AVOIDING ZERO-SUM SEQUENCES OF PRESCRIBED LENGTH OVER THE INTEGERS[1]

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Abstract

Let t and k be positive integers, and let $I_k = \{i \in \mathbb{Z} :\ -k \leq i \leq k\}$. Let $\mathsf{s}'_t(I_k)$ be the smallest positive integer ℓ such that every zero-sum sequence over I_k with at least ℓ elements contains a zero-sum subsequence with exactly t elements. If no such ℓ exists, then let $\mathsf{s}'_t(I_k) = \infty$. We prove that $\mathsf{s}'_t(I_k)$ is finite if and only if every integer in $[1, D(I_k)]$ divides t, where $D(I_k) = \max\{2, 2k-1\}$ is the Davenport constant of I_k. Moreover, we prove that if $\mathsf{s}'_t(I_k)$ is finite, then $t + k(k-1) \leq \mathsf{s}'_t(I_k) \leq t + (2k-2)(2k-3)$. We also show that $\mathsf{s}'_t(I_k) = t + k(k-1)$ holds for $k \leq 3$ and conjecture that this equality holds for $k \geq 1$.

1. Introduction and Main Results

We shall follow the notation in [16], by Grynkiewicz. Let $\mathbb{N}$ be the set of positive integers. Let G_0 be a subset of an abelian group G. A sequence over G_0 is an

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unordered list of terms in G_0, where repetition is allowed. The set of all sequences over G_0 is denoted by $\mathcal{F}(G_0)$. A sequence with no term is called *trivial* or *empty*. If S is a sequence with terms s_i, $i \in [1, n]$, we write $S = s_1 \cdot \ldots \cdot s_n = \prod_{i=1}^{n} s_i$. We say that R is a *subsequence* of S if any term in R is also in S. If R and T are subsequences of S such that $S = R \cdot T$, then R is the *complementary* sequence of T in S, and vice versa. We also write $T = S \cdot R^{-1}$ and $R = S \cdot T^{-1}$. For every sequence $S = s_1 \cdot \ldots \cdot s_n$ over G_0,

- the *opposite sequence of* S is $-S = (-s_1) \cdot \ldots \cdot (-s_n)$;
- the *length* of S is $|S| = n$;
- the *sum* of S is $\sigma(S) = s_1 + \ldots + s_n$;
- the *subsequence-sum* of S is $\Sigma(S) = \{\sigma(R) : \ R \text{ is a subsequence of } S\}$.

For any sequence R over G_0 and any integer $d \geq 0$,

$$R^{[0]} \text{ is the trivial sequence, and } R^{[d]} = \underbrace{R \cdot \ldots \cdot R}_{d} \text{ for } d > 0.$$

A sequence with sum 0 is called *zero-sum*. The set of all zero-sum sequences over G_0 is denoted by $\mathcal{B}(G_0)$. A zero-sum sequence is called *minimal* if it does not contain a proper zero-sum subsequence. The *Davenport constant* of G_0, denoted by $D(G_0)$, is the maximum length of a minimal zero-sum sequence over G_0. The research on zero-sum theory is quite extensive when G is a finite abelian group (e.g., see [5, 8, 10, 11] and the references therein). However, there is less activity when G is infinite (e.g., see [3, 6] and the references therein). The study of the case $G = \mathbb{Z}^r$ was explicitly suggested by Baeth and Geroldinger [1] due to their relevance to direct-sum decompositions of modules. Baeth, Geroldinger, Grynkiewicz, and Smertnig [2] studied the Davenport constant of $G_0 \subseteq \mathbb{Z}^r$. The Davenport constant of an interval in $\mathbb{Z}$ was first determined (see Theorem 1) by Lambert [17] (also see [7, 20, 21] for related work.) Plagne and Tringali [18] considered the Davenport constant of the Cartesian product of intervals in $\mathbb{Z}$.

For $x, y \in \mathbb{Z}$ with $x \leq y$, let $[x, y] = \{i \in \mathbb{Z} : \ x \leq i \leq y\}$. For $k \in \mathbb{N}$, let $I_k = [-k, k]$.

Theorem 1 (Lambert [17]). *If* $k \in \mathbb{N}$*, then* $D(I_k) = \max\{2, 2k - 1\}$.

For G finite and $G_0 \subseteq G$, let $\mathsf{s}_t(G_0)$ be the smallest integer $\ell \in \mathbb{N}$ such that any sequence in $\mathcal{F}(G_0)$ of length at least ℓ contains a zero-sum subsequence of length t. If $t = \exp(G)$, then $\mathsf{s}_t(G_0)$ is called the *Erdös–Ginzburg–Ziv constant*, and it is denoted by $\mathsf{s}(G)$. Erdös, Ginzburg, and Ziv [8] proved that $\mathsf{s}(\mathbb{Z}_n) = 2n - 1$. Reiher [19] proved that $\mathsf{s}(\mathbb{Z}_p \oplus \mathbb{Z}_p) = 4p - 3$ for any prime p. In general, if G has rank 2, say $G = Z_{n_1} \oplus Z_{n_2}$ with $n_2 \geq n_1 \geq 1$ and $n_1 \mid n_2$, then $\mathsf{s}(G) = 2n_1 + 2n_2 - 3$

(see [14, Theorem 5.8.3]). For groups of higher rank, we refer the reader to the paper of Fan, Gao, and Zhong [9]. Recently, Gao, Han, Peng, and Sun [13] proved that for any integer $k \geq 2$ and any finite G with exponent $n = \exp(G)$, if $n - |G|/n$ is large enough, then $\mathsf{s}_{kn}(G) = kn + D(G) - 1$.

