REVITALIZED AUTOMATIC PROOFS: DEMONSTRATIONS

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Abstract

We consider three problems from recent issues of the American Mathematical Monthly involving different versions of the Catalan triangle. Our main results offer generalizations of these identities and demonstrate automated proofs with additional twists, and on occasion we furnish a combinatorial proof.

1. Introduction

The impetus for this paper comes from Problem 11844 [6], Problem 11899 [9] and Problem 11916 [7] of the American Mathematical Monthly journal, plus the following identities that came up in our study:

$$\binom{n+m}{2n}\sum_{k=0}^{n} k\binom{2n}{n+k}^2\binom{2m}{m+k} = \frac{n}{2}\binom{2m}{m+n}\binom{2n}{n}\sum_{j=0}^{m-1}\binom{n+j}{n}\binom{n+j}{n-1}, \tag{1.1}$$

and

$$\binom{n+m}{m}\sum_{k=0}^{n} k\binom{2n}{n+k}\binom{2m}{m+k}^{2} = \frac{n}{2}\binom{2n}{n}\binom{2m}{m}\sum_{j=0}^{m-1}\binom{n+j}{n}\binom{m+j}{m-1}. \quad (1.2)$$

The purpose of our work here is to present certain generalizations and to provide *automatic proofs* as well as alternative techniques. Our demonstration of the Wilf-Zeilberger style of proof [8] exhibits the power of this methodology, especially where we supplemented it with novel adjustments whenever a direct implementation lingers.

A class of d-fold binomial sums of the type

$$R(n) = \sum_{k_1,\ldots,k_d}\prod_{i=1}^{d}\binom{2n}{n+k_i}|f(k_1,\ldots,k_d)|$$

has been investigated by several authors; see for example [2] and references therein. For a given function f, one interpretation is this: $4^{-dn}R(n)$ is the expectation of $|f|$ if one starts at the origin and takes $2n$ random steps of length $\pm\frac{1}{2}$ in each of the d dimensions, thus arriving at the point $(k_1,\ldots,k_d)\in\mathbb{Z}^d$ with probability

$$4^{-dn}\prod_{i=1}^{d}\binom{2n}{n+k_i}.$$

The organization of the paper is as follows. In Section 2, Problems 11844, 11916 and some generalized identities are proved. Section 3 resolves Problem 11899 and highlights a combinatorial proof together with q-analogues of related identities. Finally, in Section 4, we conclude with further generalizations.

2. The First Set of Main Results

Let's fix some nomenclature. The set of all integers is $\mathbb{Z}$, and the set of non-negative integers is $\mathbb{N}$. Define $B_{n,k} = \frac{k}{n}\binom{2n}{n-k} = \frac{k}{n}\binom{2n}{n+k}$ for $1 \le k \le n$, a variant of the Catalan triangle. On the other hand, the numbers $\binom{2n}{n-k} - \binom{2n}{n-k-1}$ form another variant of the Catalan triangle and these numbers count lattice paths (NE $= (1,1)$ and SE $= (1,-1)$ steps) from $(0,0)$ to $(2n,2k)$ that may touch but otherwise stay above the x-axis.

We adopt the usual convention that empty sums and empty products evaluate to 0 and 1, respectively, and that $\binom{n}{k} = 0$ whenever $k < 0$ or $k > n$. Moreover, in our function notation, we sometimes omit variables that are not telescoping variables.

Let $Q_{a,b} = \binom{a+b}{a}$. When considering a triple product of the numbers $B_{n,k}$, on occasion we use the following handy reformulation

$$\frac{abc\,Q_{a,b}Q_{b,c}Q_{c,a}}{Q_{a,a}Q_{b,b}Q_{c,c}}B_{a,k}B_{b,k}B_{c,k} = k^3\binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}. \tag{2.1}$$

Our first result solves Problem 11844 of the Monthly [6] as mentioned in the Introduction.

