\to x_0} \hat{\varphi}(x) = \Delta^{-D_1}_{\varepsilon_2\ldots\varepsilon_n[d_{n+1}-1]0[d_{n+3}-1]0\ldots} = \Delta^{-D_1}_{\varepsilon_2\ldots\varepsilon_{n-1}[\varepsilon_n-1]0[d_{n+2}-1]0[d_{n+4}-1]0\ldots}.$$

Indeed, the existence of the left-hand and right-hand finite limits at each point follows from monotonicity and boundedness of the mapping.

Consider the problem of continuity of $\hat{\varphi}$ at the point

$$x_1 = \Delta^{-D}_{c[d_2-1]0[d_4-1]0\ldots} = \Delta^{-D}_{[c-1]0[d_3-1]0[d_5-1]0\ldots} = x_2, \ c \neq 0.$$

The endpoints of the interval ∇^{-D}_c are the jump points of $\hat{\varphi}$ because

$$\hat{\varphi}(x_1) = \Delta^{-D_1}_{[d_2-1]0[d_4-1]0\ldots} \neq \Delta^{-D_1}_{0[d_3-1]0[d_5-1]0\ldots} = \hat{\varphi}(x_2).$$

□

Theorem 4. *If the mapping $\hat{\varphi}$ has a derivative at the point $x = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}$, then*

$$(\hat{\varphi}(x))' = -d_1.$$

Proof. Suppose that $\hat{\varphi}$ has a derivative at the point x_0. Let (x_n) be a sequence of $\Delta^{-D}_{\varepsilon_1(x_0)\varepsilon_2(x_0)\ldots\varepsilon_n(x_0)\varepsilon_{n+1}(x)\ldots}$. Here $\varepsilon_k(x) \neq \varepsilon_k(x_0)$ for all $k > n$. Then

$$\begin{aligned}
(\hat{\varphi}(x))' &= \lim_{\Delta x \to 0} \frac{\hat{\varphi}(x) - \hat{\varphi}(x_0)}{x - x_0} = \lim_{\Delta x \to 0} \frac{\Delta^{-D}_{\varepsilon_2(x)\varepsilon_3(x)\ldots} - \Delta^{-D}_{\varepsilon_2(x_0)\varepsilon_3(x_0)\ldots}}{\Delta x} \\
&= - \lim_{\varepsilon_n(x) \to \varepsilon_n(x_0)} \frac{\sum_{i=2}^{\infty} \frac{(-1)^i \varepsilon_i(x)}{d_2 \cdots d_i} - \sum_{i=2}^{\infty} \frac{(-1)^i \varepsilon_i(x_0)}{d_2 \cdots d_i}}{\sum_{j=1}^{\infty} \frac{(-1)^j \varepsilon_j(x)}{d_1 d_2 \cdots d_j} - \sum_{j=1}^{\infty} \frac{(-1)^j \varepsilon_j(x_0)}{d_1 d_2 \cdots d_j}} \\
&= - \lim_{\varepsilon_n(x) \to \varepsilon_n(x_0)} \frac{\sum_{j=1}^{\infty} \frac{(-1)^j \varepsilon_j(x)}{d_1 d_2 \cdots d_j} - \sum_{j=1}^{\infty} \frac{(-1)^j \varepsilon_j(x_0)}{d_1 d_2 \cdots d_j} + \frac{\varepsilon_1(x) - \varepsilon_1(x_0)}{d_1}}{\sum_{j=1}^{\infty} \frac{(-1)^j \varepsilon_j(x)}{d_1 d_2 \cdots d_j} - \sum_{j=1}^{\infty} \frac{(-1)^j \varepsilon_j(x_0)}{d_1 d_2 \cdots d_j}} d_1 = -d_1.
\end{aligned}$$

□

Corollary 2. *The derivative of* $\hat{\varphi}^k$ *does not exist at an arbitrary nega-D-rational point* $\Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{n-1}\varepsilon_n[d_{n+1}-1]0[d_{n+3}-1]0\ldots}$ *when* $k > n - 1$.

5. Representations of Rational and Irrational Numbers

The main statements of this section are analogous to the main results of the paper [8].

Theorem 5. *A rational number* $x = \frac{p}{q}$ *from* $[-1 + a_0, a_0]$ *has a finite expansion by series* (2) *if and only if there exists a number* n_0 *such that* $d_1 d_2 \cdots d_{n_0} \equiv 0 \pmod{q}$.

Corollary 3. *There exist sequences* (d_n) *such that every rational number has the finite nega-D-expansion. Consider the following examples:*

$$\sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon_n}{2 \cdot 3 \cdot \ldots \cdot (n+1)}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon_n}{2 \cdot 4 \cdot \ldots \cdot 2n}.$$

Lemma 14. *The equality*

$$\Delta^{-(d_n)}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} + \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} + \Delta^{-D}_{(1)} = 0$$

holds for each $n \in \mathbb{N}$.

Corollary 4. *There exist alternating Cantor series* (3) *such that a number of the form* $\Delta^{-(d_n)}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n(0)}$ *is an irrational number.*

The following propositions are equivalent.

Theorem 6. *A number* $x_0 \in [-1 + a_0, a_0]$ *is a rational number if and only if there exist* $k \in \mathbb{Z}_0$ *and* $t \in \mathbb{N}$ *such that*

$$\hat{\varphi}^k(x) = \hat{\varphi}^t(x).$$

Theorem 7. *A number $x_0 \in [-1+a_0, a_0]$ is a rational number if and only if there exist $k \in \mathbb{Z}_0$ and $t \in \mathbb{N}$ $(k < t)$ such that*

$$\Delta^{-D}_{\underbrace{0\ldots0}_{k}\varepsilon_{k+1}\varepsilon_{k+2}\varepsilon_{k+3}\ldots} = (-1)^{t-k} d_{k+1}d_{k+2}\cdots d_t \Delta^{-D}_{\underbrace{0\ldots0}_{t}\varepsilon_{t+1}\varepsilon_{t+2}\varepsilon_{t+3}\ldots}.$$

Theorem 8. *A number x is a rational number if and only if the sequence $(\hat{\varphi}^k(x))$, where $k = 0, 1, 2, \ldots,$ contains at least two identical terms.*

6. Foundations of the Metric Theory

Let (d_n) be a fixed sequence of positive integers, $d_n > 1$. Let $c_1, c_2, \ldots, c_m$ be an ordered tuple of integers such that $c_i \in \{0, 1, \ldots, d_i - 1\}$ for $i = \overline{1, m}$.