Observe that if G is torsion-free and $G_0 \subseteq G$, then for any nonzero element $g \in G_0$ and for any $d \in \mathbb{N}$, the sequence $g^{[d]} \in \mathcal{F}(G_0)$ does not contain a zero-sum subsequence. Thus, we will work with the following analogue of $\mathsf{s}_t(G_0)$.

Definition 1. [3] For any subset $G_0 \subseteq G$, let $\mathsf{s}'_t(G_0)$ be the smallest integer $\ell \in \mathbb{N}$ such that any sequence in $\mathcal{B}(G_0)$ of length at least ℓ contains a zero-sum subsequence of length t. If no such ℓ exists, then let $\mathsf{s}'_t(G_0) = \infty$.

If $t = \exp(G)$ is finite, then we denote $\mathsf{s}_t(G_0)$ by $\mathsf{s}(G)$. Let $r \in \mathbb{N}$ and assume that $G \cong \mathbb{Z}_n^r$. We say that G has *Property D* if, for every sequence $S \in \mathcal{F}(G)$ of length $\mathsf{s}(G) - 1$ that does not admit a zero-sum subsequence of length n, there exists some sequence $T \in \mathcal{F}(G)$ such that $S = T^{[n-1]}$. Zhong found the following interesting connections between $\mathsf{s}(G)$ and $\mathsf{s}'(G)$. (See the Appendix for their proofs.)

Lemma 1 (Zhong [22]). *Let G be a finite abelian group.*
(i) If $\gcd(\mathsf{s}(G) - 1, \exp(G)) = 1$, then $\mathsf{s}'(G) = \mathsf{s}(G)$.
(ii) Let $G \cong \mathbb{Z}_n^r$, where $n \geq 3$ and $r \geq 2$. Suppose that $c \in \mathbb{N}$, $\mathsf{s}(G) = c(n-1) + 1$, and G has Property D. If $\gcd(\mathsf{s}(G) - 1, n) = c$, then $\mathsf{s}'(G) < \mathsf{s}(G)$.

Remark 1 (Zhong [22]).
(i) If $G \cong \mathbb{Z}_n^2$ with n odd, then $\mathsf{s}'(G) = \mathsf{s}(G)$.
(ii) If $G \cong \mathbb{Z}_{2^h}^2$ with $h \geq 2$, then $\mathsf{s}'(G) = \mathsf{s}(G) - 1$.

In this paper, we prove the following results about $\mathsf{s}'_t(I_k)$, where $I_k = [-k, k]$.

Theorem 2. *Let $k, t \in \mathbb{N}$.*
(i) $\mathsf{s}'_t(I_k)$ is finite, then every integer in $[1, D(I_k)]$ divides t.
(ii) If every integer in $[1, D(I_k)]$ divides t, then

$$t + k(k-1) \leq \mathsf{s}'_t(I_k) \leq t + (2k-2)(2k-3).$$

Corollary 1. *Let $t \in \mathbb{N}$ and $k \in \{1, 2, 3\}$. Then $\mathsf{s}'_t(I_k) = t + k(k-1)$ if and only if every integer in $[1, D(I_k)]$ divides t.*

Conjecture 1. Corollary 1 holds for any $k \in \mathbb{N}$.

2. Proofs of the Main Results

For the rest of this paper, we assume that $k, t \in \mathbb{N}$. For any integers a and b, we denote $\gcd(a, b)$ by (a, b). We use the abbreviations z.s.s and z.s.s$_b$ for *zero-sum sequence(s)* and *zero-sum subsequence(s)*, respectively.

[3] This formulation was suggested to us by Geroldinger and Zhong [15].

The following lemma gives a lower bound for $\mathsf{s}'_t(I_k)$.

Lemma 2. *If* $U = k \cdot (-1)^{[k]}$ *and* $V = (k-1) \cdot (-1)^{[k-1]}$*, then* $S = U^{[\frac{t}{k+1}-1]} \cdot V^{[k]}$ *and* $R = U^{[k-1]} \cdot V^{[\frac{t}{k}-1]}$ *are z.s.s that do not contain a z.s.s*$_\flat$ *of length* t*. Thus,* $\mathsf{s}'_t(I_k) \geq t + k(k-1)$.

Proof. We prove the lemma for S only since the proof for R is similar. By contradiction, assume that S contains a z.s.s$_\flat$ of length t. Since $\sigma(S) = 0$, it follows that S also contains a z.s.s$_\flat$ S' of length $|S| - t = k(k-1) - 1$. Moreover, S' can be written as $S' = k^{[a]} \cdot (k-1)^{[b]} \cdot (-1)^{[c]}$ for some nonnegative integers a, b, and c. Hence $\sigma(S') = ak + b(k-1) - c = 0$ and $a + b + c = |S'| = k^2 - k - 1$. Thus,

$$(a+1)(k+1) = k(k-b).$$

Since $a, b, k \geq 0$, we have $0 < k - b \leq k$. Since $(k, k+1) = 1$, we obtain that $k+1$ divides $k - b$, which is a contradiction. Thus, $\mathsf{s}'_t(I_k) \geq |S| + 1 = t + k(k-1)$. $\square$

Example 1. If $k = 3$, then $S = (3 \cdot -1 \cdot -1 \cdot -1)^{[14]} \cdot (2 \cdot -1 \cdot -1)^{[3]}$ is a z.s.s of length 65 over $[-3, 3]$ which does not contain a z.s.s$_\flat$ of length $t = 60$.

Lemma 3. *Let* $a, b, x \in \mathbb{N}$*. If* $S = a^{[\frac{b}{(a,b)}]} \cdot (-b)^{[\frac{a}{(a,b)}]}$ *is a z.s.s, then the length of any z.s.s*$_\flat$ *of* $S^{[x]}$ *is a multiple of* $|S|$.

