AVOIDING ZERO-SUM SEQUENCES OF PRESCRIBED LENGTH OVER THE INTEGERS\(^{1}\)

C. Augspurger  
Department of Mathematics, Illinois State University, Normal, Illinois  
cdaugsp@ilstu.edu

M. Minter  
Department of Mathematics, Illinois State University, Normal, Illinois  
msminte@ilstu.edu

K. Shoukry  
Department of Mathematics, Illinois State University, Normal, Illinois  
keshouk@ilstu.edu

P. Sissokho\(^{2}\)  
Department of Mathematics, Illinois State University, Normal, Illinois  
psissok@ilstu.edu

K. Voss  
Department of Mathematics, Illinois State University, Normal, Illinois  
kvoss@ilstu.edu

Received: 3/29/16, Revised: 9/21/16, Accepted: 4/20/17, Published: 5/24/17

Abstract  
Let \(t\) and \(k\) be positive integers, and let \(I_k = \{i \in \mathbb{Z} : -k \leq i \leq k\}\). Let \(s'_k(I_k)\) be the smallest positive integer \(\ell\) such that every zero-sum sequence over \(I_k\) with at least \(\ell\) elements contains a zero-sum subsequence with exactly \(t\) elements. If no such \(\ell\) exists, then let \(s'_k(I_k) = \infty\). We prove that \(s'_k(I_k)\) is finite if and only if every integer in \([1, D(I_k)]\) divides \(t\), where \(D(I_k) = \max\{2, 2k - 1\}\) is the Davenport constant of \(I_k\). Moreover, we prove that if \(s'_k(I_k)\) is finite, then \(t + k(k - 1) \leq s'_k(I_k) \leq t + (2k - 2)(2k - 3)\). We also show that \(s'_k(I_k) = t + k(k - 1)\) holds for \(k \leq 3\) and conjecture that this equality holds for \(k \geq 1\).

1. Introduction and Main Results  

We shall follow the notation in [16], by Grynkiewicz. Let \(\mathbb{N}\) be the set of positive integers. Let \(G_0\) be a subset of an abelian group \(G\). A sequence over \(G_0\) is an

\(^{1}\)This research was made possible through a course called Introduction to Undergraduate Research which is sponsored by the Mathematics Department of Illinois State University. This course was taught by P. Sissokho in Spring 2015, and the following students were enrolled in it: C. Augspurger, M. Minter, K. Shoukry, and K. Voss.

\(^{2}\)Corresponding author.
unordered list of terms in $G_0$, where repetition is allowed. The set of all sequences over $G_0$ is denoted by $F(G_0)$. A sequence with no term is called trivial or empty. If $S$ is a sequence with terms $s_i$, $i \in [1, n]$, we write $S = s_1 \cdots s_n = \prod_{i=1}^{n} s_i$. We say that $R$ is a subsequence of $S$ if any term in $R$ is also in $S$. If $R$ and $T$ are subsequences of $S$ such that $S = R \cdot T$, then $R$ is the complementary sequence of $T$ in $S$, and vice versa. We also write $T = S \cdot R^{-1}$ and $R = S \cdot T^{-1}$. For every sequence $S = s_1 \cdots s_n$ over $G_0$,

- the opposite sequence of $S$ is $-S = (-s_1) \cdots (-s_n)$;
- the length of $S$ is $|S| = n$;
- the sum of $S$ is $\sigma(S) = s_1 + \cdots + s_n$;
- the subsequence-sum of $S$ is $\Sigma(S) = \{ \sigma(R) : R$ is a subsequence of $S \}$.

For any sequence $R$ over $G_0$ and any integer $d \geq 0$,

$$R^{[0]} = \text{the trivial sequence, and } R^{[d]} = \underbrace{R \cdot \cdots \cdot R}_{d \text{ times}} \text{ for } d > 0.$$  

A sequence with sum 0 is called zero-sum. The set of all zero-sum sequences over $G_0$ is denoted by $S(G_0)$. A zero-sum sequence is called minimal if it does not contain a proper zero-sum subsequence. The Davenport constant of $G_0$, denoted by $D(G_0)$, is the maximum length of a minimal zero-sum sequence over $G_0$. The research on zero-sum theory is quite extensive when $G$ is a finite abelian group (e.g., see [5, 8, 10, 11] and the references therein). However, there is less activity when $G$ is infinite (e.g., see [3, 6] and the references therein). The study of the case $G = \mathbb{Z}^r$ was explicitly suggested by Baeth and Geroldinger [1] due to their relevance to direct-sum decompositions of modules. Baeth, Geroldinger, Grynkiewicz, and Smertnig [2] studied the Davenport constant of $G_0 \subseteq \mathbb{Z}^r$. The Davenport constant of an interval in $\mathbb{Z}$ was first determined (see Theorem 1) by Lambert [17] (also see [7, 20, 21] for related work.) Plagne and Tringali [18] considered the Davenport constant of the Cartesian product of intervals in $\mathbb{Z}$.