Definition 5. *A nega-D-cylinder of rank m with base $c_1c_2\ldots c_m$* is a set $\Delta^{-D}_{c_1c_2\ldots c_m}$ formed by all numbers of the segment $[-1+a_0, a_0]$ with nega-D-representations in which the first m digits coincide with $c_1, c_2, \ldots, c_m$, respectively, i.e.,

$$\Delta^{-D}_{c_1c_2\ldots c_m} = \left\{x : x = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}, \varepsilon_j = c_j, j = \overline{1, m}\right\}.$$

Lemma 15. *A nega-D-cylinder is a closed interval, i.e.,*

$$\Delta^{-D}_{c_1c_2\ldots c_m} = \begin{cases} \left[g_m + \frac{(-1)^m}{d_1d_2\cdots d_m}(a_m - 1), g_m + \frac{(-1)^m}{d_1d_2\cdots d_m}a_m\right] & \textit{if } m \textit{ is even} \\ \\ \left[g_m + \frac{(-1)^m}{d_1d_2\cdots d_m}a_m, g_m + \frac{(-1)^m}{d_1d_2\cdots d_m}(a_m - 1)\right] & \textit{if } m \textit{ is odd,} \end{cases}$$

where

$$a_m = \sup \sum_{j=1}^{\infty} \frac{(-1)^j \varepsilon_{m+j}}{d_{m+1}d_{m+2}\cdots d_{m+j}}, \; g_m = \sum_{i=1}^{m} \frac{(-1)^i c_i}{d_1 d_2 \cdots d_i}.$$

Proof. Let m be even and $x \in \Delta^{-D}_{c_1c_2\ldots c_m}$, i.e.,

$$x = \sum_{i=1}^{m} \frac{(-1)^i c_i}{d_1d_2\cdots d_i} + \sum_{j=m+1}^{\infty} \frac{(-1)^j \varepsilon_j}{d_1d_2\cdots d_j},$$

where $\varepsilon_j \in \{0, 1, \ldots, d_j - 1\}$; then

$$x^{'} = g_m - \sum_{k=1}^{\infty} \frac{d_{m+2k-1} - 1}{d_1d_2\cdots d_{m+2k-1}} \le x \le g_m + \sum_{k=1}^{\infty} \frac{d_{m+2k} - 1}{d_1d_2\cdots d_{m+2k}} = x^{''}.$$

Hence $x \in [x^{'}, x^{''}]$ and $\Delta^{-D}_{c_1c_2\ldots c_m} \subseteq [x^{'}, x^{''}]$.

Since the equalities

$$\sum_{j=1}^{\infty} \frac{d_{m+2j}-1}{d_1 d_2 \cdots d_{m+2j}} = \frac{1}{d_1 d_2 \cdots d_m} \sup \sum_{j=1}^{\infty} \frac{(-1)^j \varepsilon_{m+j}}{d_{m+1} d_{m+2} \cdots d_{m+j}}$$

and

$$-\sum_{j=1}^{\infty} \frac{d_{m+2j-1}-1}{d_1 d_2 \cdots d_{m+2j-1}} = \frac{1}{d_1 d_2 \cdots d_m} \inf \sum_{j=1}^{\infty} \frac{(-1)^j \varepsilon_{m+j}}{d_{m+1} d_{m+2} \cdots d_{m+j}}$$

hold, we have $x \in \Delta^{-D}_{c_1 c_2 \ldots c_m}$ and $x^{'}, x^{''} \in \Delta^{-D}_{c_1 c_2 \ldots c_m}$. □

Lemma 16. *Nega-D-cylinders have the following properties:*

1.

$$\inf \Delta^{-D}_{c_1 c_2 \ldots c_m} = \begin{cases} g_m - \frac{1}{d_1 d_2 \cdots d_m} \sum_{j=1}^{\infty} \frac{d_{m+2j-1}-1}{d_{m+1} d_{m+2} \cdots d_{m+2j-1}} & \text{if } m \text{ is even} \\ g_m - \frac{1}{d_1 d_2 \cdots d_m} \sum_{j=1}^{\infty} \frac{d_{m+2j}-1}{d_{m+1} d_{m+2} \cdots d_{m+2j}} & \text{if } m \text{ is odd.} \end{cases}$$

2.

$$\sup \Delta^{-D}_{c_1 c_2 \ldots c_m} = \begin{cases} g_m + \frac{1}{d_1 d_2 \cdots d_m} \sum_{j=1}^{\infty} \frac{d_{m+2j}-1}{d_{m+1} d_{m+2} \cdots d_{m+2j}} & \text{if } m \text{ is even} \\ g_m + \frac{1}{d_1 d_2 \cdots d_m} \sum_{j=1}^{\infty} \frac{d_{m+2j-1}-1}{d_{m+1} d_{m+2} \cdots d_{m+2j-1}} & \text{if } m \text{ is odd.} \end{cases}$$

3.

$$|\Delta^{-D}_{c_1 c_2 \ldots c_m}| = \frac{1}{d_1 d_2 \cdots d_m}.$$

4.

$$\Delta^{-D}_{c_1 c_2 \ldots c_m c} \subset \Delta^{-D}_{c_1 c_2 \ldots c_m}.$$

5.

$$\Delta^{-D}_{c_1 c_2 \ldots c_m} = \bigcup_{c=0}^{d_{m+1}-1} \Delta^{-D}_{c_1 c_2 \ldots c_m c}.$$

6.

$$\lim_{m \to \infty} |\Delta^{-D}_{c_1 c_2 \ldots c_m}| = 0.$$

7.

$$\frac{|\Delta^{-D}_{c_1 c_2 \ldots c_m c_{m+1}}|}{|\Delta^{-D}_{c_1 c_2 \ldots c_m}|} = \frac{1}{d_{m+1}}.$$

8.

$$\begin{cases}\sup\Delta^{-D}_{c_1c_2\ldots c_mc} = \inf\Delta^{-D}_{c_1c_2\ldots c_m[c+1]} & \textit{if } m \textit{ is odd}\\ \sup\Delta^{-D}_{c_1c_2\ldots c_m[c+1]} = \inf\Delta^{-D}_{c_1c_2\ldots c_mc} & \textit{if } m \textit{ is even,}\end{cases}$$

where $c \neq d_{m+1} - 1$.

