A GENERALIZATION OF SCHAUZ AND BRINK'S RESTRICTED-VARIABLE VERSION OF CHEVALLEY'S THEOREM

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Abstract

Let $f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}) \in \mathbb{F}_q[t_1, \ldots, t_r][x_1, \ldots, x_s]$ be polynomials in $\mathbf{x}$ with coefficients in $\mathbb{F}_q[t_1, \ldots, t_r]$. Suppose that $f_j(\mathbf{0}) = 0$ $(1 \le j \le k)$, and let $\{0\} \subset A_l \subseteq \mathbb{F}_q$ $(1 \le l \le s)$. Provided that the number of variables s is large enough in terms of q, r, the cardinalities of the sets $|A_l|$, and the degrees of the polynomials $f_j(\mathbf{x})$, there exists a non-zero common solution to the system of equations $f_j(\mathbf{x}) = 0$ $(1 \le j \le k)$, where each x_l $(1 \le l \le s)$ is a polynomial in $\mathbf{t}$ with coefficients in A_l. We also establish similar results for systems of congruences over Dedekind domains and systems of inequalities over $\mathbb{F}_q((1/t))$.

Introduction

Let $\mathbb{F}_q$ denote the finite field with q elements, where q is a power of a prime. In 1935, Chevalley [4] proved the following theorem, which confirmed a conjecture of Artin [2] from the same year.

Theorem 1. *Let $f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}) \in \mathbb{F}_q[x_1, \ldots, x_s]$ be polynomials satisfying $f_j(\mathbf{0}) = 0$ for all $1 \le j \le k$. Then, provided that*

$$s > \sum_{j=1}^{k} \deg f_j(\mathbf{x}),$$

there exists a non-zero common solution to the system of equations $f_j(\mathbf{x}) = 0$ $(1 \le j \le k)$ with $x_l \in \mathbb{F}_q$ for $1 \le l \le s$.

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Schauz [10, Corollary 3.5] in 2008 and Brink [3, Theorem 1] in 2011 independently proved the following "restricted variable" extension of Chevalley's Theorem, in which they required each x_l of a non-zero solution $\mathbf{x} = (x_1, \ldots, x_s)$ to belong to some specified subset $A_l \subseteq \mathbb{F}_q$ for $1 \leq l \leq s$.

Theorem 2. *Let $f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}) \in \mathbb{F}_q[x_1, \ldots, x_s]$ be polynomials satisfying $f_j(\mathbf{0}) = 0$ for all $1 \leq j \leq k$. For all $1 \leq l \leq s$, let A_l be a subset of $\mathbb{F}_q$ with $\{0\} \subset A_l \subseteq \mathbb{F}_q$. Then, provided that*

$$\sum_{l=1}^{s}(|A_l| - 1) > (q-1)\sum_{j=1}^{k} \deg f_j(\mathbf{x}),$$

there exists a non-zero common solution to the system of equations $f_j(\mathbf{x}) = 0$ $(1 \leq j \leq k)$ with $x_l \in A_l$ for $1 \leq l \leq s$.

The proofs by Schauz and Brink are applications of the Combinatorial Nullstellensatz (see [1]). If $A_l = \mathbb{F}_q$ $(1 \leq l \leq s)$, we obtain Chevalley's Theorem, which is best possible in the sense that, in the language of Theorem 1, there are systems of equations of the required form with no non-zero solutions that satisfy $s = \deg f_1(\mathbf{x}) + \cdots + \deg f_k(\mathbf{x})$. By bootstrapping off of Theorem 2, we are able to obtain the following theorem.