For $x, y \in \mathbb{Z}$ with $x \leq y$, let $[x, y] = \{ i \in \mathbb{Z} : x \leq i \leq y \}$. For $k \in \mathbb{N}$, let $I_k = [-k, k]$.

**Theorem 1 (Lambert [17]).** If $k \in \mathbb{N}$, then $D(I_k) = \max\{ 2, 2k - 1 \}$.

For $G$ finite and $G_0 \subseteq G$, let $s_t(G_0)$ be the smallest integer $t \in \mathbb{N}$ such that any sequence in $F(G_0)$ of length at least $t$ contains a zero-sum subsequence of length $t$. If $t = \exp(G)$, then $s_t(G_0)$ is called the Erdős–Ginzburg–Ziv constant, and it is denoted by $s_t(G)$. Erdős, Ginzburg, and Ziv [8] proved that $s(\mathbb{Z}_n) = 2n - 1$. Reiner [19] proved that $s(\mathbb{Z}_p \oplus \mathbb{Z}_q) = 4p - 3$ for any prime $p$. In general, if $G$ has rank 2, say $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$ with $n_2 \geq n_1 \geq 1$ and $n_1 | n_2$, then $s(G) = 2n_1 + 2n_2 - 3$. 


(see [14, Theorem 5.8.3]). For groups of higher rank, we refer the reader to the paper of Pan, Gao, and Zhong [9]. Recently, Gao, Han, Peng, and Sun [13] proved that for any integer \( k \geq 2 \) and any finite \( G \) with exponent \( n = \exp(G) \), if \( n - |G|/n \) is large enough, then \( s_{k,n}(G) = kn + D(G) - 1 \).

Observe that if \( G \) is torsion-free and \( G_0 \subseteq G \), then for any nonzero element \( g \in G_0 \) and for any \( d \in \mathbb{N} \), the sequence \( g^{id} \in \mathcal{F}(G_0) \) does not contain a zero-sum subsequence. Thus, we will work with the following analogue of \( s_t(G_0) \).

**Definition 1.** \(^3\) For any subset \( G_0 \subseteq G \), let \( s_t'(G_0) \) be the smallest integer \( \ell \in \mathbb{N} \) such that any sequence in \( \mathcal{B}(G_0) \) of length at least \( \ell \) contains a zero-sum subsequence of length \( t \). If no such \( \ell \) exists, then let \( s_t'(G_0) = \infty \).

If \( t = \exp(G) \) is finite, then we denote \( s_t(G_0) \) by \( s_t(G) \). Let \( r \in \mathbb{N} \) and assume that \( G \cong \mathbb{Z}_r^n \). We say that \( G \) has Property \( D \) if, for every sequence \( S \in \mathcal{F}(G) \) of length \( s_t(G) - 1 \) that does not admit a zero-sum subsequence of length \( n \), there exists some sequence \( T \in \mathcal{F}(G) \) such that \( S = T^{[n-1]} \). Zhong found the following interesting connections between \( s_t(G) \) and \( s_t'(G) \). (See the Appendix for their proofs.)

**Lemma 1 (Zhong [22]).** Let \( G \) be a finite abelian group.

(i) If \( \gcd(s_t(G) - 1, \exp(G)) = 1 \), then \( s_t'(G) = s_t(G) \).

(ii) Let \( G \cong \mathbb{Z}_n^r \), where \( n \geq 3 \) and \( r \geq 2 \). Suppose that \( c \in \mathbb{N} \), \( s_t(G) = c(n-1) + 1 \), and \( G \) has Property \( D \). If \( \gcd(s_t(G) - 1, n) = c \), then \( s_t'(G) < s_t(G) \).

**Remark 1 (Zhong [22]).**

(i) If \( G \cong \mathbb{Z}_n^2 \) with \( n \) odd, then \( s_t'(G) = s_t(G) \).

(ii) If \( G \cong \mathbb{Z}_{2h}^2 \) with \( h \geq 2 \), then \( s_t'(G) = s_t(G) - 1 \).

In this paper, we prove the following results about \( s_t'(I_k) \), where \( I_k = [-k,k] \).

**Theorem 2.** Let \( k, t \in \mathbb{N} \).

(i) \( s_t'(I_k) \) is finite, then every integer in \( [1, D(I_k)] \) divides \( t \).

(ii) If every integer in \( [1, D(I_k)] \) divides \( t \), then

\[
 t + k(k-1) \leq s_t'(I_k) \leq t + (2k-2)(2k-3).
\]

**Corollary 1.** Let \( t \in \mathbb{N} \) and \( k \in \{1, 2, 3\} \). Then \( s_t'(I_k) = t + k(k-1) \) if and only if every integer in \( [1, D(I_k)] \) divides \( t \).