9.

$$\Delta^{-D}_{c_1c_2\ldots c_m} \cap \Delta^{-D}_{e_1e_2\ldots e_m} = \begin{cases}\Delta^{-D}_{c_1c_2\ldots c_m} & \textit{if } e_i = c_i \textit{ for } i = \overline{1,m}\\ \varnothing & \textit{if } \exists i\ (i<m) \textit{ such that } c_i \neq e_i\\ \varnothing & \textit{if } \exists i \textit{ such that } c_i \neq e_i,\ c_m \neq e_m - 1.\end{cases}$$

Here $e_m \neq 0$ *in the last case.*

10.

$$\bigcap_{m=1}^{\infty} \Delta^{-D}_{c_1c_2\ldots c_m} = x = \Delta^{-D}_{c_1c_2\ldots c_m\ldots}.$$

Proof. Properties 1 and 2 follow immediately from the definition of $\Delta^{-D}_{c_1c_2\ldots c_m}$. *Property 3* is a corollary of these properties. *Properties 6 and 7* follow from *Property 3.*

Property 4. Let m be even. Let us prove that the conditions

$$\begin{cases}\inf\Delta^{-D}_{c_1c_2\ldots c_mc} \geq \inf\Delta^{-D}_{c_1c_2\ldots c_m}\\ \sup\Delta^{-D}_{c_1c_2\ldots c_mc} \leq \sup\Delta^{-D}_{c_1c_2\ldots c_m}\end{cases}$$

hold. In fact,

$$\begin{aligned}&\inf\Delta^{-D}_{c_1c_2\ldots c_mc} - \inf\Delta^{-D}_{c_1c_2\ldots c_m} = g_m + \frac{(-1)^{m+1}c}{d_1d_2\cdots d_{m+1}}\\ &+ \frac{(-1)^{m+1}}{d_1d_2\cdots d_{m+1}}\left(\frac{d_{m+3}-1}{d_{m+2}d_{m+3}} + \frac{d_{m+5}-1}{d_{m+2}\cdots d_{m+5}} + \ldots\right) - g_m\\ &- \frac{(-1)^m}{d_1d_2\cdots d_m}\left(-\frac{d_{m+1}-1}{d_{m+1}} - \frac{d_{m+3}-1}{d_{m+1}d_{m+2}d_{m+3}} - \frac{d_{m+5}-1}{d_{m+1}\cdots d_{m+5}} - \ldots\right)\\ &= \frac{d_{m+1}-1-c}{d_1d_2\cdots d_{m+1}} \geq 0.\end{aligned}$$

If the condition $c = d_{m+1} - 1$ holds, then the last inequality is an equality. As above,

$$\begin{aligned}&\sup\Delta^{-D}_{c_1\ldots c_m} - \sup\Delta^{-D}_{c_1\ldots c_mc} = \frac{(-1)^m}{d_1d_2\cdots d_m}\left(\frac{d_{m+2}-1}{d_{m+1}d_{m+2}} + \frac{d_{m+4}-1}{d_{m+1}\cdots d_{m+4}} + \ldots\right)\\ &+ g_m - g_m - \frac{(-1)^{m+1}c}{d_1\cdots d_{m+1}} - \frac{(-1)^{m+1}}{d_1\cdots d_{m+1}}\left(-\frac{d_{m+2}-1}{d_{m+2}} - \frac{d_{m+4}-1}{d_{m+2}\cdots d_{m+4}} - \ldots\right)\\ &= \frac{c}{d_1d_2\cdots d_{m+1}} \geq 0.\end{aligned}$$

Here the last inequality is an equality whenever the condition $c = 0$ holds.

Similarly, the last-mentioned system of inequalities is true in the case of odd m.

Property 5 follows from *Property 4* and the definition of $\Delta^{-D}_{c_1c_2\ldots c_m}$.

Property 8. Let m be odd. Then

$$\sup \Delta^{-D}_{c_1c_2\ldots c_m c} - \inf \Delta^{-D}_{c_1c_2\ldots c_m[c+1]} = \frac{(-1)^{m+1}c}{d_1d_2\cdots d_{m+1}} + \frac{(-1)^{m+1}}{d_1d_2\cdots d_{m+1}}a_{m+1}$$
$$-\frac{c+1}{d_1d_2\cdots d_{m+1}}(-1)^{m+1} - \frac{(-1)^{m+1}}{d_1d_2\cdots d_{m+1}}(a_{m+1}-1) = 0.$$

Let m be even. Then

$$\sup \Delta^{-D}_{c_1c_2\ldots c_m[c+1]} - \inf \Delta^{-D}_{c_1c_2\ldots c_m c} = \frac{(-1)^{m+1}}{d_1d_2\cdots d_{m+1}}(c+1)$$
$$+\frac{(-1)^{m+1}}{d_1d_2\cdots d_{m+1}}(a_{m+1}-1) - \frac{(-1)^{m+1}c}{d_1d_2\cdots d_{m+1}} - \frac{(-1)^{m+1}}{d_1d_2\cdots d_{m+1}}a_{m+1} = 0.$$

Property 9 follows from *properties 1, 2, and 8.*

Property 10. From *Property 4* it follows that

$$\Delta^{-D}_{c_1} \subset \Delta^{-D}_{c_1c_2} \subset \Delta^{-D}_{c_1c_2c_3} \subset \ldots \subset \Delta^{-D}_{c_1c_2\ldots c_n} \subset \ldots.$$

Since the last lemma and Cantor's intersection theorem are true, we obtain

$$\bigcap_{n=1}^{\infty} \Delta^{-D}_{c_1c_2\ldots c_n} = x = \Delta^{-D}_{c_1c_2\ldots c_n\ldots}.$$

□

7. Simplest Metric Problems

Let k be a fixed positive integer, c be a fixed digit from A_{d_k}. Consider the following set

$$\Delta^k_c = \{x : x = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{k-1}c\varepsilon_{k+1}\ldots}\}.$$

Lemma 17. *The set Δ^k_c $(k > 1)$ is the union of nega-D-cylinders of rank k.*

Proof. Let $k = 1$. Then it is easy to see that $\Delta^k_c = \Delta^{-D}_c$.