Lemma 1. *For non-negative integers $m \geq n$, we have*

$$\sum_{k=0}^{n}(m-2k)\binom{m}{k}^3 = (m-n)\binom{m}{n}\sum_{j=0}^{m-n-1}\binom{n+j}{n}\binom{n+j}{m-n-1}. \tag{2.2}$$

Proof. We apply the method of Wilf-Zeilberger [8]. This techniques works, in the present case, after multiplying (2.2) through with $(-1)^m$. Let

$$F_1(m,k) = (-1)^m(m-2k)\binom{m}{k}^3,$$

the resulting summand on the left-hand side of (2.2), fix n, and let $f_1(m) = \sum_{k=0}^{n} F_1(m,k)$. Now, introduce the companion function

$$G_1(m,k) = -F_1(m,k)\cdot\frac{(2m-k+2)k^3}{(m-2k)(m-k+1)^3}$$

and check that $F_1(m+1,k) - F_1(m,k) = G_1(m,k+1) - G_1(m,k)$. Telescoping gives

$$\begin{aligned} f_1(m+1) - f_1(m) &= \sum_{k=0}^{n} F_1(m+1,k) - \sum_{k=0}^{n} F_1(m,k) = \sum_{k=0}^{n}[G_1(m,k+1) - G_1(m,k)] \\ &= G_1(m,n+1) - 0 = (-1)^{m+1}\binom{m}{n}^3(2m-n+1). \end{aligned}$$

Let

$$F_2(m,j) = (-1)^m(m-n)\binom{m}{n}\binom{n+j}{n}\binom{n+j}{m-n-1},$$

the summand on the right-hand side of (2.2) and let $f_2(m) = \sum_{j=0}^{m-n-1} F_2(m,j)$. Introduce

$$G_2(m,j) = F_2(m,j)\cdot\frac{j(m-2n-j-1)}{(m-n)^2}$$

and check that $F_2(m+1,j) - F_2(m,j) = G_2(m,j+1) - G_2(m,j)$. Summing

$0 \le j \le m-n$ and telescoping, we arrive at

$$\begin{aligned} f_2(m+1) - f_2(m) &= \sum_{j=0}^{m-n} F_2(m+1,j) - \sum_{j=0}^{m-n} F_2(m,j) + F_2(m,m-n) \\ &= \sum_{j=0}^{m-n} [G_2(m,j+1) - G_2(m,j)] + F_2(m,m-n) \\ &= G_2(m,m-n+1) - 0 + F_2(m,m-n) \\ &= (-1)^{m+1}\binom{m}{n}^3 (2m-n+1)\,, \end{aligned}$$

the same recurrence as for f_1, both holding for $m \ge n$. The initial values are equal: $f_1(n) = 0$ because the terms in the sum cancel symmetrically, and $f_2(n)$ is an empty sum. Hence $f_1(m) = f_2(m)$ for $m \ge n$ and the lemma is established. $\square$

Theorem 1. *For nonnegative integers r, s and $m \ge n$, we have*

$$\sum_{k=0}^{n} \frac{(m-2k)\binom{m+r+s}{m,r,s}\binom{m}{k}\binom{m+2r}{k+r}\binom{m+2s}{k+s}}{\binom{m+2r}{m+r}\binom{m+2s}{m+s}\binom{m+s}{n+s}} = (m-n) \sum_{j=0}^{m-n+r-1} \binom{n+j}{n}\binom{n+j+s}{m-n+s-1}. \tag{2.3}$$

Proof. Again we use the Wilf-Zeilberger method, actually twice. Multiply through equation (2.3) by $\binom{m+s}{n+s}$ and let

$$F_1(r,k) = \frac{(m-2k)\binom{m+r+s}{m,r,s}\binom{m}{k}\binom{m+2r}{k+r}\binom{m+2s}{k+s}}{\binom{m+2r}{m+r}\binom{m+2s}{m+s}},$$

the resulting summand on the new left-hand side of (2.3). Fix n, m, s and and let $f_1(r) = \sum_{k=0}^{n} F_1(r,k)$. Now, introduce the companion function