**Conjecture 1.** Corollary 1 holds for any \( k \in \mathbb{N} \).

2. Proofs of the Main Results

For the rest of this paper, we assume that \( k, t \in \mathbb{N} \). For any integers \( a \) and \( b \), we denote \( \gcd(a,b) \) by \( (a,b) \). We use the abbreviations \( z.ss \) and \( z.s.s \) for zero-sum sequence(s) and zero-sum subsequence(s), respectively.

\(^3\)This formulation was suggested to us by Geroldinger and Zhong [15].
The following lemma gives a lower bound for $s(I_k)$.

**Lemma 2.** If $U = k \cdot (-1)^{[k]}$ and $V = (k - 1) \cdot (-1)^{\lfloor k - 1 \rfloor}$, then $S = U^{\lfloor \frac{k}{k-1} \rfloor} \cdot V^{[k]}$ and $R = U^{\lfloor k - 1 \rfloor} \cdot V^{\lfloor k - 1 \rfloor}$ are z.s.s. that do not contain a z.s.s. of length $t$. Thus, $s(I_k) \geq t + k(k - 1)$.

**Proof.** We prove the lemma for $S$ only since the proof for $R$ is similar. By contradiction, assume that $S$ contains a z.s.s. of length $t$. Since $\sigma(S) = 0$, it follows that $S$ also contains a z.s.s. $S'$ of length $|S| - t = k(k - 1) - 1$. Moreover, $S'$ can be written as $S' = k^a \cdot (k - 1)^b \cdot (-1)^c$ for some nonnegative integers $a$, $b$, and $c$. Hence $\sigma(S') = ak + b(k - 1) - c = 0$ and $a + b + c = |S'| = k^2 - k - 1$. Thus,

$$(a + 1)(k + 1) = k(k - 1).$$

Since $a, b, k \geq 0$, we have $0 < k - b \leq k$. Since $(k, k + 1) = 1$, we obtain that $k + 1$ divides $k - b$, which is a contradiction. Thus, $s(I_k) \geq |S| + 1 = t + k(k - 1)$. \qed

**Example 1.** If $k = 3$, then $S = (3 \cdot -1 \cdot -1 \cdot -1)^{[14]} \cdot (2 \cdot -1 \cdot -1)^{[2]}$ is a z.s.s of length 65 over $[\{-3, 3\}]$ which does not contain a z.s.s. of length $t = 60$.

**Lemma 3.** Let $a, b, x \in \mathbb{N}$. If $S = a^{\left\lfloor \frac{x}{a+b} \right\rfloor} \cdot (-b)^{\left\lfloor \frac{x}{a+b} \right\rfloor}$ is a z.s.s, then the length of any z.s.s. of $S^x$ is a multiple of $|S|$.

**Proof.** Let $S'$ be a z.s.s. of $S^x$. Since the terms of $S$ are $a$ and $-b$, there exist nonnegative integers $h$ and $r$ such that $S' = a^h \cdot (-b)^r$ and

$$\sigma(S') = ha - rb = 0 \Rightarrow h = \frac{a}{(a, b)} = r = \frac{b}{(a, b)}. \quad (1)$$

Since $(\frac{b}{(a, b)}, \frac{b}{(a, b)}) = 1$, we obtain $\frac{b}{(a, b)}$ divides $h$ and $\frac{a}{(a, b)}$ divides $r$. Thus, $h = p\frac{b}{(a, b)}$ and $r = q\frac{a}{(a, b)}$ for some integers $p$ and $q$. Substituting $h$ and $r$ back into (1) yields $p = q$. Thus,

$$|S'| = h + r = p\frac{b}{(a, b)} + q\frac{a}{(a, b)} = p|S|. \quad \square$$

**Lemma 4.** If $s(I_k)$ is finite, then every odd integer in $[1, D(I_k)]$ divides $t$.

**Proof.** The lemma is trivial for $k = 1$. If $k \geq 2$, then Theorem 1 implies that $D(I_k) = 2k - 1$. Let $\ell = 2c - 1$ be an odd integer in $[3, D(I_k)]$, and consider the minimal z.s.s $S = c^{[c-1]} \cdot (-c + 1)^{[c]}$. If $x \in \mathbb{N}$, then Lemma 3 implies that for any z.s.s. $R$ of $S^x$, $|R|$ divides $S_c = 2c - 1 = \ell$. Thus, if $\ell \mid t$, then there is no z.s.s. of $S^x$ whose length is equal to $t$. Since $x$ is arbitrary, it follows that $s(I_k)$ can be arbitrarily large. This proves the lemma by contrapositive. \qed
To prove the upper bound in Theorem 2(ii), we will use Lemma 5 which is a direct application of a well-known fact: “Any sequence of \( n \) integers contains a nonempty subsequence whose sum is divisible by \( n \).