Let $k = 2$. Then

$$\Delta^2_c = \Delta^{-D}_{0c} \cup \Delta^{-D}_{1c} \cup \Delta^{-D}_{2c} \cup \ldots \cup \Delta^{-D}_{[d_1-1]c}.$$

Let $k = n$. Then

$$\Delta^n_c = \Delta^{-D}_{\underbrace{00\ldots00}_{n-1}c} \cup \Delta^{-D}_{\underbrace{00\ldots01}_{n-1}c} \cup \ldots \cup \Delta^{-D}_{[d_1-1][d_2-1]\ldots[d_{n-1}-1]c}.$$

□

Lemma 18. *The Lebesgue measure of* Δ^k_c *is equal to* $\frac{1}{d_k}$.

Proof. It is easy to see that

$$\lambda(\Delta^k_c) = \sum_{c_1=0}^{d_1-1} \cdots \sum_{c_{k-1}=0}^{d_{k-1}-1} |\Delta^{-D}_{c_1c_2\ldots c_{k-1}c}| = \frac{1}{d_k} \sum_{c_1=0}^{d_1-1} \cdots \sum_{c_{k-1}=0}^{d_{k-1}-1} |\Delta^{-D}_{c_1c_2\ldots c_{k-1}}| = \frac{1}{d_k}.$$

□

Corollary 5. *The Lebesgue measure of* $\Delta^k_{\overline{c}} = \left\{x : x = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}, \varepsilon_k \neq c\right\}$ *is equal to* $1 - \frac{1}{d_k}$.

Lemma 19. *The diameter of the set* Δ^k_c *is calculated by the following formula*

$$d(\Delta^k_c) = \frac{d_1d_2\cdots d_k - d_k + 1}{d_1d_2\cdots d_k}.$$

Proof. Let k be even,

$$a_k = \sup \sum_{j=1}^{\infty} \frac{(-1)^j \varepsilon_{k+j}}{d_{k+1}d_{k+2}\cdots d_{k+j}}.$$

Then

$$\begin{aligned}
d(\Delta^k_c) &= \max \sum_{i=1}^{k-1} \frac{(-1)^i \varepsilon_i}{d_1d_2\cdots d_i} + \frac{(-1)^k c}{d_1d_2\cdots d_k} + \frac{(-1)^k}{d_1d_2\cdots d_k} a_k \\
&\quad - \min \sum_{i=1}^{k-1} \frac{(-1)^i \varepsilon_i}{d_1d_2\cdots d_i} - \frac{(-1)^k c}{d_1d_2\cdots d_k} - \frac{(-1)^k}{d_1d_2\cdots d_k}(a_k - 1) \\
&= \left(\frac{d_2-1}{d_1d_2} + \frac{d_4-1}{d_1\cdots d_4} + \cdots + \frac{d_{k-2}-1}{d_1\cdots d_{k-2}}\right) \\
&\quad + \left(\frac{d_1-1}{d_1} + \frac{d_3-1}{d_1d_2d_3} + \cdots + \frac{d_{k-1}-1}{d_1\cdots d_{k-1}}\right) + \frac{(-1)^k}{d_1\cdots d_k} \\
&= 1 - \frac{1}{d_1\cdots d_{k-1}} + \frac{(-1)^k}{d_1d_2\cdots d_k} = \frac{d_1d_2\cdots d_k - d_k + 1}{d_1\cdots d_k}.
\end{aligned}$$

Let k be odd. Then

$$\begin{aligned}
d(\Delta^k_c) &= \left(\frac{d_2-1}{d_1d_2} + \frac{d_4-1}{d_1\cdots d_4} + \cdots + \frac{d_{k-1}-1}{d_1\cdots d_{k-1}}\right) \\
&\quad - \left(-\frac{d_1-1}{d_1} - \frac{d_3-1}{d_1d_2d_3} - \cdots - \frac{d_{k-2}-1}{d_1\cdots d_{k-2}}\right) + \frac{(-1)^{k+1}}{d_1d_2\cdots d_k} \\
&= 1 - \frac{1}{d_1d_2\cdots d_{k-1}} + \frac{1}{d_1d_2\cdots d_k} = \frac{d_1d_2\cdots d_k - d_k + 1}{d_1\cdots d_k}.
\end{aligned}$$

□

Let $(c_1, c_2, \dots, c_m)$ and $(k_1, k_2, \dots, k_m)$ be fixed tuples of positive integers such that $c_i \in A_{d_{k_i}}$, $i = \overline{1, m}$, $0 < k_1 < k_2 < \ldots < k_m$. Consider the following set

$$\Delta^{k_1 k_2 \ldots k_m}_{c_1 c_2 \ldots c_m} = \{x : x = \Delta^{-D}_{\varepsilon_1 \ldots \varepsilon_{k_1 - 1} c_1 \varepsilon_{k_1+1} \ldots \varepsilon_{k_2-1} c_2 \ldots \varepsilon_{k_m - 1} c_m \varepsilon_{k_m+1} \varepsilon_{k_m+2} \ldots}\}.$$

Lemma 20. *The Lebesgue measure of the set* $\Delta^{k_1 k_2 \ldots k_m}_{c_1 c_2 \ldots c_m}$ *is calculated by the formula*

$$\lambda\left(\Delta^{k_1 k_2 \ldots k_m}_{c_1 c_2 \ldots c_m}\right) = \prod_{i=1}^{m} \frac{1}{d_{k_i}}.$$

Proof. It is easy to see that

$$\lambda\left(\Delta^{k_1 k_2}_{c_1 c_2}\right) = \frac{1}{d_{k_2}} \cdot \frac{d_{k_2 - 1}}{d_{k_2 - 1}} \cdot \ldots \cdot \frac{d_{k_1+1}}{d_{k_1+1}} |\Delta^{k_1}_{c_1}| = \frac{1}{d_{k_2}} \cdot \frac{1}{d_{k_1}} = \lambda\left(\Delta^{k_1}_{c_1}\right) \cdot \lambda\left(\Delta^{k_2}_{c_2}\right).$$