$$G_1(r,k) = F_1(r,k) \cdot \frac{k(s+k)}{(m-2k)(m+r-k+1)}$$

and (routinely) check that $F_1(r+1,k) - F_1(r,k) = G_1(r,k+1) - G_1(r,k)$. Telescoping gives, for $r \ge 0$,

$$\begin{aligned} f_1(r+1) - f_1(r) &= \sum_{k=0}^{n} F_1(r+1,k) - \sum_{k=0}^{n} F_1(r,k) = \sum_{k=0}^{n} [G_1(r,k+1) - G_1(r,k)] \\ &= G_1(r,n+1) - 0 = (m-n)\binom{m+s}{n+s}\binom{m+r}{n}\binom{m+r+s}{m-n+s-1}. \end{aligned}$$

Let

$$f_2(r) = (m-n)\binom{m+s}{n+s} \sum_{j=0}^{m-n+r-1} \binom{n+j}{n}\binom{n+j+s}{m-n+s-1},$$

the entire summand on the right-hand side of (2.3). By cancelling identical terms we obtain, for $r \geq 0$,

$$f_2(r+1) - f_2(r) = (m-n)\binom{m+s}{n+s}\binom{m+r}{n}\binom{m+r+s}{m-n+s-1}.$$

It remains to verify the initial condition $f_1(0) = f_2(0)$; that is,

$$\sum_{k=0}^{n} \frac{(m-2k)\binom{m+s}{m}\binom{m}{k}^2\binom{m+2s}{k+s}}{\binom{m+2s}{m+s}} = (m-n)\binom{m+s}{n+s}\sum_{j=0}^{m-n-1}\binom{n+j}{n}\binom{n+j+s}{m-n+s-1}. \tag{2.4}$$

Denote the summand on the left-hand side of (2.4) by $F_2(s,k)$ and its sum by $f_2(s) = \sum_{k=0}^{n} F_2(s,k)$. Now, introduce the companion function

$$G_2(s,k) = F_2(s,k) \cdot \frac{k^2}{(m-2k)(m+s-k+1)}$$

and verify that $F_2(s+1,k) - F_2(s,k) = G_2(s,k+1) - G_2(s,k)$. Telescoping gives

$$\begin{aligned} f_2(s+1) - f_2(s) &= \sum_{k=0}^{n} F_2(s+1,k) - \sum_{k=0}^{n} F_2(s,k) = \sum_{k=0}^{n}[G_2(s,k+1) - G_2(s,k)] \\ &= G_2(s,n+1) - 0 = (m-n)\binom{m+s}{n+s+1}\binom{m+s}{n}\binom{m}{n}. \end{aligned}$$

Let

$$F_3(s,j) = (m-n)\binom{m+s}{n+s}\binom{n+j}{n}\binom{n+j+s}{m-n+s-1},$$

the summand on the right-hand side of (2.4), and let $f_3(s) = \sum_{j=0}^{m-n-1} F_3(s,j)$. Introduce

$$G_3(s,j) = F_3(s,j) \cdot \frac{j(2n-m+j+1)}{(n+s+1)(m-n+s)}$$

and check that $F_3(s+1,j) - F_3(s,j) = G_3(s,j+1) - G_3(s,j)$. Summing and telescoping, we get

$$\begin{aligned} f_3(s+1) - f_3(s) &= \sum_{j=0}^{m-n-1} F_3(s+1,j) - \sum_{j=0}^{m-n-1} F_3(s,j) = \sum_{j=0}^{m-n-1}[G_3(s,j+1) - G_3(s,j)] \\ &= G_3(s,m-n) - 0 = (m-n)\binom{m+s}{n+s+1}\binom{m+s}{n}\binom{m}{n}. \end{aligned}$$

The initial condition $f_2(0) = f_3(0)$ is precisely the content of Lemma 1. $\square$

The next statement covers Problem 11916 [7] as an immediate application of Theorem 1.