Proof. Let S' be a z.s.s$_\flat$ of $S^{[x]}$. Since the terms of S are a and $-b$, there exist nonnegative integers h and r such that $S' = a^h \cdot (-b)^r$ and

$$\sigma(S') = ha - rb = 0 \Rightarrow h\frac{a}{(a,b)} = r\frac{b}{(a,b)}. \qquad (1)$$

Since $\left(\frac{b}{(a,b)}, \frac{b}{(a,b)}\right) = 1$, we obtain $\frac{b}{(a,b)}$ divides h and $\frac{a}{(a,b)}$ divides r. Thus, $h = p\frac{b}{(a,b)}$ and $r = q\frac{a}{(a,b)}$ for some integers p and q. Substituting h and r back into (1) yields $p = q$. Thus,

$$|S'| = h + r = p\frac{b}{(a,b)} + q\frac{a}{(a,b)} = p|S|. \qquad \square$$

Lemma 4. *If* $\mathsf{s}'_t(I_k)$ *is finite, then every odd integer in* $[1, D(I_k)]$ *divides* t.

Proof. The lemma is trivial for $k = 1$. If $k \geq 2$, then Theorem 1 implies that $D(I_k) = 2k - 1$. Let $\ell = 2c - 1$ be an odd integer in $[3, D(I_k)]$, and consider the minimal z.s.s $S = c^{[c-1]} \cdot (-c+1)^{[c]}$. If $x \in \mathbb{N}$, then Lemma 3 implies that for any z.s.s$_\flat$ R of $S^{[x]}$, $|R|$ divides $|S| = 2c - 1 = \ell$. Thus, if $\ell \nmid t$, then there is no z.s.s$_\flat$ of $S^{[x]}$ whose length is equal to t. Since x is arbitrary, it follows that $\mathsf{s}'_t(I_k)$ can be arbitrarily large. This proves the lemma by contrapositive. $\square$

To prove the upper bound in Theorem 2(ii), we will use Lemma 5 which is a direct application of a well-known fact: "Any sequence of n integers contains a nonempty subsequence whose sum is divisible by n".

Lemma 5. *Let $\beta \in \mathbb{N}$ and $X \in \mathcal{F}(\mathbb{Z})$. If $|X| \geq \beta$, then there exists a factorization $X = X_0 \cdot X_1 \cdot \ldots \cdot X_r$ such that:*

(i) $|X_0| \leq \beta - 1$ and $\beta \nmid \sigma(R)$ for any nonempty subsequence R of X_0;

(ii) $|X_j| \leq \beta$ and $\beta \mid \sigma(X_j)$ for all $j \in [1, r]$.

We will also use the following lemmas.

Lemma 6. *Assume that $k \geq 2$ and that every integer in $[1, D(I_k)]$ divides t. Let S be a z.s.s over $I_k = [-k, k]$ that does not contain a z.s.s$_b$ of length t. Let $S = S_1 \cdot \ldots \cdot S_h$ be a factorization into minimal z.s.s$_b$ S_i, $i \in [1, h]$. If $|S| \geq t + k(k-1)$, then there exists some length β such that:*

$$n_\beta = |\{S_i : \ |S_i| = \beta, \ i \in [1, h]\}| > (2k-2)(2k-3).$$

Proof. Recall that (a, b) denotes $\gcd(a, b)$. It is easy to see that

$$(2k-3, 2k-2) = (2k-2, 2k-1) = (2k-3, 2k-1) = 1. \tag{2}$$

Since $k \geq 2$ and every integer in $[1, D(I_k)] = [1, 2k-1]$ is a factor of t, it follows from (2) that $t = p(2k-1)(2k-2)(2k-3)$, for some $p \in \mathbb{N}$. By definition, we have $\max_{i \in [1,h]} |S_i| \leq D(I_k) = 2k-1$. Thus, it follows from the pigeonhole principle that there exists some length β such that:

$$n_\beta \geq \frac{t + k(k-1)}{\max_{i \in [1,h]} |S_i|} \geq \frac{t + k(k-1)}{2k-1} > p(2k-2)(2k-3). \qquad \square$$

Lemma 7. *Assume that $k \geq 2$ and that every integer in $[1, D(I_k)]$ divides t. Let S be a z.s.s over $I_k = [-k, k]$ of length $|S| \geq t + k(k-1)$ such that S does not contain a z.s.s$_b$ of length t. Let $S = S_1 \cdot \ldots \cdot S_h$ be a factorization into minimal z.s.s$_b$ S_i, $i \in [1, h]$. Let $L = \{|S_i| : \ i \in [1, h]\}$, $n_\ell = |\{S_i : \ |S_i| = \ell, \ i \in [1, h]\}|$, and $\alpha = \max_{\ell \in L} \ell$. If there exists $\beta \in L$ such that $n_\beta \geq \alpha - 1$, then*

$$|S| \leq t - \beta + (\beta - 1) \max_{\ell \in L \setminus \{\beta\}} \ell.$$

Remark 2. By Lemma 6, there exists $\beta \in L$ such that $n_\beta > (2k-2)(2k-3)$. Moreover, $\alpha = \max_{\ell \in L} \ell \leq D(I_k) \leq (2k-2)(2k-3) + 1$ for $k \geq 2$. Thus, $n_\beta \geq \alpha$, i.e., the hypothesis of Lemma 7 always holds.