Clearly,

$$\begin{aligned}\lambda\left(\Delta^{k_1 k_2 \ldots k_m}_{c_1 c_2 \ldots c_m}\right) &= \frac{1}{d_{k_m}} \left|\Delta^{k_1 k_2 \ldots k_{m-1}}_{c_1 c_2 \ldots c_{m-1}}\right| = \frac{1}{d_{k_m}} \cdot \frac{1}{d_{k_{m-1}}} \left|\Delta^{k_1 k_2 \ldots k_{m-2}}_{c_1 c_2 \ldots c_{m-2}}\right| = \ldots \\ &= \frac{1}{d_{k_m}} \cdot \frac{1}{d_{k_{m-1}}} \cdot \ldots \cdot \frac{1}{d_{k_1}} = \frac{1}{d_{k_1} d_{k_2} \cdots d_{k_m}}.\end{aligned}$$

□

Corollary 6. *Sets of the form* $\Delta^{k_1 k_2 \ldots k_m}_{c_1 c_2 \ldots c_m}$ *are metrically independent, i.e.,*

$$\lambda\left(\Delta^{k_1 k_2 \ldots k_m}_{c_1 c_2 \ldots c_m}\right) = \lambda\left(\bigcap_{i=1}^{m} \Delta^{k_i}_{c_i}\right) = \prod_{i=1}^{m} \lambda\left(\Delta^{k_i}_{c_i}\right).$$

Lemma 21. *The diameter* $d\left(\Delta^{k_1 k_2 \ldots k_m}_{c_1 c_2 \ldots c_m}\right)$ *of the set* $\Delta^{k_1 k_2 \ldots k_m}_{c_1 c_2 \ldots c_m}$ *is calculated by the formula*

$$d\left(\Delta^{k_1 k_2 \ldots k_m}_{c_1 c_2 \ldots c_m}\right) = 1 - \sum_{i=1}^{m} \frac{d_{k_i} - 1}{d_1 d_2 \cdots d_{k_i}}.$$

Proof. Let $K = \{k_1, k_2, \dots, k_m\}$ and $l = 1, 2, \ldots$. Then

$$\begin{aligned}&\sup \Delta^{k_1 k_2 \ldots k_m}_{c_1 c_2 \ldots c_m} - \inf \Delta^{k_1 k_2 \ldots k_m}_{c_1 c_2 \ldots c_m} \\ &= \sum_{2l = j \notin K, j < k_m} \frac{d_j - 1}{d_1 d_2 \cdots d_j} + \frac{(-1)^{k_m}}{d_1 d_2 \cdots d_{k_m}} \sum_{p=1}^{\infty} \frac{d_{k_m + 2p} - 1}{d_{k_m+1} \cdots d_{k_m+2p}} \\ &+ \frac{(-1)^{k_m}}{d_1 d_2 \cdots d_{k_m}} \sum_{p=1}^{\infty} \frac{d_{k_m+2p} - 1}{d_{k_m+1} \cdots d_{k_m+2p}} \\ &- \sum_{2l+1 = j \notin K, j < k_m} \frac{1 - d_j}{d_1 d_2 \cdots d_j} - \frac{(-1)^{k_m}}{d_1 d_2 \cdots d_{k_m}} \sum_{p=1}^{\infty} \frac{1 - d_{k_m+2p+1}}{d_{k_m+1} \cdots d_{k_m+2p+1}} \\ &= \sum_{0 < j < k_m, j \notin K} \frac{d_j - 1}{d_1 d_2 \cdots d_j} + \frac{1}{d_1 d_2 \cdots d_{k_m}} = 1 - \sum_{i=1}^{m} \frac{d_{k_i} - 1}{d_1 d_2 \cdots d_{k_i}}.\end{aligned}$$

□

8. Alternating Cantor Series and the Hausdorff-Besicovitch Dimension Faithfulness

Let Φ_1 be the family of all closed intervals, Φ_2 be the family of all possible rank cylinders $\Delta^{-D}_{c_1c_2...c_n}$, and E be an arbitrary subset of $[a_0-1, a_0]$.

Theorem 9. *If a sequence (d_n) is bounded, then the family Φ_2 of coverings of $[a_0-1, a_0]$ is faithful for the Hausdorff-Besicovitch dimension calculation.*

Proof. Let us find conditions for (d_n) such that the inequality

$$m^{\alpha}_{\varepsilon}(E, \Phi_1) \le m^{\alpha}_{\varepsilon}(E, \Phi_2)$$

holds for $\Phi_2 \subset \Phi_1$.

Let u be an arbitrary closed interval of covering of E, k be the minimal positive integer such that u does not contain nega-D-cylinders $\Delta^{-D}_{c_1c_2...c_{k-1}}$ of rank $k-1$. Then u belongs to not more than d_k cylinders of rank k but u contains a cylinder of rank $k+1$. Hence,

$$m^{\alpha}_{\varepsilon}(E, \Phi_1) \le m^{\alpha}_{\varepsilon}(E, \Phi_2) \le d_k d_{k+1} m^{\alpha}_{\varepsilon}(E, \Phi_1),$$

where

$$m^{\alpha}_{\varepsilon}(E, \Phi) = \inf_{d(E_j)\le\varepsilon} \sum_j d^{\alpha}(E_j)$$

for a fixed $\varepsilon > 0$, a fixed $\alpha > 0$, and covering of E by sets E_j with diameters $d(E_j) \le \varepsilon$.