Corollary 1. *Let a, b and c be non-negative integers. Then, the function*

$$U(a,b,c) = a\binom{a+b}{a}\sum_{j=0}^{c-1}\binom{a+j}{a}\binom{b+j}{b-1}$$

is symmetric; that is, $U(\sigma(a),\sigma(b),\sigma(c)) = U(a,b,c)$ for any σ in the symmetric group $\mathfrak{S}_3$.

Proof. If $n = a, m = 2a, r = b-a, s = c-a$, the left-hand side of (2.3) times $\binom{m+s}{n+s}$ turns into (in the second step, reindex $k \to a-k$)

$$\begin{aligned} LHS &= \frac{\binom{b+c}{2a,b-a,c-a}}{\binom{2b}{b+a}\binom{2c}{c+a}}\sum_{k=0}^{a}(2a-2k)\binom{2a}{k}\binom{2b}{k+b-a}\binom{2c}{k+c-a} \\ &= 2\frac{(a+b)!(b+c)!(c+a)!}{(2a)!(2b)!(2c)!}\sum_{k=0}^{a}k\binom{2a}{a-k}\binom{2b}{b-k}\binom{2c}{c-k} \\ &= \frac{2Q_{a,b}Q_{b,c}Q_{c,a}}{Q_{a,a}Q_{b,b}Q_{c,c}}\sum_{k=0}^{a}k\binom{2a}{a+k}\binom{2b}{b+k}\binom{2c}{c+k} \end{aligned}$$

and the right-hand side of (2.3) multipled by $\binom{m+s}{n+s} = \binom{a+c}{c}$ simplifies to

$$RHS = a\binom{a+c}{c}\sum_{j=0}^{b-1}\binom{a+j}{a}\binom{c+j}{c-1} = aQ_{c,a}\sum_{j=0}^{b-1}\binom{a+j}{a}\binom{c+j}{c-1}.$$

Therefore, we obtain

$$\frac{Q_{a,b}Q_{b,c}Q_{c,a}}{Q_{a,a}Q_{b,b}Q_{c,c}}\sum_{k=0}^{a}k\binom{2a}{a+k}\binom{2b}{b+k}\binom{2c}{c+k} = \frac{aQ_{c,a}}{2}\sum_{j=0}^{b-1}\binom{a+j}{a}\binom{c+j}{c-1}. \quad (2.5)$$

The following obvious symmetrization

$$\sum_{k=0}^{a}k\binom{2a}{a+k}\binom{2b}{b+k}\binom{2c}{c+k} = \sum_{k=0}^{\min\{a,b,c\}}k\binom{2a}{a+k}\binom{2b}{b+k}\binom{2c}{c+k}$$

implies that the left-hand side of the identity in (2.5) has to be symmetric. The assertion follows from the symmetry inherited by the right-hand side of the same equation (2.5). $\square$

Example 1. In equation (2.5), the special case $a = n,\ b = c = m$ becomes (1.1) while $a = b = n,\ c = m$ recovers (1.2).

Corollary 2. *Preserve notations from Corollary 1. For $a, b, c \in \mathbb{N}$ and any $\sigma \in \mathfrak{S}_3$, we have*

$$\sum_{k=0}^{a} k\binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k} = \frac{\sigma(a)Q_{\sigma(a),\sigma(b)}}{2}\sum_{j=0}^{\sigma(c)-1}\binom{\sigma(a)+j}{\sigma(a)}\binom{\sigma(b)+j}{\sigma(b)-1}. \tag{2.6}$$

Proof. First, rewrite the summand on the left-hand side of (2.5) as follows:

$$k\frac{Q_{a,b}Q_{b,c}Q_{c,a}}{Q_{a,a}Q_{b,b}Q_{c,c}}\binom{2a}{a+k}\binom{2b}{b+k}\binom{2c}{c+k} = \frac{k\ (a+b)!(b+c)!(c+a)!}{(a+k)!(a-k)!(b+k)!(b-k)!(c+k)!(c-k)!}$$
$$= k\binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k},$$

which is precisely the left-hand side of (2.6). Now, apply the identity in (2.5) and the statement of Corollary 1. □