**Lemma 5.** Let \( \beta \in \mathbb{N} \) and \( X \in F(\mathbb{Z}) \). If \( |X| \geq \beta \), then there exists a factorization \( X = X_0 \cdot X_1 \cdots \cdot X_r \) such that:

(i) \( |X_0| \leq \beta - 1 \) and \( \beta \mid \sigma(R) \) for any nonempty subsequence \( R \) of \( X_0 \);

(ii) \( |X_j| \leq \beta \) and \( \beta \mid \sigma(X_j) \) for all \( j \in [1, r] \).

We will also use the following lemmas.

**Lemma 6.** Assume that \( k \geq 2 \) and that every integer in \( [1, D(I_k)] \) divides \( t \). Let \( S \) be a z.s.s over \( I_k = [-k, k] \) that does not contain a z.s.s of length \( t \). Let \( S = S_1 \cdots S_h \) be a factorization into minimal z.s.s \( S_i, i \in [1, h] \). If \( |S| \geq t + k(k-1) \), then there exists some length \( \beta \) such that:

\[
n_\beta = |\{S_i : |S_i| = \beta, i \in [1, h]\}| > (2k-2)(2k-3).
\]

**Proof.** Recall that \( (a, b) \) denotes \( \gcd(a, b) \). It is easy to see that

\[
(2k - 3, 2k - 2) = (2k - 2, 2k - 1) = (2k - 3, 2k - 1) = 1. \tag{2}
\]

Since \( k \geq 2 \) and every integer in \( [1, D(I_k)] = [1, 2k - 1] \) is a factor of \( t \), it follows from (2) that \( t = p(2k - 1)(2k - 2)(2k - 3), \) for some \( p \in \mathbb{N} \). By definition, we have \( \max_{i \in [1, k]} |S_i| \leq D(I_k) = 2k - 1 \). Thus, it follows from the pigeonhole principle that there exists some length \( \beta \) such that:

\[
n_\beta \geq \frac{t + k(k-1)}{\max_{i \in [1, h]} |S_i|} \geq \frac{t + k(k-1)}{2k-1} > (2k-2)(2k-3).
\]

**Lemma 7.** Assume that \( k \geq 2 \) and that every integer in \( [1, D(I_k)] \) divides \( t \). Let \( S \) be a z.s.s over \( I_k = [-k, k] \) of length \( |S| \geq t + k(k-1) \) such that \( S \) does not contain a z.s.s of length \( t \). Let \( S = S_1 \cdots S_h \) be a factorization into minimal z.s.s \( S_i, i \in [1, h] \). Let \( L = \{|S_i| : i \in [1, h]\} \), \( n_\ell = |\{S_i : |S_i| = \ell, i \in [1, h]\}| \), and \( \alpha = \max_{\ell \in L} \ell \). If there exists \( \beta \in L \) such that \( n_\beta \geq \alpha - 1 \), then

\[
|S| \leq t - \beta + (\beta - 1) \max_{\ell \in L \setminus \{\beta\}} \ell.
\]

**Remark 2.** By Lemma 6, there exists \( \beta \in L \) such that \( n_\beta > (2k-2)(2k-3) \). Moreover, \( \alpha = \max_{\ell \in L} \ell \leq D(I_k) = (2k-2)(2k-3) + 1 \) for \( k \geq 2 \). Thus, \( n_\beta \geq \alpha \), i.e., the hypothesis of Lemma 7 always holds.
Proof of Lemma 6. By hypothesis, there exists \( \beta \in L \) such that \( n_\beta \geq \alpha - 1 \). Given a factorization \( S = S_1 \cdots S_h \) into minimal z.s.s.s. \( S_i, i \in [1, h] \), consider the following sequence of lengths in \( L \setminus \{ \beta \} \):
\[
X = \prod_{i=1, |S_i| \neq \beta}^h |S_i| = \prod_{\ell \in L \setminus \{ \beta \}} \ell^{\ell(n_\ell)}.
\]
By Lemma 5, there exists a factorization \( X = X_0 \cdot X_1 \cdots X_r \) such that:
\[
|X_0| \leq \beta - 1 \text{ and } \beta \mid \sigma(R) \text{ for any nonempty subsequence } R \text{ of } X_0;
\]
\[
|X_j| \leq \beta \text{ and } \beta \mid \sigma(X_j) \text{ for all } j \in [1, r].
\]
Thus,
\[
\sigma(X_j) = \sum_{x \in X_j} x \leq |X_j| \cdot \max_{x \in X_j} x \leq \beta \alpha \text{ for all } j \in [1, r].
\]
To summarize, it follows from the hypothesis of the lemma, (4), and (5) that
\[
\beta \mid t, n_\beta \geq \alpha - 1, \beta \mid \sigma(X_j), \text{ and } \sigma(X_j) \leq \alpha \beta \text{ for all } j \in [1, r].
\]
Thus, if
\[
\beta n_\beta + \sum_{j=1}^r \sigma(X_j) \geq t,
\]
then there exists a nonnegative integer \( n'_\beta \leq n_\beta \) and a subset \( Q \subseteq [1, r] \) such that
\[
\beta n'_\beta + \sum_{q \in Q} \sigma(X_q) = t.
\]
Then \( S \) would contain a z.s.s.s. of length \( t \) obtained by concatenating \( n'_\beta \) z.s.s.s. of \( S \) of length \( \beta \) and all the z.s.s.s. of \( S \) whose lengths form the subsequence \( \prod_{q \in Q} X_q \) of \( X \). This contradicts the hypothesis of the theorem. Thus, the following inequality holds:
\[
\beta n_\beta + \sum_{j=1}^r \sigma(X_j) < t.
\]
Since \( \beta \) divides both \( t \) and \( \sum_{j=1}^r \sigma(X_j) \), it follows from (6) that
\[
\beta n_\beta + \sum_{j=1}^r \sigma(X_j) \leq t - \beta.
\]
Thus, it follows from (7) and the definitions of \( X \) and \( X_j \) (\( 0 \leq j \leq r \)) that
\[
|S| = \sum_{\ell \in L} \ell n_\ell = \beta n_\beta + \sigma(X) = \beta n_\beta + \sum_{j=1}^r \sigma(X_j) + \sigma(X_0) \leq t - \beta + \sigma(X_0).
\]
Next, it follows from (3) and (8) that