Theorem 3. *Let $f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}) \in \mathbb{F}_q[t_1, \ldots, t_r][x_1, \ldots, x_s]$ be polynomials in $\mathbf{x}$ with coefficients in $\mathbb{F}_q[\mathbf{t}]$ such that $f_j(\mathbf{0}) = 0$ for $1 \leq j \leq k$. For all $1 \leq l \leq s$, let A_l be a subset of $\mathbb{F}_q$ with $\{0\} \subset A_l \subseteq \mathbb{F}_q$. Then, provided that*

$$\sum_{l=1}^{s}(|A_l| - 1) > (q-1)\sum_{j=1}^{k} (\deg f_j(\mathbf{x}))^{r+1},$$

there exists a non-zero common solution to the system of equations $f_j(\mathbf{x}) = 0$ $(1 \leq j \leq k)$, where each x_l $(1 \leq l \leq s)$ is a polynomial in $\mathbf{t}$ with coefficients in A_l.

When $A_l = \mathbb{F}_q$ $(1 \leq l \leq s)$, this theorem recovers a result of Lang [8, Corollary to Theorem 6]. We now state two corollaries, which may be of interest to number theorists, that follow directly from setting each set A_l in the theorem above to be $\{0, 1\}$ or $\{0, \pm 1\}$.

Corollary 1. *Let $f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}) \in \mathbb{F}_q[t_1, \ldots, t_r][x_1, \ldots, x_s]$ be polynomials in $\mathbf{x}$ with coefficients in $\mathbb{F}_q[\mathbf{t}]$ such that $f_j(\mathbf{0}) = 0$ for $1 \leq j \leq k$. Then, provided that*

$$s > (q-1)\sum_{j=1}^{k} (\deg f_j(\mathbf{x}))^{r+1},$$

there exists a non-zero common solution to the system of equations $f_j(\mathbf{x}) = 0$ $(1 \leq j \leq k)$, where each x_l $(1 \leq l \leq s)$ is a sum of distinct monomials in $\mathbf{t}$.

Corollary 2. *Let $f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}) \in \mathbb{F}_q[t_1, \ldots, t_r][x_1, \ldots, x_s]$ be polynomials in $\mathbf{x}$ with coefficients in $\mathbb{F}_q[\mathbf{t}]$ such that $f_j(\mathbf{0}) = 0$ for $1 \leq j \leq k$. Then, provided that*

$$s > \frac{q-1}{2} \sum_{j=1}^{k} (\deg f_j(\mathbf{x}))^{r+1},$$

there exists a non-zero common solution to the system of equations $f_j(\mathbf{x}) = 0$ $(1 \leq j \leq k)$, where each x_l $(1 \leq l \leq s)$ can be written as sums and differences of distinct monomials in $\mathbf{t}$.

In Section 2 we use a result of Clark [6] in which the coefficient ring of the polynomials is a Dedekind domain instead of a finite field $\mathbb{F}_q$. In Section 3, we prove the existence of solutions to systems of polynomial inequalities in $\mathbb{F}_q((1/t))$ and more generally $\mathbb{F}_q((1/t_1)) \cdots ((1/t_r))$, where the solutions have coefficients restricted to subsets of $\mathbb{F}_q$. The method of proof is nearly identical in all of these cases, but proofs are provided for completeness. The arguments used are similar to the exposition found in the proof of Theorem 1.4 in [9].

1. Proof of Theorem 3

In this section, we prove Theorem 3 via induction on r. Note that when $r = 0$, the statement follows from Theorem 2. Suppose that the theorem holds for some particular value of $r \in \mathbb{N} \cup \{0\}$. Let $f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}) \in \mathbb{F}_q[t_1, \ldots, t_{r+1}][x_1, \ldots, x_s]$ be polynomials in $\mathbf{x}$ with coefficients in $\mathbb{F}_q[t_1, \ldots, t_{r+1}]$ such that $f_j(\mathbf{0}) = 0$ $(1 \leq j \leq k)$ and such that

$$\sum_{l=1}^{s} (|A_l| - 1) > (q-1) \sum_{j=1}^{k} d_j^{r+2}, \tag{1}$$

where $d_j = \deg f_j(\mathbf{x})$.