Theorem 2. *Let $e(a,b,c) = ab + bc + ca$. For non-negative integers a, b and c, we have*

$$\sum_{k=0}^{a} k^3\binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k} = \frac{b^2c^2Q_{b,c}}{2}\sum_{j=0}^{a-1}\frac{e(a,b,c)\binom{b+j}{b}\binom{c+j}{c}}{e(j,b,c)\cdot e(j+1,b,c)}. \tag{2.7}$$

Proof. Once again use the Wilf-Zeilberger method. First, divide through by $e := e(a,b,c)$ to denote the summand on the left-hand side of (2.7) by $F_1(a,k)$ and its sum by $f_1(a) = \sum_{k=0}^{a} F_1(a,k)$. Now, introduce the companion function

$$G_1(a,k) = -F_1(a,k)\cdot\frac{((e+b+c)k^2-(e+b+c)k+abc+bc)(b+k)(c+k)}{2k^3(a+1-k)\cdot(e+b+c)}$$

and check that $F_1(a+1,k) - F_1(a,k) = G_1(a,k+1) - G_1(a,k)$. Keeping in mind that $F_1(a,a+1) = 0$ and telescoping gives

$$f_1(a+1) - f_1(a) = \sum_{k=0}^{a+1} F_1(a+1,k) - \sum_{k=0}^{a+1} F_1(a,k) = \sum_{k=0}^{a+1}[G_1(a,k+1) - G_1(a,k)]$$
$$= G_1(a,a+2) - G_1(a,0) = 0 - G_1(a,0) = \frac{b^2c^2Q_{a,b}Q_{b,c}Q_{c,a}}{2e(e+b+c)}.$$

Notice k^3 in the denominator of $G_1(a,k)$ disappears because there is k^3 in the numerator of $F_1(a,k)$, hence $G_1(a,0)$ makes sense. This difference formula for $f_1(a+1) - f_1(a)$ leads to

$$f_1(a) = \frac{b^2c^2Q_{b,c}}{2}\cdot\sum_{j=0}^{a-1}\frac{\binom{b+j}{a}\binom{c+j}{c}}{(jb+bc+cj)\cdot(jb+bc+cj+b+c)}$$

which is the required conclusion. □

Remark 1. In [5], Miana, Ohtsuka and Romero obtained two identities for the sum $\sum_{k=0}^{n} B_{n,k}^3$. From Theorem 2 and (2.1), one obtains an identity for the sum $\sum_{k=0}^{a} B_{a,k}B_{b,k}B_{c,k}$.

Remark 2. Corollary 2 and Theorem 2 exhibit formulas for $\sum_k k(\cdots)$ and $\sum_k k^3(\cdots)$. It appears that similar (albeit complicated) results are possible for sums of the type $\sum_k k^p(\dots)$ whenever p is an odd positive integer (but not when p is even).

We offer a 4-parameter generalization of Corollary 2.

Theorem 3. *For non-negative integers a, b, c and d, we have*

$$\sum_{k=0}^{a} k\binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+d}{c+k}\binom{d+a}{d+k} = \frac{bQ_{b,c}Q_{c,d}Q_{b+c+d,a}}{2Q_{a,c}}\sum_{j=0}^{a-1}\frac{Q_{b,j}Q_{c-1,j+1}Q_{d-1,j+1}}{Q_{b+c+d,j+1}}.$$

Proof. The proof goes along the same arguments used for Corollary 2 and Theorem 2. □

Remark 3. It is interesting to compare our results against Corollary 4.1 of [3]. Although these results are similar, there are differences: in our case the right-hand sides are less involved while those of [3] are more general. See also Corollary 4.2 and Theorem 4.3 of [5]. The examples below explore some specific cases.