Proof of Lemma 6. By hypothesis, there exists $\beta \in L$ such that $n_\beta \geq \alpha - 1$. Given a factorization $S = S_1 \cdot \ldots \cdot S_h$ into minimal z.s.s$_\flat$ S_i, $i \in [1, h]$, consider the following sequence of lengths in $L \setminus \{\beta\}$:

$$X = \prod_{i=1,\, |S_i| \neq \beta}^{h} |S_i| = \prod_{\ell \in L \setminus \{\beta\}} \ell^{[n_\ell]}.$$

By Lemma 5, there exists a factorization $X = X_0 \cdot X_1 \ldots X_r$ such that:

$$|X_0| \leq \beta - 1 \text{ and } \beta \nmid \sigma(R) \text{ for any nonempty subsequence } R \text{ of } X_0; \tag{3}$$

$$|X_j| \leq \beta \text{ and } \beta \mid \sigma(X_j) \text{ for all } j \in [1, r]. \tag{4}$$

Thus,

$$\sigma(X_j) = \sum_{x \in X_j} x \leq |X_j| \cdot \max_{x \in X_j} x \leq \beta\alpha \text{ for all } j \in [1, r]. \tag{5}$$

To summarize, it follows from the hypothesis of the lemma, (4), and (5) that

$$\beta \mid t,\ n_\beta \geq \alpha - 1,\ \beta \mid \sigma(X_j), \text{ and } \sigma(X_j) \leq \alpha\beta \text{ for all } j \in [1, r].$$

Thus, if

$$\beta n_\beta + \sum_{j=1}^{r} \sigma(X_j) \geq t,$$

then there exists a nonnegative integer $n'_\beta \leq n_\beta$ and a subset $Q \subseteq [1, r]$ such that

$$\beta n'_\beta + \sum_{q \in Q} \sigma(X_q) = t.$$

Then S would contain a z.s.s$_\flat$ of length t obtained by concatenating n'_β z.s.s$_\flat$ of S of length β and all the z.s.s$_\flat$ of S whose lengths form the subsequence $\prod_{q \in Q}^{h} X_q$ of X. This contradicts the hypothesis of the theorem. Thus, the following inequality holds:

$$\beta n_\beta + \sum_{j=1}^{r} \sigma(X_j) < t. \tag{6}$$

Since β divides both t and $\sum_{j=1}^{r} \sigma(X_j)$, it follows from (6) that

$$\beta n_\beta + \sum_{j=1}^{r} \sigma(X_j) \leq t - \beta. \tag{7}$$

Thus, it follows from (7) and the definitions of X and X_j $(0 \leq j \leq r)$ that

$$|S| = \sum_{\ell \in L} \ell n_\ell = \beta n_\beta + \sigma(X) = \beta n_\beta + \sum_{j=1}^{r} \sigma(X_j) + \sigma(X_0) \leq t - \beta + \sigma(X_0). \tag{8}$$

Next, it follows from (3) and (8) that

$$|S| \le t - \beta + \sigma(X_0) \le t - \beta + |X_0| \max_{\ell \in L \setminus \{\beta\}} \ell \le t - \beta + (\beta - 1) \max_{\ell \in L \setminus \{\beta\}} \ell. \qquad \square$$

Proof of Theorem 2. We first prove part (i). Suppose that $\mathsf{s}'_t(I_k)$ is finite. Then it follows from Lemma 4 that every odd integer in $[1, D(I_k)]$ divides t. Thus, it remains to show that if a is an even integer in $[1, D(I_k)]$, then $a \mid t$.

Case 1: $a = 2^e$ for some $e \in \mathbb{N}$.

Lemma 3 implies that for any $p \in \mathbb{N}$, the sequence $S = (1 \cdot -1)^{[p]}$ is a z.s.s whose z.s.s♭ have lengths that are divisible by 2. Therefore, if $2 \nmid t$, then $\mathsf{s}'_t(I_k) \ge |S| = 2p$, where p can be chosen to be arbitrarily large. Thus, $2 \mid t$ if $\mathsf{s}'_t(I_k)$ is finite.

Now assume that $e > 1$. Since the gcd of two numbers divides their difference, $(a/2-1, a/2+1) \le 2$. Since $a/2-1$ and $a/2+1$ are both odd, $(a/2-1, a/2+1) = 1$. Lemma 4 implies that for any $p \in \mathbb{N}$, the sequence $S^{[p]}$ with $S = (a/2-1)^{[a/2+1]} \cdot (-a/2-1)^{[a/2-1]}$ is a z.s.s whose z.s.s♭ have lengths that are divisible by $|S| = a$. Thus, if $a \nmid t$, we can construct arbitrarily long z.s.s over $I_k = [-k, k]$ that do not contain z.s.s♭ of length t, because p can be chosen to be arbitrarily large. Thus, $a \mid t$ if $\mathsf{s}'_t(I_k)$ is finite.

Case 2: a is not a power of 2.

Thus, $a = 2^e j$, where e and j are nonnegative integers and j is odd. By Lemma 4, $j \mid t$, and if follows from Case 1 that $2^e \mid t$. Since j is odd, $(2^e, j) = 1$. Since 2^e and j are factors of t, it follows that $2^e j \mid t$.

The above cases and Lemma 4 imply that every integer in $[1, D(I_k)]$ divides t.

Since the lower bound of $\mathsf{s}'_t(I_t)$ in Theorem 2(ii) follows from Lemma 2, it remains to prove its upper bound. Recall that every integer in $[1, D(I_k)]$ divides t. Let S be an arbitrary z.s.s over $I_k = [-k, k]$ that does not contain a z.s.s♭ of length t.

If $k = 1$, then it follows from Theorem 1 that $D(I_k) = 2$. Thus, $2 \mid t$ and $|S| = x_1 + 2x_2$ for some nonnegative integers x_1 and x_2. If $|S| \ge t$, then $x_1 \ge 2$ or $x_2 \ge t/2$ (because $2 \mid t$). This implies that there exist nonnegative integers $x'_1 \le x_1$ and $x'_2 \le x_2$ such that $x'_1 + 2x'_2 = t$. Thus, $S' = 0^{[x'_1]} \cdot (1 \cdot -1)^{[x'_2]}$ is a z.s.s♭ of S of length t, which is a contradiction since S does not contain a z.s.s♭ of length t. Hence $|S| \le t - 1$, and $\mathsf{s}'_t(I_k) \le |S| + 1 = t$.