If we had a non-zero common solution $\mathbf{x}$ to the system of equations $f_j(\mathbf{x}) = 0$ $(1 \leq j \leq k)$, where each x_l $(1 \leq l \leq s)$ is a polynomial in $t_1, \ldots, t_{r+1}$ with coefficients in A_l, then we would be able to write

$$x_l = \sum_{i=0}^{m} a_{l,i} t_{r+1}^i, \tag{2}$$

where $a_{l,i} \in \mathbb{F}_q[t_1, \ldots, t_r]$ and m is some nonnegative integer. We turn this reasoning around and use formula (2) as an ansatz. Namely, treat the $a_{l,i}$ as indeterminates, and transform the equations $f_j(\mathbf{x}) = 0$ into equations involving the variables $\mathbf{a} = (a_{l,i})$ by using (2) to substitute for each x_l. We will then use the inductive hypothesis to find a nonzero solution $\mathbf{a} = (a_{l,i})$ with $a_{l,i} \in \mathbb{F}_q[t_1, \ldots, t_r]$. Note that if each $a_{l,i}$

with $0 \le i \le m$ is a polynomial in $t_1, \dots, t_r$ with coefficients in A_l, then x_l defined by (2) is a polynomial in $t_1, \dots, t_{r+1}$ with coefficients in A_l.

Let n denote the highest power of t_{r+1} that occurs amongst the coefficients of the polynomials $f_1(\mathbf{x}), \dots, f_k(\mathbf{x})$. For each $1 \le j \le k$, by applying (2), we can write

$$f_j(\mathbf{x}) = \sum_{w=0}^{md_j+n} p_{j,w}(\mathbf{a}) t_{r+1}^w,$$

where each $p_{j,w}(\mathbf{a})$ is a polynomial of degree at most d_j in variables $a_{l,i}$ $(1 \le l \le s, 0 \le i \le m)$ with coefficients in $\mathbb{F}_q[t_1, \dots, t_r]$. Note that $\mathbf{a} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$. Hence, $p_{j,w}(\mathbf{0}) = 0$ for all $1 \le j \le k$ and $0 \le w \le md_j + n$.

Applying our inductive hypothesis to the system

$$p_{j,w}(\mathbf{a}) = 0 \qquad (1 \le j \le k,\ 0 \le w \le md_j + n),$$

there exists a non-zero common solution to the system of equations $f_j(\mathbf{x}) = 0$ $(1 \le j \le k)$, where each x_l $(1 \le l \le s)$ is a polynomial in $t_1, \dots, t_{r+1}$ with coefficients in A_l, provided that

$$\sum_{l=1}^{s} (m+1)(|A_l| - 1) > (q-1) \sum_{j=1}^{k} (md_j + n + 1) d_j^{r+1}.$$

Dividing both sides of this inequality by $m+1$, we obtain

$$\sum_{l=1}^{s} (|A_l| - 1) > (q-1) \sum_{j=1}^{k} \frac{md_j + n + 1}{m+1} d_j^{r+1}.$$

By (1), the above inequality will hold for sufficiently large values of m. This completes the proof of the theorem.

2. Extension to Dedekind Domains

Many authors have proved variants and generalizations of Chevalley's theorem; see [5] for a brief overview. For example, Schauz [10, Theorem 8.4], Wilson [12], and Brink [3, Theorem 2] proved a variant of the restricted variable extension of Chevalley's theorem where the coefficient ring $\mathbb{F}_q$ of the polynomials $f_j(\mathbf{x})$ is replaced by $\mathbb{Z}$ and the conditions $f_j(\mathbf{x}) = 0$ are replaced by conguences $f_j(\mathbf{x}) \equiv 0 \bmod p^{v_j}$ for a fixed prime p. Brink [3] also stated that the coefficient ring could taken to be the ring of integers in a number field, and a proof in this case was given by Clark, Forrow, and Schmitt [7, Theorem 3.1]. The referee brought to our attention a recent result of Clark [6, Theorem 1.7] that generalizes this line of study to Dedekind domains. We now state a formulation of Clark's result that we will then generalize.