Note that $d_k d_{k+1} \le (\max_n\{d_n\})^2 < \infty$ whenever (d_n) is bounded. Indeed,

$$0 < \lambda_1 = \frac{1}{\max_n\{d_n\}} \le \frac{|\Delta^{-D}_{c_1c_2...c_n i}|}{|\Delta^{-D}_{c_1c_2...c_n}|} = \frac{1}{d_{n+1}} \le \frac{1}{2} = \lambda_2 < 1,$$

where λ_1 and λ_2 are fixed numbers for an arbitrary $n \in \mathbb{N}$. It is true if and only if a sequence (d_n) is bounded. □

9. Sets of Incomplete Sums

Let (d_n) be a fixed sequence of positive integers, $d_n > 1$, and let (ε_n) be a fixed sequence of digits. Then we have a fixed number of the form

$$\Delta^{-D}_{\varepsilon_1\varepsilon_2...\varepsilon_n...} = s_0 = \sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon_n}{d_1 d_2 \cdots d_n}. \tag{6}$$

Let (A'_n) be a sequence of $A'_n = \{0, \varepsilon_n\}$ and

$$L'_{s_0} = A'_1 \times A'_2 \times A'_n \times \ldots = \{\delta : \delta = (\delta_1, \delta_2, \ldots, \delta_n, \ldots), \delta_n \in A'_n\}.$$

Definition 6. A number of the form

$$s = s(\delta) = \sum_{n=1}^{\infty} \frac{(-1)^n \delta_n}{d_1 d_2 \cdots d_n}, \tag{7}$$

where $\delta = (\delta_n) \in L'_{s_0}$, is called *an incomplete sum of alternating Cantor series* (6).

By M_{s_0} denote the set of all incomplete sums of alternating Cantor series (6), i.e.,

$$M_{s_0} = \{x : x = \Delta^{-D}_{\delta_1\delta_2\ldots\delta_n\ldots}, (\delta_n) \in L'_{s_0}\}.$$

It is obvious that

$$M_{s_0} \subset \left[-\sum_{k=1}^{\infty} \frac{\varepsilon_{2k-1}}{d_1 d_2 \cdots d_{2k-1}}, \sum_{k=1}^{\infty} \frac{\varepsilon_{2k}}{d_1 d_2 \cdots d_{2k}}\right] = I_0^{M_{s_0}} \text{ for } s_0 = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}$$

and $M_{s_0} = \{0\}$ for $s_0 = 0$. Moreover,

$$\bigcup_{s_0} M_{s_0} = \left[-\sum_{k=1}^{\infty} \frac{d_{2k-1}}{d_1 d_2 \cdots d_{2k-1}}, \sum_{k=1}^{\infty} \frac{d_{2k}}{d_1 d_2 \cdots d_{2k}}\right].$$

We begin with definitions.

Definition 7. A cylinder of rank m with base $c_1c_2\ldots c_m$ is a set of the following form

$$\Delta^{M_{s_0}}_{c_1c_2\ldots c_m} = \left\{x : x = \sum_{i=1}^{n} \frac{(-1)^i c_i}{d_1 d_2 \cdots d_i} + \sum_{j=n+1}^{\infty} \frac{(-1)^j \delta_j}{d_1 d_2 \cdots d_j}\right\},$$

where $c_1, c_2, \ldots, c_n$ are fixed numbers from $A'_1, A'_2, \ldots, A'_n$, respectively, and $\delta_j \in A'_j$.

Definition 8. A cylindrical closed interval (interval) $I^{M_{s_0}}_{c_1c_2\ldots c_n}$ ($\nabla^{M_{s_0}}_{c_1c_2\ldots c_n}$) of rank n with base $c_1c_2\ldots c_n$ is a closed interval (interval) whose endpoints coincide with endpoints of $\Delta^{M_{s_0}}_{c_1c_2\ldots c_n}$.

The following properties of cylindrical sets follow from Definition 7:

1.
$$\inf \Delta^{M_{s_0}}_{c_1c_2\ldots c_n} = \begin{cases} \Delta^{-D}_{c_1c_2\ldots c_n 0\varepsilon_{n+2}0\varepsilon_{n+4}\ldots} & \text{if } n \text{ is odd} \\ \Delta^{-D}_{c_1c_2\ldots c_n\varepsilon_{n+1}0\varepsilon_{n+3}0\varepsilon_{n+5}\ldots} & \text{if } n \text{ is even.} \end{cases}$$

2.
$$\sup \Delta^{M_{s_0}}_{c_1c_2\ldots c_n} = \begin{cases} \Delta^{-D}_{c_1c_2\ldots c_n\varepsilon_{n+1}0\varepsilon_{n+3}0\varepsilon_{n+5}\ldots} & \text{if } n \text{ is odd} \\ \Delta^{-D}_{c_1c_2\ldots c_n 0\varepsilon_{n+2}0\varepsilon_{n+4}\ldots} & \text{if } n \text{ is even.} \end{cases}$$

3.
$$\lim_{n\to\infty} d(\Delta^{M_{s_0}}_{c_1c_2\ldots c_n}) = \lim_{n\to\infty} \Delta^D_{\underbrace{0\ldots0}_{n}\varepsilon_{n+1}\varepsilon_{n+2}\ldots} = 0.$$

4. If $\varepsilon_{n+1} \neq 0$, then $\Delta^{M_{s_0}}_{c_1c_2\ldots c_n} = \Delta^{M_{s_0}}_{c_1c_2\ldots c_n 0} \cup \Delta^{M_{s_0}}_{c_1c_2\ldots c_n\varepsilon_{n+1}}$.

5.
$$\Delta^{M_{s_0}}_{c_1c_2\ldots c_n} \subset I^{M_{s_0}}_{c_1c_2\ldots c_n} \subset \Delta^{-D}_{c_1c_2\ldots c_n},$$
$$M_{s_0} \subset \bigcup_{c_i\in A_i^{'}, i=\overline{1,n}} \Delta^{M_{s_0}}_{c_1c_2\ldots c_n} \subset \bigcup_{c_i\in A_i^{'}, i=\overline{1,n}} I^{M_{s_0}}_{c_1c_2\ldots c_n}.$$

6.
$$\left|\Delta^{-D}_{c_1c_2\ldots c_n} \setminus (\Delta^{-D}_{c_1c_2\ldots c_n 0} \cup \Delta^{-D}_{c_1c_2\ldots c_n\varepsilon_{n+1}})\right| = \begin{cases} \frac{d_{n+1}-2}{d_1d_2\cdots d_{n+1}} & \text{if } \varepsilon_{n+1} > 0 \\ \frac{d_{n+1}-1}{d_1d_2\cdots d_{n+1}} & \text{if } \varepsilon_{n+1} = 0. \end{cases}$$

Here by $|\cdot|$ denote the length of an interval and by $\setminus$ denote the difference of sets.