$$|S| \leq t - \beta + \sigma(X_0) \leq t - \beta + |X_0| \max_{t \leq k \setminus \{\beta\}} \ell \leq t - \beta + (\beta - 1) \max_{t \leq k \setminus \{\beta\}} \ell.$$  

\[ \square \]

Proof of Theorem 2. We first prove part (i). Suppose that $s'_a(I_k)$ is finite. Then it follows from Lemma 4 that every odd integer in $[1, D(I_k)]$ divides $t$. Thus, it remains to show that if $a$ is an even integer in $[1, D(I_k)]$, then $a | t$.

Case 1: $a = 2^e$ for some $e \in \mathbb{N}$.

Lemma 3 implies that for any $p \in \mathbb{N}$, the sequence $S = (1 \cdot -1)^{|p|}$ is a z.s.s whose z.s.s have lengths that are divisible by 2. Therefore, if $2 \nmid t$, then $s'_a(I_k) \geq |S| = 2p$, where $p$ can be chosen to be arbitrarily large. Thus, $2 | t$ if $s'_a(I_k)$ is finite.

Now assume that $e > 1$. Since the gcd of two numbers divides their difference, $(a/2 - 1, a/2 + 1) \leq 2$. Since $a/2 - 1$ and $a/2 + 1$ are both odd, $(a/2 - 1, a/2 + 1) = 1$. Lemma 4 implies that for any $p \in \mathbb{N}$, the sequence $S[p]$ with $S = (a/2 - 1)\{a/2 + 1\}$.

Thus, if $a \nmid t$, we can construct arbitrarily long z.s.s over $I_k = [-k, k]$ that do not contain z.s.s of length $t$, because $p$ can be chosen to be arbitrarily large. Thus, $a | t$ if $s'_a(I_k)$ is finite.

Case 2: $a$ is not a power of 2.

Thus, $a = 2^e j$, where $e$ and $j$ are nonnegative integers and $j$ is odd. By Lemma 4, $j | t$, and if follows from Case 1 that $2^n | t$. Since $j$ is odd, $(2^n, j) = 1$. Since $2^n$ and $j$ are factors of $t$, it follows that $2^n j | t$.

The above cases and Lemma 4 imply that every integer in $[1, D(I_k)]$ divides $t$.

Since the lower bound of $s'_a(I_k)$ in Theorem 2(ii) follows from Lemma 2, it remains to prove its upper bound. Recall that every integer in $[1, D(I_k)]$ divides $t$. Let $S$ be an arbitrary z.s.s over $I_k = [-k, k]$ that does not contain a z.s.s of length $t$.

If $k = 1$, then it follows from Theorem 1 that $D(I_k) = 2$. Thus, $2 | t$ and $|S| = x_1 + 2x_2$ for some nonnegative integers $x_1$ and $x_2$. If $|S| \geq t$, then $x_1 \geq 2$ or $x_2 \geq t/2$ (because $2 | t$). This implies that there exist nonnegative integers $x_1' \leq x_1$ and $x_2' \leq x_2$ such that $x'_1 + 2x'_2 = t$. Thus, $S' = 0[x'_1] \cdot (1 \cdot -1)[x'_2]$ is a z.s.s of $S$ of length $t$, which is a contradiction since $S$ does not contain a z.s.s of length $t$. Hence $|S| \leq t - 1$, and $s'_a(I_k) \leq |S| + 1 = t$.