Theorem 4. *Let R be a Dedekind domain with maximal ideal $\mathfrak{p}$ and finite residue field $R/\mathfrak{p} \cong \mathbb{F}_q$. Let*

$$f_1(\mathbf{x}), \dots, f_k(\mathbf{x}) \in R[x_1, \dots, x_s]$$

be polynomials in $\mathbf{x}$ with $f_j(\mathbf{0}) \equiv 0 \bmod \mathfrak{p}^{v_j} R$ and $d_j = \deg f_j(\mathbf{x})$ for $1 \leq j \leq k$. Let $v_1, \dots, v_k \in \mathbb{Z}^+$, and let $A_1, \dots, A_s$ be subsets of R such that for each l $(1 \leq l \leq s)$, the elements of A_l are pairwise incongruent modulo $\mathfrak{p}$ and $0 \in A_l$. If

$$\sum_{l=1}^{s} (|A_l| - 1) > \sum_{j=1}^{k} d_j (q^{v_j} - 1),$$

then there exists a non-zero solution to the system of congruences

$$f_j(\mathbf{x}) \equiv 0 \bmod \mathfrak{p}^{v_j} R \qquad (1 \leq j \leq k)$$

with $x_l \in A_l$ for $1 \leq l \leq s$.

By performing a similar argument to the proof of Theorem 3, we arrive at the following theorem.

Theorem 5. *Let R be a Dedekind domain with maximal ideal $\mathfrak{p}$ and finite residue field $R/\mathfrak{p} \cong \mathbb{F}_q$. Let*

$$f_1(\mathbf{x}), \dots, f_k(\mathbf{x}) \in R[t_1, \dots, t_r][x_1, \dots, x_s]$$

be polynomials in $\mathbf{x}$ with coefficients in $R[\mathbf{t}]$ satisfying $f_j(\mathbf{0}) \equiv 0 \bmod \mathfrak{p}^{v_j} R[t_1, \dots, t_r]$ and $d_j = \deg f_j(\mathbf{x})$ for $1 \leq j \leq k$. Let $v_1, \dots, v_k \in \mathbb{Z}^+$, and let $A_1, \dots, A_s$ be subsets of R such that for each l $(1 \leq l \leq s)$, the elements of A_l are pairwise incongruent modulo $\mathfrak{p}$ and $0 \in A_l$. Then, provided that

$$\sum_{l=1}^{s} (|A_l| - 1) > \sum_{j=1}^{k} d_j^{r+1} (q^{v_j} - 1),$$

there exists a non-zero common solution to the system of congruences

$$f_j(\mathbf{x}) \equiv 0 \bmod \mathfrak{p}^{v_j} R[t_1, \dots, t_r] \qquad (1 \leq j \leq k),$$

where each x_l $(1 \leq l \leq s)$ is a polynomial in $\mathbf{t}$ with coefficients in A_l.

Proof. We proceed via induction on r. Note that when $r = 0$, the statement follows from Theorem 4. Suppose that the theorem holds for some particular value of $r \in \mathbb{N} \cup \{0\}$. Let $f_1(\mathbf{x}), \dots, f_k(\mathbf{x}) \in R[t_1, \dots, t_{r+1}][x_1, \dots, x_s]$ be polynomials in $\mathbf{x}$ with coefficients in $R[t_1, \dots, t_{r+1}]$, such that for each $1 \leq j \leq k$, we have $f_j(\mathbf{0}) \equiv 0 \bmod \mathfrak{p}^{v_j} R[t_1, \dots, t_{r+1}]$. Let $d_j = \deg f_j(\mathbf{x})$, and assume