Now assume $k \ge 2$. Since S was arbitrarily chosen, if $|S| \le t + k(k-1) - 1$, then

$$\mathsf{s}'_t(I_k) \le |S| + 1 \le t + k(k-1) \le t + (2k-2)(2k-3),$$

which yields the upper bound in Theorem 2(ii). Thus, we may assume that $|S| \ge t + k(k-1)$. Let $S = S_1 \cdot \ldots \cdot S_h$ be a factorization into minimal z.s.s♭ S_i, $i \in [1, h]$. Let $L = \{|S_i| : \ i \in [1, h]\}$, $n_\ell = |\{S_i : \ |S_i| = \ell, \ i \in [1, h]\}|$, and $\alpha = \max_{\ell \in L} \ell$. If $\alpha = 1$, then any term of S is equal to 0, which is a contradiction since $|S| \ge t + k(k-1)$ and S does not contain a z.s.s♭ of length t. So, we may assume that $\alpha \ge 2$. Then Remark 2 implies that there exists $\beta \in L$ such that $n_\beta \ge \alpha - 1$. If $\beta = \alpha$, then

Lemma 7 yields

$$|S| \le t - \alpha + (\alpha - 1) \max_{\ell \in L \setminus \{\alpha\}} \ell \le t - \alpha + (\alpha - 1)^2.$$

If $1 \le \beta \le \alpha - 1$, then Lemma 7 also yields

$$\begin{aligned} |S| &\le t + \max_{1 \le \beta \le \alpha - 1} \left(-\beta + (\beta - 1) \max_{\ell \in L \setminus \{\beta\}} \ell \right) \\ &\le t + \max_{1 \le \beta \le \alpha - 1} (-\beta + (\beta - 1)\alpha) \\ &= t + (-(\alpha - 1) + (\alpha - 2)\alpha) \\ &= t - \alpha + (\alpha - 1)^2. \end{aligned}$$

So in all cases, we obtain

$$|S| \le t - \alpha + (\alpha - 1)^2 \le t - (2k - 1) + (2k - 2)^2, \tag{9}$$

where we used the fact $\alpha \le D(I_k) = 2k - 1$. Since S was chosen to be an arbitrary z.s.s over $I_k = [-k, k]$ which does not contain a z.s.s$_\flat$ of length t,

$$\mathsf{s}'_t(I_k) \le |S| + 1 \le t - (2k - 1) + (2k - 2)^2 + 1 = t + (2k - 2)(2k - 3). \qquad \square$$

Proof of Corollary 1. For $k \in \{1, 2\}$, the corollary holds since the upper and lower bounds of $\mathsf{s}'_t(I_k)$ given by Theorem 2 are both equal to $t + k(k - 1)$.

For $k = 3$, it also follows from Theorem 2 that $t + 6 \le \mathsf{s}'_t(I_3) \le t + 12$. Thus, it remains to show that if S is an arbitrary z.s.s over I_3 which does not contain a z.s.s$_\flat$ of length t, then $|S| \ne t + d$ for all $d \in [6, 11]$.

Consider a factorization $S = S_1 \cdot \ldots \cdot S_h$ into minimal z.s.s$_\flat$ S_i, $i \in [1, h]$. Let $L = \{|S_i| :\ i \in [1, h]\}$, $n_\ell = |\{S_i :\ |S_i| = \ell,\ i \in [1, h]\}|$, and $\alpha = \max_{\ell \in L} \ell$. Thus, $\alpha \le D(I_3) = 5$. If $\alpha \le 4$, then Lemma 7 yields

$$|S| \le t + \max_{1 \le \alpha \le 4} \left((\alpha - 1)^2 - \alpha \right) = t + (4 - 1)^2 - 4 = t + 5.$$

Thus, we may assume that $\alpha = \max\limits_{\ell \in L} \ell = 5$ for any factorization of S.

If $\beta \in \{1, 2\}$ and $n_\beta \ge 4$, then Lemma 7 yields

$$|S| \le t + \max_{\beta \in \{1,2\}} ((\beta - 1)\alpha - \beta) = t + (2 - 1)5 - 2 = t + 3.$$

Next, suppose that R is a z.s.s$_\flat$ of S with length at least 4. Then $R \cdot -R$ can be trivially factorized into $|W|$ z.s.s$_\flat$ of length 2, where $|W| \ge 4$. This yields a new factorization $S = S'_1 \cdot \ldots \cdot S'_h$ with $n_2 \ge 4$, which implies that $|S| < t + 5$ by the above analysis upon setting $\beta = 2$.

Also note that if $n_\ell \ge t/\ell$ for some $\ell \in L$, then we obtain a z.s.s$_\flat$ of S of length t by concatenating t/ℓ z.s.s$_\flat$ of length ℓ in S. This would contradict the definition of S. Thus, we can assume that $n_\ell \le t/\ell - 1$ for all $\ell \in L$, where $L \subseteq [1, 5]$.

To recapitulate, we may assume that for any factorization $S = S_1 \cdot \ldots \cdot S_h$, with $S_L = \prod_{i=1}^{h} |S_i|$ and $n_\ell = |\{S_i : \ |S_i| = \ell, \ i \in [1,h]\}|$, the following properties hold:

(i) $S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 3^{[n_3]} \cdot 2^{[n_2]} \cdot 1^{[n_1]}$, where $0 \le n_\ell \le t/\ell - 1$ for $\ell \in [1,5]$, $n_5 \ge 1$, and $n_1, n_2 \le 3$;

(ii) there is a one-to-one correspondence between the subsequences S'_L of S_L and the z.s.s♭ S' of S with length $|S'| = \sigma(S'_L)$;

(iii) if R is z.s.s over I_3 such that $|R| \ge 4$, then R and $-R$ cannot both be subsequences of S;

(iv) if R is a minimal z.s.s♭ of S such that $|R| = 5$, then $R = 3^{[2]} \cdot (-2)^{[3]}$. (This follows from (iii) and the fact $A = 3^{[2]} \cdot (-2)^{[3]}$ and $-A$ are the only minimal z.s.s of length 5 over $I_3 = [-3,3]$. Thus, if $-A$ is the only z.s.s♭ of S, then we can analyze $-S$ instead of S.)