Now assume $k \geq 2$. Since $S$ was arbitrarily chosen, if $|S| \leq t + k(k - 1) - 1$, then

$$s'_a(I_k) \leq |S| + 1 \leq t + k(k - 1) \leq t + (2k - 2)(2k - 3),$$

which yields the upper bound in Theorem 2(ii). Thus, we may assume that $|S| \geq t + k(k - 1)$. Let $S = S_1 \ldots S_h$ be a factorization into minimal z.s.s $S_i, i \in [1, h]$. Let $L = \{|S_i| : i \in [1, h]\}$, $u_\ell = \{|S_i| : |S_i| = \ell, i \in [1, h]\}$, and $\alpha = \max_{\ell \in L} \ell$. If $\alpha = 1$, then any term of $S$ is equal to 0, which is a contradiction since $|S| \geq t + k(k - 1)$ and $S$ does not contain a z.s.s of length $t$. So, we may assume that $\alpha \geq 2$. Then Remark 2 implies that there exists $\beta \in L$ such that $n_\beta \geq \alpha - 1$. If $\beta = \alpha$, then
Lemma 7 yields
\[ |S| \leq t - \alpha + (\alpha - 1) \max_{\ell \in L \setminus \{\alpha\}} \ell \leq t - \alpha + (\alpha - 1)^2. \]

If \(1 \leq \beta \leq \alpha - 1\), then Lemma 7 also yields
\[
|S| \leq t + \max_{1 \leq \beta \leq \alpha - 1} \left( -\beta + (\beta - 1) \max_{\ell \in L \setminus \{\beta\}} \ell \right) \\
\leq t + \max_{1 \leq \beta \leq \alpha - 1} (-\beta + (\beta - 1)\alpha) \\
= t + (-\alpha + 1) + (\alpha - 2)\alpha \\
= t - \alpha + (\alpha - 1)^2.
\]

So in all cases, we obtain
\[
|S| \leq t - \alpha + (\alpha - 1)^2 \leq t - (2k - 1) + (2k - 2)^2, \tag{9}
\]
where we used the fact \(\alpha \leq D(I_k) = 2k - 1\). Since \(S\) was chosen to be an arbitrary z.s.s over \(I_k = [-k, k]\) which does not contain a z.s.s. of length \(t\),
\[
s'_1(I_k) \leq |S| + 1 \leq t - (2k - 1) + (2k - 2)^2 + 1 = t + (2k - 2)(2k - 3). \quad \Box
\]

**Proof of Corollary 1.** For \(k \in \{1, 2\}\), the corollary holds since the upper and lower bounds of \(s'_1(I_k)\) given by Theorem 2 are both equal to \(t + k(k - 1)\).

For \(k = 3\), it also follows from Theorem 2 that \(t + 6 \leq s'_1(I_3) \leq t + 12\). Thus, it remains to show that if \(S\) is an arbitrary z.s.s over \(I_3\) which does not contain a z.s.s. of length \(t\), then \(|S| \neq t + d\) for all \(d \in [6, 11]\).

Consider a factorization \(S = S_1 \cdot \ldots \cdot S_h\) into minimal z.s.s. \(S_i\), \(i \in [1, h]\). Let \(L = \{|S_i| : i \in [1, h]\}\), \(n_\ell = \{|S_i| : |S_i| = \ell, i \in [1, h]\}\), and \(\alpha = \max_{\ell \in L} \ell\). Thus, \(\alpha \leq D(I_3) = 5\). If \(\alpha \leq 4\), then Lemma 7 yields
\[
|S| \leq t + \max_{1 \leq \alpha \leq 4} (\alpha(\alpha - 1)^2 - \alpha) = t + (4 - 1)^2 - 4 = t + 5.
\]

Thus, we may assume that \(\alpha = \max \ell = 5\) for any factorization of \(S\).

If \(\beta \in \{1, 2\}\) and \(n_\beta \geq 4\), then Lemma 7 yields
\[
|S| \leq t + \max_{\beta \in \{1, 2\}} ((\beta - 1)\alpha - \beta) = t + (2 - 1)5 - 2 = t + 3.
\]

Next, suppose that \(R\) is a z.s.s. of \(S\) with length at least 4. Then \(R \cdot -R\) can be trivially factorized into \(|W|\) z.s.s. of length 2, where \(|W| \geq 4\). This yields a new factorization \(S = S'_1 \cdot \ldots \cdot S'_h\) with \(n_2 \geq 4\), which implies that \(|S| < t + 5\) by the above analysis upon setting \(\beta = 2\).