$$\sum_{l=1}^{s} (|A_l| - 1) > \sum_{j=1}^{k} d_j^{r+2} (q^{v_j} - 1). \tag{3}$$

Let n denote the highest power of t_{r+1} that occurs amongst the coefficients of the polynomials $f_1(\mathbf{x}), \ldots, f_k(\mathbf{x})$. Let m be a parameter to be chosen later, and for each $1 \leq l \leq s$, use the ansatz

$$x_l = \sum_{i=0}^{m} a_{l,i} t_{r+1}^i,$$

where the $a_{l,i}$ are for the moment indeterminates and will later be elements of $R[t_1, \ldots, t_r]$. Note that if each $a_{l,i}$ with $0 \leq i \leq m$ is a polynomial in $t_1, \ldots, t_r$ with coefficients in A_l, then x_l is a polynomial in $t_1, \ldots, t_{r+1}$ with coefficients in A_l.

For each $1 \leq j \leq k$, upon substituting $\sum_{i=0}^{m} a_{l,i} t_{r+1}^i$ for x_l $(1 \leq l \leq s)$ in the congruence

$$f_j(\mathbf{x}) \equiv 0 \bmod \mathfrak{p}^{v_j} R[t_1, \ldots, t_{r+1}],$$

we have

$$\sum_{w=0}^{md_j+n} p_{j,w}(\mathbf{a}) t_{r+1}^w \equiv 0 \bmod \mathfrak{p}^{v_j} R[t_1, \ldots, t_{r+1}],$$

where each $p_{j,w}(\mathbf{a})$ is a polynomial of degree at most d_j in variables $a_{l,i}$ $(1 \leq l \leq s, 0 \leq i \leq m)$ with coefficients in $R[t_1, \ldots, t_r]$. Note that $f_j(\mathbf{0}) \equiv 0 \bmod \mathfrak{p}^{v_j} R[t_1, \ldots, t_{r+1}]$ implies $p_{j,w}(\mathbf{0}) \equiv 0 \bmod \mathfrak{p}^{v_j} R[t_1, \ldots, t_r]$ for all $1 \leq j \leq k$ and $0 \leq w \leq md_j + n$.

Applying our inductive hypothesis to the system of congruences

$$p_{j,w}(\mathbf{a}) \equiv 0 \bmod \mathfrak{p}^{v_j} R[t_1, \ldots, t_r] \qquad (1 \leq j \leq k, 0 \leq w \leq md_j + n),$$

there exists a non-zero common solution to this system where each $a_{l,i}$ is a polynomial in $t_1, \ldots, t_r$ with coefficients in A_l, provided that

$$\sum_{l=1}^{s} (m+1)(|A_l| - 1) > \sum_{j=1}^{k} (md_j + n + 1) d_j^{r+1} (q^{v_j} - 1).$$

Dividing both sides of the above inequality by $m + 1$, we obtain

$$\sum_{l=1}^{s} (|A_l| - 1) > \sum_{j=1}^{k} \frac{md_j + n + 1}{m+1} d_j^{r+1} (q^{v_j} - 1).$$

By (3), the above inequality holds for m sufficiently large. This completes the proof of the theorem. $\square$

3. Inequalities

Let $\mathbb{F}_q((1/t))$ be the completion of $\mathbb{F}_q(t)$ at the infinite place. Every non-zero element α in $\mathbb{F}_q((1/t))$ can be written as $\alpha = \sum_{-\infty < i \leq n} a_i t^i$, where each a_i is an element in $\mathbb{F}_q$ and $a_n \neq 0$. In this case, we define ord α to be n, and we adopt the convention that ord $0 = -\infty$. Note that for a non-zero polynomial x in $\mathbb{F}_q[t]$, we have $\deg x = \text{ord } x$. The argument in the proof of Theorem 3 can be used to prove the existence of solutions to systems of polynomial inequalities in $\mathbb{F}_q((1/t))$, where the solutions have coefficients restricted to subsets of $\mathbb{F}_q$.