Example 2. Set $a = b = c = n$ in Corollary 2. The outcome is

$$\sum_{k=0}^{n} k\binom{2n}{n+k}^3 = \frac{1}{2}\binom{2n}{n}\sum_{j=0}^{n} j\binom{n+j-1}{n-1}^2.$$

Example 3. Set $a = b = c = n$ in Theorem 2. The outcome is

$$\sum_{k=0}^{n} k^3\binom{2n}{n+k}^3 = \frac{3}{2}n^4\binom{2n}{n}\sum_{j=0}^{n-1}\frac{\binom{n+j}{n}^2}{(n+2j)(n+2j+2)}.$$

Example 4. Set $a = b = c = d = n$ in Theorem 3. The outcome is

$$\sum_{k=0}^{n} k\binom{2n}{n+k}^4 = \frac{1}{2}\binom{4n}{n}\binom{2n}{n}\sum_{j=0}^{n} j\binom{n+j-1}{n-1}^3\binom{3n+j}{3n}^{-1}.$$

3. The Second Set of Main Results

We start with a q-identity and its ordinary counterpart will allow us to prove one of the Monthly problems which was alluded to in the Introduction. Let's recall some notation. The q-analogue of the integer n is given by $[n]_q = \frac{1-q^n}{1-q}$, the factorial by $[n]_q! = \prod_{i=1}^{n} \frac{1-q^i}{1-q}$ and the binomial coefficients by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Lemma 2. *For a free parameter q and a positive integer n, we have*

$$\sum_{k=0}^{n} \binom{2n+1}{n-k}_q \left[\binom{2n}{n-k}_q - \binom{2n}{n-k-1}_q \right] q^{k(k+1)} = q^n \binom{2n}{n}_q^2.$$

Proof. Let $G(n,k) = \binom{2n}{n+k}_q^2 \cdot q^{n+k^2}$. Now, check that

$$\begin{aligned}\binom{2n+1}{n-k}_q \left[\binom{2n}{n-k}_q - \binom{2n}{n-k-1}_q \right] q^{k(k+1)} &= \binom{2n+1}{n-k}_q^2 \frac{1-q^{2k+1}}{1-q^{2n+1}} q^{n+k^2} \\ &= G(n,k) - G(n,k+1)\end{aligned}$$

and then sum over $0 \le k \le n$ to obtain $G(n,0) = q^n \binom{2n}{n}_q^2$. □

We now demonstrate a combinatorial argument for the special case $q = 1$ of Lemma 2.

Lemma 3. *For non-negative integers n, we have*

$$\sum_{k=0}^{n} \binom{2n+1}{n-k} \left[\binom{2n}{n-k} - \binom{2n}{n-k-1} \right] = \binom{2n}{n}^2. \tag{3.1}$$

Proof. The first factor in the summand on the left side of (3.1) counts paths of $2n+1$ steps, consisting of upsteps $(1,1)$ or downsteps $(1,-1)$, that start at the origin and end at height $2k+1$. The second factor is the generalized Catalan number that counts *nonnegative* (i.e., first quadrant) paths of $2n$ up/down steps that end at height $2k$. By concatenating the first path and the reverse of the second, we see that the left side counts the set X_n of paths of $2n+1$ upsteps and $2n$ downsteps that avoid the x-axis for $x > 2n$, i.e. avoid $(2n+2,0), (2n+4,0), \ldots, (4n,0)$.

Now $\binom{2n}{n}$ is the number of *balanced* paths of length $2n$ (i.e., n upsteps and n downsteps), but it is also the number of nonnegative $2n$-paths and, for $n \ge 1$, twice the number of positive (= nonnegative, no-return) $2n$-paths (see [4], for example). So, the right side of (3.1) counts the set Y_n of pairs (P,Q) of nonnegative $2n$-paths. Here is a bijection ϕ from X_n to Y_n. A path $P \in X_n$ ends at height 1 and so its last upstep from the x-axis splits it into $P = BUD$ where B is a balanced path and D is a Dyck path of length $\ge 2n$ since P avoids the x-axis for $x > 2n$. Write D as QR where R is of length $2n$.