Also note that if \(n_\ell \geq t/\ell\) for some \(\ell \in L\), then we obtain a z.s.s. of \(S\) of length \(t\) by concatenating \(t/\ell\) z.s.s. of length \(\ell\) in \(S\). This would contradict the definition of \(S\). Thus, we can assume that \(n_\ell \leq t/\ell - 1\) for all \(\ell \in L\), where \(L \subseteq [1, 5]\).
To recapitulate, we may assume that for any factorization $S = S_1 \cdot \ldots \cdot S_h$, with $S_L = \prod_{i=1}^{h} |S_i|$ and $n_{t} = |\{S_i : |S_i| = \ell, i \in [1, h]\}|$, the following properties hold:

(i) $S_L = 5^{n_5} \cdot 4^{n_4} \cdot 3^{n_3} \cdot 2^{n_2} \cdot 1^{n_1}$, where $0 \leq n_{t} \leq t/\ell - 1$ for $\ell \in [1, 5]$, $n_5 \geq 1$, and $n_1, n_2 \leq 3$;

(ii) there is a one-to-one correspondence between the subsequences $S_L'$ of $S_L$ and the z.s.s. $S'$ of $S$ with length $|S'| = \sigma(S_L')$;

(iii) if $R$ is z.s.s over $I_3$ such that $|R| \geq 4$, then $R$ and $-R$ cannot both be subsequences of $S$;

(iv) if $R$ is a minimal z.s.s of $S$ such that $|R| = 5$, then $R = 3^{[2]} \cdot (-2)^{[3]}$. (This follows from (iii) and the fact $A = 3^{[2]} \cdot (-2)^{[3]}$ and $-A$ are the only minimal z.s.s of length 5 over $I_3 = [-3, 3]$. Thus, if $-A$ is the only z.s.s of $S$, then we can analyze $-S$ instead of $S$.)

**Claim 1:** If $5 \cdot 3^2$ is a subsequence of $S_L$, then $|S| \neq t + d$ for all $d \in [6, 11]$.

If $n_4 + n_2 + n_1 \geq 1$, then either $5 \cdot 4 \cdot 3^2$, or $5 \cdot 3^2 \cdot 2$, or $5 \cdot 3^2 \cdot 1$ is a subsequence of $S_L$, which implies that $\Sigma(S_L)$ contains all the integers in $[6, 11]$. In this case, $|S| \neq t + d$ for $d \in [6, 11]$, since $S$ does not contain a z.s.s of length $t$ by hypothesis. Thus, we may assume that $n_4 = n_2 = n_1 = 0$, which implies that $S_L = 5^{[n_5]} \cdot 3^{[n_3]}$. If $n_5 \leq 1$, then $|S| = \sigma(S_L) = 5n_5 + 3n_3 \leq 5 + 3(t/3 - 1) < t + 5$. Thus, we may assume that $S_L = 5^{[n_5]} \cdot 3^{[n_3]}$, where $n_5 \geq 2$ and $n_3 \geq 4$.

Then $\Sigma(S_L)$ contain all the integers in $[6, 11] \setminus \{7\}$; and so $|S| \neq t + d$ for $d \in [6, 11] \setminus \{7\}$. It remains to show that $|S| \neq t + 7$. Note that the only minimal z.s.s of length 3 over $[-3, 3]$ are (up to sign) $B_1 = 2 \cdot (-1)^{[2]}$ and $B_2 = 3 \cdot -2 = -1$. Since $5 \cdot 3^2$ is a subsequence of $S_L$, the assumptions (i)–(iv) imply that $S' = A \cdot X \cdot Y \cdot Z \cdot W$ is a subsequence of $S$, where $A = 3^{[2]} \cdot (-2)^{[3]}$ and $X, Y, Z, W \in \{-B_1, B_1, -B_2, B_2\}$. By inspecting the sequence $S'$ for all possible choices of $X, Y, Z,$ and $W$, we see that $S'$ admits a z.s.s, of length 7. For instance, if $X = Y = Z = B_2$, then

$$S' = A \cdot B_2^{[3]} \cdot W = A^{[2]} \cdot 3 \cdot (-1)^{[3]} \cdot W$$

contains the subsequence $3 \cdot (-1)^{[3]} \cdot W$, which is a z.s.s of length $4 + |W| = 7$. Hence, $|S| \neq t + 7$. Thus, $|S| \neq t + d$ for all $d \in [6, 11]$.