Theorem 6. *Let $f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}) \in \mathbb{F}_q((1/t))[x_1, \ldots, x_s]$ be polynomials in $\mathbf{x}$ with coefficients in $\mathbb{F}_q((1/t))$ such that $f_j(\mathbf{0}) = 0$ for $1 \leq j \leq k$. Let $d_j = \deg f_j(\mathbf{x})$ for $1 \leq j \leq k$, and for $1 \leq l \leq s$, let A_l be a subset of $\mathbb{F}_q$ with $\{0\} \subset A_l \subseteq \mathbb{F}_q$. Then, for any integer τ, provided that*

$$\sum_{l=1}^{s}(|A_l| - 1) > (q-1)\sum_{j=1}^{k} d_j^2,$$

there exists a non-zero common solution to the system of inequalities

$$\text{ord } f_j(\mathbf{x}) \leq \tau \qquad (1 \leq j \leq k),$$

where each x_l $(1 \leq l \leq s)$ is a polynomial in t with coefficients in A_l. Furthermore, when n denotes the highest power of t that occurs amongst the coefficients of the polynomials $f_1(\mathbf{x}), \ldots, f_k(\mathbf{x})$, there exists a non-zero solution of this type satisfying

$$\max_{1\leq l\leq s} \deg x_l \leq \max\left\{ \frac{(q-1)\sum\limits_{j=1}^{k}(n-\tau-d_j)d_j}{\sum\limits_{l=1}^{s}(|A_l|-1) - (q-1)\sum\limits_{j=1}^{k} d_j^2}, \frac{\tau-n}{d_1}, \ldots, \frac{\tau-n}{d_k} \right\}.$$

Note that in the case where $A_l = \mathbb{F}_q$ $(1 \leq l \leq s)$, the result matches that of Wooley and the second author [11, Page 728]. Furthermore, this result implies the $r = 1$ case of Theorem 3.

Proof. Let m be a parameter to be chosen later that satisfies $\tau \leq md_j + n$ for all $1 \leq j \leq k$, and for each $1 \leq l \leq s$, write

$$x_l = \sum_{i=0}^{m} a_{l,i} t^i,$$

where $a_{l,i} \in \mathbb{F}_q$. Note that if each $a_{l,i}$ with $0 \leq i \leq m$ is in A_l, then x_l is a polynomial in t with coefficients in A_l.

For each $1 \leq j \leq k$, we can now write

$$f_j(\mathbf{x}) = \sum_{-\infty < w \leq md_j+n} p_{j,w}(\mathbf{a}) t^w,$$

where each $p_{j,w}(\mathbf{a})$ is a polynomial of degree at most d_j in variables $a_{l,i}$ $(1 \le l \le s, 0 \le i \le m)$ with coefficients in $\mathbb{F}_q$. Note that $\mathbf{a} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$. Hence, $p_{j,w}(\mathbf{0}) = 0$ for all $1 \le j \le k$ and $w \le md_j + n$.

The system of inequalities

$$\text{ord } f_j(\mathbf{x}) \le \tau \qquad (1 \le j \le k)$$

is equivalent to the system of equations

$$p_{j,w}(\mathbf{a}) = 0 \qquad (1 \le j \le k, \tau < w \le md_j + n).$$

By Theorem 2, there exists a non-zero common solution with $a_{l,i} \in A_l$ $(1 \le l \le s,\ 0 \le i \le m)$ to this system provided that

$$\sum_{l=1}^{s}(m+1)(|A_l|-1) > (q-1)\sum_{j=1}^{k}(md_j + n - \tau)d_j.$$

Dividing both sides of this inequality by $m+1$, we obtain

$$\sum_{l=1}^{s}(|A_l|-1) > (q-1)\sum_{j=1}^{k}\frac{md_j + n - \tau}{m+1}d_j, \tag{4}$$

and by taking m sufficiently large, we find that it is sufficient to require that

$$\sum_{l=1}^{s}(|A_l|-1) > (q-1)\sum_{j=1}^{k}d_j^2. \tag{5}$$

Whenever (5) holds, the inequality in (4) is satisfied provided that

$$m+1 > \frac{(q-1)\sum_{j=1}^{k}(n-\tau-d_j)d_j}{\sum_{l=1}^{s}(|A_l|-1) - (q-1)\sum_{j=1}^{k}d_j^2}.$$