Claim 1: *If $5 \cdot 3^{[4]}$ is a subsequence of S_L, then $|S| \ne t + d$ for all $d \in [6,11]$.*

If $n_4 + n_2 + n_1 \ge 1$, then either $5 \cdot 4 \cdot 3^{[4]}$, or $5 \cdot 3^{[4]} \cdot 2$, or $5 \cdot 3^{[4]} \cdot 1$ is a subsequence of S_L, which implies that $\Sigma(S_L)$ contains all the integers in $[6,11]$. In this case, $|S| \ne t + d$ for $d \in [6,11]$, since S does not contain a z.s.s♭ of length t by hypothesis. Thus, we may assume that $n_4 = n_2 = n_1 = 0$, which implies that $S_L = 5^{[n_5]} \cdot 3^{[n_3]}$. If $n_5 \le 1$, then $|S| = \sigma(S_L) = 5n_5 + 3n_3 \le 5 + 3(t/3 - 1) < t + 5$. Thus, we may assume that $S_L = 5^{[n_5]} \cdot 3^{[n_3]}$, where $n_5 \ge 2$ and $n_3 \ge 4$.

Then $\Sigma(S_L)$ contain all the integers in $[6,11] \setminus \{7\}$; and so $|S| \ne t + d$ for $d \in [6,11] \setminus \{7\}$. It remains to show that $|S| \ne t + 7$. Note that the only minimal z.s.s of length 3 over $[-3,3]$ are (up to sign) $B_1 = 2 \cdot (-1)^{[2]}$ and $B_2 = 3 \cdot -2 \cdot -1$. Since $5 \cdot 3^{[4]}$ is a subsequence of S_L, the assumptions (i)–(iv) imply that $S' = A \cdot X \cdot Y \cdot Z \cdot W$ is a subsequence of S, where $A = 3^{[2]} \cdot (-2)^{[3]}$ and $X, Y, Z, W \in \{-B_1, B_1, -B_2, B_2\}$. By inspecting the sequence S' for all possible choices of X, Y, Z, and W, we see that S' admits a z.s.s♭ of length 7. For instance, if $X = Y = Z = B_2$, then

$$S' = A \cdot B_2^{[3]} \cdot W = A^{[2]} \cdot 3 \cdot (-1)^{[3]} \cdot W$$

contains the subsequence $3 \cdot (-1)^{[3]} \cdot W$, which is a z.s.s♭ of length $4 + |W| = 7$. Hence, $|S| \ne t + 7$. Thus, $|S| \ne t + d$ for all $d \in [6,11]$.

Claim 2: *If $5 \cdot 4^{[2]} \cdot 3$ is a subsequence of S_L, then $|S| \ne t + d$ for all $d \in [6,11]$.*

If $n_3 \ge 2$ or $n_1 + n_2 \ge 1$, then either $5 \cdot 4^{[2]} \cdot 3^{[2]}$, or $5 \cdot 4^{[2]} \cdot 3 \cdot 2$, or $5 \cdot 4^{[2]} \cdot 3 \cdot 1$ is a subsequence of S_L, which implies that $\Sigma(S_L)$ contains all the integers in $[6,11]$. In this case, $|S| \ne t + d$ for $d \in [6,11]$, since S does not contain a z.s.s♭ of length t by hypothesis. Thus, we may assume that $n_3 = 1$ and $n_1 = n_2 = 0$, which implies that $S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 3$. If $n_5 \le 1$, then

$$|S| = \sigma(S_L) = 5n_5 + 4n_4 + 3 \le 5 + 4(t/4 - 1) + 3 < t + 5.$$

Thus, we may assume that $S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 3$, where $n_5 \geq 2$ and $n_4 \geq 2$.

Thus, $5^{[2]} \cdot 4^{[2]} \cdot 3$ is a subsequence of S_L, which implies that $\Sigma(S_L)$ contain all the integers in $[7, 11]$. Thus, $|S| \neq t + d$ for $d \in [7, 11]$. It remains to show that $|S| \neq t + 6$. Note that the only minimal z.s.s of length 4 over $[-3, 3]$ are (up to sign) $C_1 = 3 \cdot (-1)^{[3]}$ and $C_2 = 3 \cdot 1 \cdot (-2)^{[2]}$. Since $5 \cdot 4^{[2]} \cdot 3$ is a subsequence of S_L, the assumptions (i)–(iv) imply that $S' = A \cdot X \cdot Y \cdot Z$ is a subsequence of S, where $A = 3^{[2]} \cdot (-2)^{[3]}$, $X, Y \in \{-C_1, C_1, -C_2, C_2\}$, and $Z \in \{-B_1, B_1, -B_2, B_2\}$. By inspecting the sequence S' for all possible choices of X, Y, and Z, we see that S' admits a z.s.s$_\flat$ of length 6. For instance, if $X = C_1$ and $Y = C_2$, then

$$S' = A \cdot C_1 \cdot C_2 \cdot Z = A \cdot (3 \cdot -1 \cdot -2)^{[2]} \cdot (1 \cdot -1) \cdot Z$$

contains the subsequence $(3 \cdot -1 \cdot -2)^{[2]}$, which is a z.s.s$_\flat$ of length 6. Hence, $|S| \neq t + 6$. Thus, $|S| \neq t + d$ for all $d \in [6, 11]$.