Lemma 22. *Let* $s_0 = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}$ *be a fixed number,* (c_n) *be an arbitrary fixed sequence from* $L^{'}_{s_0}$*; then the following are true:*

1.
$$\bigcap_{i=1}^{\infty} \Delta^{-D}_{c_1c_2\ldots c_n} = \bigcap_{i=1}^{\infty} \Delta^{M_{s_0}}_{c_1c_2\ldots c_n} = \Delta^{-D}_{c_1c_2\ldots c_n\ldots}.$$

2.
$$\Delta^{M_{s_0}}_{c_1c_2\ldots c_n} = \Delta^{-D}_{c_1c_2\ldots c_n} \cap M_{s_0},$$

where
$$M_{s_0} = \bigcap_{n=1}^{\infty} \left(\bigcup_{c_i\in A_i^{'}, i=\overline{1,n}} \Delta^{M_{s_0}}_{c_1c_2\ldots c_n} \right)$$

and $A_i^{'} = \{0, \varepsilon_i\}$ *for all positive integers* i*.*

Proof. 1. Let $x = \Delta^{-D}_{c_1c_2\ldots c_n\ldots}$. From the definition of $\Delta^{M_{s_0}}_{c_1c_2\ldots c_n}$ it follows that $\Delta^{-D}_{c_1c_2\ldots c_n\ldots} = x \in \Delta^{M_{s_0}}_{c_1c_2\ldots c_n}$. Therefore $x \in \bigcap_{n=1}^{\infty} I^{M_{s_0}}_{c_1c_2\ldots c_n}$. The first statement follows from Property 5 of $\Delta^{M_{s_0}}_{c_1c_2\ldots c_n}$.

2. Let $x \in M_{s_0}$. Then x belongs to a certain cylinder $\Delta^{M_{s_0}}_{c_1c_2\ldots c_n} \subset \Delta^{-D}_{c_1c_2\ldots c_n}$. Let us consider the set $\Delta^{-D}_{c_1c_2\ldots c_n} \cap M_{s_0}$. Numbers of the form $\Delta^{-D}_{c_1c_2\ldots c_n\delta_{n+1}\delta_{n+2}\ldots\delta_{n+k}\ldots}$, where $\delta_{n+k} \in A^{'}_{n+k}$, are elements of this set. Consequently,
$$\Delta^{M_{s_0}}_{c_1c_2\ldots c_n} \subset (\Delta^{-D}_{c_1c_2\ldots c_n} \cap M_{s_0}).$$

Also, if $x \in (\Delta^{-D}_{c_1c_2\ldots c_n} \cap M_{s_0})$, then $x \in \Delta^{M_{s_0}}_{c_1c_2\ldots c_n}$. ∎

Theorem 10. *The set M_{s_0} of incomplete sums of alternating Cantor series* (6) *is:*

1. *the one-element set $\{0\}$ whenever $s_0 = 0$;*
2. *a finite set whenever the condition $\varepsilon_n \neq 0$ holds for a finite number of n in $s_0 = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}$;*
3. *the segment $\left[-\frac{2}{3}, \frac{1}{3}\right]$ whenever $d_n = const = 2$ for all $n \in \mathbb{N}$ and $s_0 = -\frac{1}{3}$;*
4. *a union of finite number of segments whenever there exists a finite number of m_i ($i = \overline{1, k_0}$, k_0 is a fixed number) that $d_{m_i} \neq 2$ and $s_0 = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{m_{k_0}}(1)}$;*
5. *an uncountable, perfect, and nowhere dense set of zero Lebesgue measure whenever $s_0 \neq 0$ and $d_n > 2$ hold for an infinite number of n.*

Proof. Since an alternating Cantor series is the nega-binary sum

$$-\frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2^2} - \frac{\varepsilon_3}{2^3} + \cdots + \frac{(-1)^n\varepsilon_n}{2^n} + \ldots,$$

where $\varepsilon_n \in \{0, 1\}$, whenever $d_n = const = 2$ and the set M_{s_0} is $C[-D, A^{'}_n]$, we see that *statements 1-4* are true.

We now prove *statement 5* is true. Let the mapping $f : M_{s_0} \to C[E, V_n]$, where $V_n = A^{'}_n$, be given by

$$x = \Delta^{-D}_{\delta_1\delta_2\ldots\delta_n\ldots} \xrightarrow{f} \sum_{n=1}^{\infty} \frac{1}{(2+\delta_1)(2+\delta_1+\delta_2)\cdots(2+\delta_1+\cdots+\delta_n)} = \Delta^{E}_{\delta_1\delta_2\ldots\delta_n\ldots}$$
$$= f(x) = y.$$

Here $\Delta^{E}_{\delta_1\delta_2\ldots\delta_n\ldots}$ is the representation by an Engel series. This mapping is not a bijection at the nega-D-rational points

$$\Delta^{-D}_{\delta_1\delta_2\ldots\delta_{k-1}\delta_k[d_{k+1}-1]0[d_{k+3}-1]0\ldots} = \Delta^{-D}_{\delta_1\delta_2\ldots\delta_{k-1}[\delta_k-1]0[d_{k+2}-1]0[d_{k+4}-1]\ldots}.$$

It is true when $s_0 = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{k-1}1[d_{k+1}-1][d_{k+2}-1][d_{k+3}-1]\ldots}$.

Since the set of nega-D-rational numbers is not more than countable in M_{s_0}, we can use one of the representations of a nega-D-rational number (e.g., the first representation) when the argument is a nega-D-rational number. Hence M_{s_0} is uncountable, since $C[E, V_n]$ is uncountable.