Therefore, there exists a non-zero solution of the desired type satisfying

$$\max_{1 \le l \le s} \deg x_l \le \frac{(q-1)\sum_{j=1}^{k}(n-\tau-d_j)d_j}{\sum_{l=1}^{s}(|A_l|-1) - (q-1)\sum_{j=1}^{k}d_j^2}. \qquad \square$$

We now generalize the argument of Theorem 6 to the more complicated case of the iterated Laurent series field $\mathbb{F}_q((1/t_1))\cdots((1/t_r))$. For $\alpha \in \mathbb{F}_q((1/t_1))\cdots((1/t_r))$, we will write $\text{ord}_{t_i}\,\alpha$ for the order of α with respect to the indeterminant t_i.

Theorem 7. *Let $f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}) \in \mathbb{F}_q((1/t_1))\cdots((1/t_r))[x_1, \ldots, x_s]$ be polynomials in $\mathbf{x}$ with coefficients in $\mathbb{F}_q((1/t_1))\cdots((1/t_r))$ such that $f_j(\mathbf{0}) = 0$ for $1 \le j \le k$. Let $d_j = \deg f_j(\mathbf{x})$ for $1 \le j \le k$, and for $1 \le l \le s$, let A_l be a subset of $\mathbb{F}_q$ with $\{0\} \subset A_l \subseteq \mathbb{F}_q$. Then, for any integers $\tau_{i,j}$ $(1 \le i \le r,\, 1 \le j \le k)$, provided that*

$$\sum_{l=1}^{s}(|A_l| - 1) > (q-1)\sum_{j=1}^{k} d_j^{r+1},$$

there exists a non-zero common solution to the system of inequalities

$$\operatorname{ord}_{t_i} f_j(\mathbf{x}) \le \tau_{i,j} \qquad (1 \le i \le r,\, 1 \le j \le k),$$

where each x_l $(1 \le l \le s)$ is a polynomial in $\mathbf{t}$ with coefficients in A_l. Furthermore, when $n_{i,j}$ denotes the highest power of t_i that occurs amongst the coefficients of the polynomial $f_j(\mathbf{x})$, there exists a non-zero solution of this type satisfying

$$\operatorname{ord}_{t_i} x_l \le m_i \quad (1 \le l \le s, 1 \le i \le r)$$

provided that

$$\tau_{i,j} < m_i d_j + n_{i,j} \qquad (1 \le i \le r,\, 1 \le j \le k)$$

and

$$\sum_{l=1}^{s}(|A_l| - 1) > (q-1)\sum_{j=1}^{k}\left(d_j \prod_{i=1}^{r} \frac{m_i d_j + n_{i,j} - \tau_{i,j}}{m_i + 1}\right).$$

Note that this theorem implies Theorem 3. Unlike other proofs in this paper, we will not use induction in this proof but instead directly apply Theorem 2 to a larger system of equations.

Proof. For each $1 \le i \le r$, let m_i be a parameter to be chosen later that satisfies $\tau_{i,j} < m_i d_j + n_{i,j}$ for all $1 \le j \le k$. For each $1 \le l \le s$, write

$$x_l = \sum_{\substack{\mathbf{u} \\ 0 \le u_i \le m_i \ (1 \le i \le r)}} a_{l,\mathbf{u}} t_1^{u_1} \cdots t_r^{u_r},$$

where $a_{l,\mathbf{u}} \in \mathbb{F}_q$. Note that if each $a_{l,\mathbf{u}}$ is in A_l, then x_l is a polynomial in $\mathbf{t}$ with coefficients in A_l.