Claim 3: *If* $5 \cdot 4^{[3]}$ *is a subsequence of* S_L*, then* $|S| \neq t + d$ *for all* $d \in [6, 11]$.

If $n_3 \geq 1$, then $5 \cdot 4^{[2]} \cdot 3$ is a subsequence of S_L, and we are back in Claim 2. Thus, we may assume that $n_3 = 0$. If $n_2 \geq 1$ or $n_1 \geq 2$, then either $5 \cdot 4^{[3]} \cdot 2$ or $5 \cdot 4^{[3]} \cdot 1^{[2]}$ is a subsequence of S_L, which implies that $\Sigma(S_L)$ contains all the integers in $[6, 11]$. In this case, $|S| \neq t + d$ for $d \in [6, 11]$, since S does not contain a z.s.s$_\flat$ of length t by hypothesis. Thus, we may further assume that $n_2 = 0$ and $n_1 \leq 1$. Thus, $S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 1^{[n_1]}$. Moreover, if $n_5 \leq 1$, then

$$|S| = \sigma(S_L) = 5n_5 + 4n_4 + n_1 \leq 5 + 4(t/4 - 1) + 1 < t + 5.$$

Thus, we may assume that $S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 1^{[n_1]}$, where $n_5 \geq 2$, $n_4 \geq 3$, and $n_1 \leq 1$. Since $5^{[2]} \cdot 4^{[3]}$ is a subsequence of S_L, it follows that $\Sigma(S_L)$ contain all the integers in $[8, 10]$. Thus, S contains a z.s.s$_\flat$ of length ℓ for each $\ell \in [8, 10]$. Hence, $|S| \neq t + d$ for all $d \in [8, 10]$. Moreover, the assumptions (i)–(iv) imply that $S' = A \cdot X \cdot Y \cdot Z$ is a subsequence of S, where $A = 3^{[2]} \cdot (-2)^{[3]}$ and $X, Y, Z \in \{-C_1, C_1, -C_2, C_2\}$. By inspecting the sequence S' for all possible choices of X, Y, and Z, we see that S' admits a z.s.s$_\flat$ of length 7. Hence, $|S| \neq t + 7$. Overall, we obtain $|S| \neq t + d$ for any $d \in [7, 10]$.

If $5 \cdot 4^{[4]}$ is a subsequence of S_L, it again follows from the assumptions (i)–(iv) that $S' = A \cdot X \cdot Y \cdot Z \cdot W$ is a subsequence of S, where $A = 3^{[2]} \cdot (-2)^{[3]}$ and $X, Y, Z, W \in \{-C_1, C_1, -C_2, C_2\}$. By inspecting the sequence S' for all possible choices of X, Y, Z, and W, we see that S' admits z.s.s$_\flat$ of lengths 6 and 11. In this case, $|S| \neq t + d$ for all $d \in [6, 11]$. Thus, we may assume that $S_L = 5^{[n_5]} \cdot 4^{[3]} \cdot 1^{[n_1]}$, where $n_5 \geq 2$ and $n_1 \leq 1$.

Now, it remains to show that $|S| \neq t + a$ for $a \in \{6, 11\}$. If $|S| = t + a$, then

$$5n_5 + 4(3) + n_1 = \sigma(S_L) = |S| = t + a, \text{ which implies that } 5n_5 = t + a - 12 - n_1.$$

This is a contradiction since $5 \mid t$ (by hypothesis) and $5 \nmid (a - 12 - n_1)$ for $a \in \{6, 11\}$ and $n_1 \in \{0, 1\}$. Thus, $|S| \neq t + d$ for all $d \in [6, 11]$.

By Claim 1–Claim 3, we may assume S also satisfies the following property:

(v) $S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 3^{[n_3]} \cdot 2^{[n_2]} \cdot 1^{[n_1]}$, where $0 \le n_\ell \le t/\ell - 1$ for all $\ell \in [1,5]$; $n_1, n_2, n_3 \le 3$; $n_4 \le 2$; $(n_4, n_3) \ne (2,1)$; and $n_5 \ge 1$.

We will use this assumption in the remaining claims.

Claim 4: *The statement $|S| \ne t+6$ holds.*

Assume that $|S| = t+6$. If $n_1 \ge 1$, then $5 \cdot 1$ is a subsequence of S_L, which implies that S contains a z.s.s$_b$ of length $5+1=6$ whose complementary sequence in S is a z.s.s$_b$ of length t. Thus, $n_1 = 0$. By a similar reasoning, we infer that $n_2 \le 2$, $n_3 \le 1$, and $n_4 n_2 = 0$. Moreover, the condition (v) implies that $n_4 \le 2$ and $(n_4, n_3) \ne (2,1)$. Thus, $|S| = \sigma(S_L) \le 5n_5 + 4\cdot 2 \le 5(t/5-1)+8 < t+6$, which is a contradiction. Thus, $|S| \ne t+6$.

Claim 5: *The statement $|S| \ne t+7$ holds.*

Assume that $|S| = t+7$. If $n_2 \ge 1$, then $5 \cdot 2$ is a subsequence of S_L, which implies that S contains a z.s.s$_b$ of length $5+2=7$ whose complementary sequence in S is a z.s.s$_b$ of length t. Thus, $n_2 = 0$. By a similar reasoning, we infer that $n_1 \le 1$, $n_4 n_3 = 0$, and $n_1 = 0$ if $n_3 \ge 2$. Moreover, the condition (v) implies that $n_3 \le 3$ and $n_4 \le 2$. Thus, $|S| = \sigma(S_L) \le 5n_5 + 3\cdot 3 \le 5(t/5-1)+9 < t+7$, which is a contradiction. Thus, $|S| \ne t+7$.