Let us prove that the set M_{s_0} is nowhere dense. Choose a cylinder $\Delta^{M_{s_0}}_{c_1c_2\ldots c_{n-1}}$ such that the condition $\varepsilon_n \neq 0$ holds for $s_0 = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}$. Consider the mutual placement of $\Delta^{M_{s_0}}_{c_1c_2\ldots c_{n-1}0}$ and $\Delta^{M_{s_0}}_{c_1c_2\ldots c_{n-1}\varepsilon_n}$. Let n be even. Then

$$\inf \Delta^{M_{s_0}}_{c_1c_2\ldots c_{n-1}\varepsilon_n} - \sup \Delta^{M_{s_0}}_{c_1c_2\ldots c_{n-1}0} = \sum_{i=1}^{n-1} \frac{(-1)^i c_i}{d_1d_2\cdots d_i} + \frac{\varepsilon_n}{d_1d_2\cdots d_n}$$

$$- \sum_{k=1}^{\infty} \frac{\varepsilon_{n+2k-1}}{d_1d_2\cdots d_{n+2k-1}} - \sum_{i=1}^{n-1} \frac{(-1)^i c_i}{d_1d_2\cdots d_i} - \sum_{k=1}^{\infty} \frac{\varepsilon_{n+2k}}{d_1d_2\cdots d_{n+2k}}$$

$$= \frac{\varepsilon_n}{d_1d_2\cdots d_n} - \sum_{k=1}^{\infty} \frac{\varepsilon_{n+k}}{d_1d_2\cdots d_{n+k}} = \frac{1}{d_1d_2\cdots d_n}\left(\varepsilon_n - \sum_{k=1}^{\infty}\frac{\varepsilon_{n+k}}{d_{n+1}\cdots d_{n+k}}\right) \geq 0.$$

That is the cylinders are left-to-right situated and the last-mentioned difference equals zero when $s_0 = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{n-1}1[d_{n+1}-1][d_{n+2}-1][d_{n+3}-1]\ldots}$. Similarly, the inequality

$$\inf \Delta^{M_{s_0}}_{c_1c_2\ldots c_{n-1}0} - \sup \Delta^{M_{s_0}}_{c_1c_2\ldots c_{n-1}\varepsilon_n} \geq 0$$

holds when n is odd. That is cylinders $\Delta^{M_{s_0}}_{c_1c_2\ldots c_n}$ are right-to-left situated. Thus for any interval belonging to $[\inf M_{s_0}, \sup M_{s_0}]$ there exists a subinterval that does not contain points from M_{s_0}, since $\Delta^{M_{s_0}}_{c_1c_2\ldots c_{n-1}0} \cap\ \Delta^{M_{s_0}}_{c_1c_2\ldots c_{n-1}\varepsilon_n} \neq \varnothing$ if and only if

$$\varepsilon_n = 0 \text{ or } s_0 = \Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{n-1}1[d_{n+1}-1][d_{n+2}-1][d_{n+3}-1]\ldots}.$$

Let us prove that M_{s_0} is a closed set without isolated points. Choose an arbitrary limit point x_0 from M_{s_0}. From the definition of a limit point it follows that for all $\varepsilon > 0$ an interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ contains at least one point (that does not coincide with x_0) from M_{s_0}. If there does not exist a unique closed interval $I^{M_{s_0}}_{\delta_1(x_0)\delta_2(x_0)\ldots\delta_n(x_0)}$ such that $x_0 \in I^{M_{s_0}}_{\delta_1(x_0)\delta_2(x_0)\ldots\delta_n(x_0)}$, then x_0 belongs to one of the adjacent to M_{s_0} intervals. Therefore there exists $\varepsilon_0 > 0$ such that $(x_0 - \varepsilon_0, x_0 + \varepsilon_0) \cap M_{s_0} = \varnothing$. In this case, x_0 is not a limit point. If there exists a closed interval $I^{M_{s_0}}_{\delta_1(x_0)\delta_2(x_0)\ldots\delta_n(x_0)}$, then

$$x_0 = \bigcap_{n=1}^{\infty} I^{M_{s_0}}_{\delta_1(x_0)\delta_2(x_0)\ldots\delta_n(x_0)} \text{ and } x_0 \in M_{s_0}.$$

Hence M_{s_0} is a closed set.

Suppose there exists a certain isolated point $x^{'} = \Delta^{-D}_{\delta_1\delta_2\ldots\delta_n\ldots}$. Then there exists $\varepsilon_0 > 0$ such that

$$(x^{'} - \varepsilon_0, x^{'} + \varepsilon_0) \cap (M_{s_0} \setminus \{x^{'}\}) = \varnothing. \tag{8}$$

Take a number m such that $d(\Delta^{M_{s_0}}_{\delta_1\delta_2\ldots\delta_m}) < \varepsilon_0$ and $\varepsilon_{m+1}(s_0) \neq 0$. Then $\Delta^{M_{s_0}}_{\delta_1\delta_2\ldots\delta_m} \subset (x^{'} - \varepsilon_0, x^{'} + \varepsilon_0)$ and

$$x^{'} \neq x = \Delta^{-D}_{\delta_1\delta_2\ldots\delta_n\sigma\delta_{m+2}\ldots} \in (x^{'} - \varepsilon_0, x^{'} + \varepsilon_0) \cap M_{s_0},$$

where

$$\sigma = \begin{cases} \varepsilon_{m+1} & \text{if } \delta_{m+1} = 0 \\ 0 & \text{if } \delta_{m+1} \neq 0. \end{cases}$$

The last condition contradicts (8). The set M_{s_0} does not contain isolated points.

Let us calculate the Lebesgue measure of M_{s_0}. Let F_k be the union of closed intervals $I^{M_{s_0}}_{c_1c_2...c_k}$ of rank k ($c_k \in A'_k$). Then $M_{s_0} \subset F_k \subset F_{k+1}$ for all $k \in \mathbb{N}$ and $\lambda(M_{s_0}) \le \lim_{k\to\infty} F_k$. Since the condition $d(\Delta^{M_{s_0}}_{c_1c_2...c_n}) = |I^{M_{s_0}}_{c_1c_2...c_n}|$ and the properties of $\Delta^{M_{s_0}}_{c_1c_2...c_n}$ hold, it follows that

$$\lambda(M_{s_0}) \le \lim_{k\to\infty}\left(2^k \cdot \Delta^D_{\underbrace{0\ldots0}_{k}\varepsilon_{k+1}(s_0)\varepsilon_{k+2}(s_0)\varepsilon_{k+3}(s_0)\ldots}\right)$$

$$= \lim_{k\to\infty}\left(\frac{2^k}{d_1d_2\cdots d_k}\cdot\sum_{i=k+1}^{\infty}\frac{\varepsilon_i(s_0)}{d_{k+1}\cdots d_i}\right) = 0. \qquad \square$$

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