For each $1 \le j \le k$, we can now write

$$f_j(\mathbf{x}) = \sum_{\substack{\mathbf{w} \\ -\infty < w_i \le m_i d_j + n_{i,j} \ (1 \le i \le r)}} p_{j,\mathbf{w}}(\mathbf{a}) t_1^{w_1} \cdots t_r^{w_r},$$

where each $p_{j,\mathbf{w}}(\mathbf{a})$ is a polynomial of degree at most d_j in variables $a_{l,\mathbf{u}}$ $(1 \le l \le s,\, 0 \le u_1 \le m_1, \ldots, 0 \le u_r \le m_r)$ with coefficients in $\mathbb{F}_q$. Note that $\mathbf{a} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$. Hence, $p_{j,\mathbf{w}}(\mathbf{0}) = 0$ for all values of j and $\mathbf{w}$.

The system of inequalities

$$\mathrm{ord}_{t_i} f_j(\mathbf{x}) \le \tau_{i,j} \qquad (1 \le i \le r,\ 1 \le j \le k),$$

is equivalent to the system of equations $p_{j,\mathbf{w}}(\mathbf{a}) = 0$ where j and $\mathbf{w}$ satisfy $1 \le j \le k$ and $\tau_{i,j} < w_i \le m_i d_j + n_{i,j}$ for $1 \le i \le r$. By Theorem 2, there exists a non-zero common solution with $a_{l,\mathbf{u}} \in A_l$ $(1 \le l \le s,\ 0 \le u_1 \le m_1, \ldots, 0 \le u_r \le m_r)$ to this system provided that

$$\sum_{l=1}^{s} \left((|A_l| - 1) \prod_{i=1}^{r} (m_i + 1) \right) > (q-1) \sum_{j=1}^{k} \left(d_j \prod_{i=1}^{r} (m_i d_j + n_{i,j} - \tau_{i,j}) \right),$$

which is equivalent to

$$\sum_{l=1}^{s} (|A_l| - 1) > (q-1) \sum_{j=1}^{k} \left(d_j \prod_{i=1}^{r} \frac{m_i d_j + n_{i,j} - \tau_{i,j}}{m_i + 1} \right). \qquad \square$$

In the case when all $n_{i,j} = n$, $\tau_{i,j} = \tau$, and $d_j = d$ in the above theorem, we obtain the following corollary.

Corollary 3. *Let $f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}) \in \mathbb{F}_q((1/t_1)) \cdots ((1/t_r))[x_1, \ldots, x_s]$ be polynomials in $\mathbf{x}$ of degree d with coefficients in $\mathbb{F}_q((1/t_1)) \cdots ((1/t_r))$ such that $f_j(\mathbf{0}) = 0$ for $1 \le j \le k$. For $1 \le l \le s$, let A_l be a subset of $\mathbb{F}_q$ with $\{0\} \subset A_l \subseteq \mathbb{F}_q$. Then, for any integer τ, provided that*

$$\sum_{l=1}^{s} (|A_l| - 1) > (q-1) k d^{r+1},$$

there exists a non-zero common solution to the system of inequalities

$$\mathrm{ord}_{t_i} f_j(\mathbf{x}) \le \tau \qquad (1 \le i \le r,\ 1 \le j \le k),$$

where each x_l $(1 \le l \le s)$ is a polynomial in $\mathbf{t}$ with coefficients in A_l. Furthermore, when n denotes the highest power of any t_i that occurs amongst the coefficients of the polynomials $f_j(\mathbf{x})$, there exists a non-zero solution of this type satisfying

$$\mathrm{ord}_{t_i} x_l \le \max \left\{ \frac{\tau + 1 - n}{d}, \frac{n - \tau - d}{\left(\frac{\sum_{l=1}^{s} (|A_l| - 1)}{(q-1)kd} \right)^{1/r} - d} \right\} \quad (1 \le l \le s, 1 \le i \le r).$$

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