Claim 6: *The statement $|S| \ne t+8$ holds.*

Assume that $|S| = t+8$. If $n_3 \ge 1$, then $5 \cdot 3$ is a subsequence of S_L, which implies that S contains a z.s.s$_b$ of length $5+3=8$ whose complementary sequence in S is a z.s.s$_b$ of length t. Thus, $n_3 = 0$. By a similar reasoning, we infer that $n_4 \le 1$, $n_2 \le 3$, $n_1 \le 2$, $n_1 n_2 = 0$, and $n_4 \ge 1$ implies that $n_2 \le 1$. Thus,

$$|S| = \sigma(S_L) \le 5n_5 + 2\cdot 3 \le 5(t/5-1)+6 < t+8,$$

which is a contradiction. Thus, $|S| \ne t+8$.

Claim 7: *The statement $|S| \ne t+9$ holds.*

Assume that $|S| = t+9$. If $n_3 \ge 1$, then S contain a z.s.s$_b$ T of length 3. Thus, $S' = S \cdot T^{-1}$ is a z.s.s of length $|S|-3 = t+6$ which does not contain a z.s.s$_b$ of length t. This contradicts Claim 4. Thus, $n_3 = 0$. Similarly, $n_2 = 0$ (by Claim 5) and $n_1 = 0$ (by Claim 6). Moreover, the condition (v) implies that $n_4 \le 2$. Thus,

$$|S| = \sigma(S_L) = 5n_5 + 4n_4 \le 5(t/5-1) + 4\cdot 2 < t+9,$$

which is a contradiction. Thus, $|S| \ne t+9$.

Claim 8: *The statement $|S| \ne t+d$ holds for $d \in \{10, 11\}$.*

Assume that $|S| = t+10$. If $n_\ell \ge 1$ for some $\ell \in [1,4]$, then S contain a z.s.s$_b$ T of length ℓ. Thus, $S' = S \cdot T^{-1}$ is a z.s.s of length $|S| - \ell = t+10-\ell$ which does not contain a z.s.s$_\flat$ of length t. Since $(|S|-\ell) \in [t+6, t+9]$, this contradicts one of the four previous claims (Claim 4–Claim 7). So we may assume that $n_\ell = 0$ for every $\ell \in [1,4]$. Thus, $|S| = \sigma(S_L) = 5n_5 \le 5(t/5-1) < t+10$, which is a contradiction. Thus, $|S| \ne t+10$.

Finally, assume that $|S| = t+11$. Since $n_5 \ge 1$, S contains a z.s.s$_\flat$ T of length 5. Thus, $S' = S \cdot T^{-1}$ is a z.s.s of length $|S|-5 = t+6$ which does not contain a z.s.s$_\flat$ of length t. This contradicts Claim 1. Thus, $|S| \ne t+11$.

In conclusion, we have shown that if S is an arbitrary z.s.s over $I_3 = [-3,3]$ which does not contain a z.s.s$_\flat$ of length t, then $|S| \ne t+d$ for $d \in [6,11]$. Thus, $\mathsf{s}'_t(I_3) = t+6$. $\square$

3. Appendix

In this section, we include Zhong's proofs of Lemma 1 and Remark 1.

Proof of Lemma 1. (i) Since $\mathsf{s}(G) \le \mathsf{s}'(G)$, it suffices to prove that $\mathsf{s}'(G) \ge \mathsf{s}(G)$. Let $S = \prod_{i=1}^{\mathsf{s}(G)-1} g_i$ be a sequence in $\mathcal{F}(G)$ of length $|S| = \mathsf{s}(G)-1$ such that S has no z.s.s$_\flat$ of length $\exp(G)$. Assume that $\sigma(S) = h$ is in G, and let $t \in \mathbb{N}$ be such that $(\mathsf{s}(G)-1)t \equiv 1 \pmod{\exp(G)}$. Thus, $(\mathsf{s}(G)-1)th = h$ in G. Define $S' = \prod_{i=1}^{\mathsf{s}(G)-1}(g_i - th)$. Since $\sigma(S') = \sigma(S) - (\mathsf{s}(G)-1)th = 0$ and S' does not contain a z.s.s$_\flat$ of length $\exp(G)$, it follows that $\mathsf{s}'(G) \ge \mathsf{s}(G)$.

(ii) Let $S \in \mathcal{B}(G)$ be such that $|S| = \mathsf{s}(G)-1$. We want to prove that S contains a z.s.s$_\flat$ of length $n = \exp(G)$. If we assume to the contrary that S does not contain a z.s.s$_\flat$ of length n, then Property D (defined on page 3) implies that there exists $T \in \mathcal{F}(G)$ such that $S = T^{[n-1]}$. Thus, $|T| = c$ and $\sigma(T) = 0$. Therefore $T^{[n/c]}$ is a z.s.s of length n, a contradiction. $\square$

Proof of Remark 1. (i) Let n be odd and $G \cong \mathbb{Z}_n^2$. Since $\mathsf{s}(G) = 4n-3$, then $\gcd(\mathsf{s}(G)-1, n) = 1$. Thus, $\mathsf{s}(G) = \mathsf{s}'(G)$ by Lemma 1(i).

(ii) Let $h \ge 2$ be an integer and $G \cong \mathbb{Z}_{2^h}^2$. Thus, $\exp(G) = 2^h$, $\mathsf{s}(G) = 4(2^h-1)+1$, $\gcd(\mathsf{s}(G)-1, \exp(G)) = 4$, and G has Property D (by [12, Theorem 3.2]). Thus, Lemma 1(ii) yields $\mathsf{s}'(G) < \mathsf{s}(G)$. Since $\gcd(\mathsf{s}(G)-2, \exp(G)) = 1$, the proof of Lemma 1$(i)$ yields $\mathsf{s}'(G) > \mathsf{s}(G)-2$. Thus, $\mathsf{s}'(G) = \mathsf{s}(G)-1$. $\square$

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