If B is empty, set $\phi(P) = (Q, \text{Reverse}(R))$, a pair of nonnegative $2n$-paths ending at the same height. If B is nonempty, then by the above remarks it is equivalent to a bicolored positive path S of the same length, say colored red or blue. If red, set $\phi(P) = (Q\,S,\ \text{Reverse}(R)) \in Y_n$ with the first path ending strictly higher than the second. If blue, set $\phi(P) = (\text{Reverse}(R),\ Q\,S) \in Y_n$ with the first path ending strictly lower than the second. It is easy to check that ϕ is a bijection from X_n to Y_n. □

As an application, we present a proof for Problem 11899 as advertised in the introduction.

Corollary 3. *For non-negative positive integer n, we have*

$$\sum_{k=0}^{n}\binom{2n}{k}\binom{2n+1}{k}+\sum_{k=n+1}^{2n+1}\binom{2n}{k-1}\binom{2n+1}{k}=\binom{4n+1}{2n}+\binom{2n}{n}^2.$$

Proof. Start by writing

$$A_1=\sum_{k=0}^{n}\binom{2n}{k}\binom{2n+1}{k}, \qquad \tilde{A}_1=\sum_{k=n+1}^{2n+1}\binom{2n}{k-1}\binom{2n+1}{k},$$
$$A_2=\sum_{k=n+1}^{2n+1}\binom{2n}{k}\binom{2n+1}{k}.$$

The required identity is

$$A_1+\tilde{A}_1=\binom{4n+1}{2n}+\binom{2n}{n}^2,$$

but reindexing gives $A_1=\tilde{A}_1$. In view of the Vandermonde-Chu identity $A_1+A_2=\binom{4n+1}{2n}$, it suffices to prove that $A_1-A_2=\binom{2n}{n}^2$, that is,

$$\sum_{k=0}^{n}\binom{2n+1}{n-k}\left[\binom{2n}{n-k}-\binom{2n}{n-k-1}\right]=\binom{2n}{n}^2$$

which is exactly what Lemma 3 shows. However, here is yet another verification: if we let $G(n,k)=\binom{2n}{n+k}^2$ then it is routine to check that

$$\binom{2n+1}{n-k}\left[\binom{2n}{n-k}-\binom{2n}{n-k-1}\right]=\binom{2n+1}{n-k}^2\frac{2k+1}{2n+1}=G(n,k)-G(n,k+1).$$

Obviously then

$$\sum_{k=0}^{n}[G(n,k)-G(n,k+1)]=G(n,0)-G(n,n+1)=G(n,0)=\binom{2n}{n}^2.$$

The proof follows. □

4. Concluding Remarks

In this section, we list binomial identities with extra parameters similar to those from the preceding sections, however their proofs are left to the interested reader because we wish to limit undue replication of our techniques.

The first result generalizes Corollary 2.

Proposition 1. *For non-negative integers a, b, c and an integer r, we have*

$$\sum_{k=1}^{a+r}(2k-r)\binom{a+b+r}{a+k}\binom{b+c+r}{b+k}\binom{c+a+r}{c+k} = (a+r)Q_{a+r,b}\sum_{j=0}^{c+r-1}\binom{a+j}{a}\binom{b+j}{b+r-1}.$$

Next, we state certain natural q-analogues of Corollary 2 and Corollary 1.

Theorem 4. *For non-negative integers a, b and c, we have*

$$\sum_{k=0}^{a}\frac{(1-q^{2k})q^{2k^2-k-1}}{1-q^a}\binom{a+b}{a+k}_q\binom{b+c}{b+k}_q\binom{c+a}{c+k}_q = \binom{a+b}{a}_q\sum_{j=0}^{c-1}q^j\binom{a+j}{a}_q\binom{b+j}{b-1}_q.$$

Corollary 4. *Let a, b and c be non-negative integers. Then, the function*

$$U_q(a,b,c) = \frac{1-q^a}{1-q}\binom{a+b}{a}_q\sum_{j=0}^{c-1}\binom{a+j}{a}_q\binom{b+j}{b-1}_q$$

is symmetric, i.e. $U_q(\sigma(a),\sigma(b),\sigma(c)) = U_q(a,b,c)$ for any σ in the symmetric groups $\mathfrak{S}_3$.

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