**Claim 2:** If $5 \cdot 4^2 \cdot 3$ is a subsequence of $S_L$, then $|S| \neq t + d$ for all $d \in [6, 11]$.

If $n_3 \geq 2$ or $n_1 + n_2 \geq 1$, then either $5 \cdot 4^2 \cdot 3^2$, or $5 \cdot 4^2 \cdot 3 \cdot 2$, or $5 \cdot 4^2 \cdot 3 \cdot 1$ is a subsequence of $S_L$, which implies that $\Sigma(S_L)$ contains all the integers in $[6, 11]$. In this case, $|S| \neq t + d$ for $d \in [6, 11]$, since $S$ does not contain a z.s.s of length $t$ by hypothesis. Thus, we may assume that $n_3 = 1$ and $n_1 = n_2 = 0$, which implies that $S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 3$. If $n_5 \leq 1$, then

$$|S| = \sigma(S_L) = 5n_5 + 4n_4 + 3 \leq 5 + 4(t/4 - 1) + 3 < t + 5.$$
Thus, we may assume that $S_L = 5^{n_5} \cdot 4^{n_4} \cdot 3$, where $n_5 \geq 2$ and $n_4 \geq 2$.

Thus, $5^{[2]} \cdot 4^{[2]} \cdot 3$ is a subsequence of $S_L$, which implies that $\Sigma(S_L)$ contain all the integers in $[7, 11]$. Thus, $|S| \neq t + d$ for $d \in [7, 11]$. It remains to show that $|S| \neq t + 6$. Note that the only minimal z.s.s of length 4 over $[-3, 3]$ are (up to sign) $C_1 = 3 \cdot (-1)^{[3]}$ and $C_2 = 3 \cdot 1 \cdot (-2)^{[2]}$. Since $5 \cdot 4^{[2]} \cdot 3$ is a subsequence of $S_L$, the assumptions (i)--(iv) imply that $S' = A \cdot X \cdot Y \cdot Z$ is a subsequence of $S$, where $A = 3^{[2]} \cdot (-2)^{[3]}$, $X, Y \in \{-C_1, C_1, -C_2, C_2\}$, and $Z \in \{-B_1, B_1, -B_2, B_2\}$. By inspecting the sequence $S'$ for all possible choices of $X, Y$, and $Z$, we see that $S'$ admits a z.s.s of length 6. For instance, if $X = C_1$ and $Y = C_2$, then

$$S' = A \cdot C_1 \cdot C_2 \cdot Z = A \cdot (3 \cdot -1 \cdot -2)^{[2]} \cdot (1 \cdot -1) \cdot Z$$

contains the subsequence $(3 \cdot -1 \cdot -2)^{[2]}$, which is a z.s.s, of length 6. Hence, $|S| \neq t + 6$. Thus, $|S| \neq t + d$ for all $d \in [6, 11]$.

**Claim 3:** If $5 \cdot 4^{[3]}$ is a subsequence of $S_L$, then $|S| \neq t + d$ for all $d \in [6, 11]$.

If $n_3 \geq 1$, then $5 \cdot 4^{[2]} \cdot 3$ is a subsequence of $S_L$, and we are back in Claim 2. Thus, we may assume that $n_3 = 0$. If $n_2 \geq 1$ or $n_1 \geq 2$, then either $5 \cdot 4^{[3]} \cdot 2$ or $5 \cdot 4^{[3]} \cdot 1^{[2]}$ is a subsequence of $S_L$, which implies that $\Sigma(S_L)$ contains all the integers in $[6, 11]$. In this case, $|S| \neq t + d$ for $d \in [6, 11]$, since $S$ does not contain a z.s.s of length $t$ by hypothesis. Thus, we may further assume that $n_2 = 0$ and $n_1 \leq 1$. Thus, $S_L = 5^{[n_3]} \cdot 4^{[n_4]} \cdot 1^{[n_1]}$. Moreover, if $n_5 \leq 1$, then

$$|S| = \sigma(S_L) = 5n_5 + 4n_4 + n_1 \leq 5 + 4(t/4 - 1) + 1 < t + 5.$$ 

Thus, we may assume that $S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 1^{[n_1]}$, where $n_5 \geq 2$, $n_4 \geq 3$, and $n_1 \leq 1$. Since $5^{[2]} \cdot 4^{[3]}$ is a subsequence of $S_L$, it follows that $\Sigma(S_L)$ contain all the integers in $[8, 10]$. Thus, $S$ contains a z.s.s of length $\ell$ for each $\ell \in [8, 10]$. Hence, $|S| \neq t + d$ for all $d \in [8, 10]$. Moreover, the assumptions (i)--(iv) imply that $S' = A \cdot X \cdot Y \cdot Z$ is a subsequence of $S$, where $A = 3^{[2]} \cdot (-2)^{[3]}$ and $X, Y, Z \in \{-C_1, C_1, -C_2, C_2\}$. By inspecting the sequence $S'$ for all possible choices of $X, Y$, and $Z$, we see that $S'$ admits a z.s.s of length 7. Hence, $|S| \neq t + 7$. Overall, we obtain $|S| \neq t + d$ for any $d \in [7, 10]$.

If $5 \cdot 4^{[4]}$ is a subsequence of $S_L$, it again follows from the assumptions (i)--(iv) that $S' = A \cdot X \cdot Y \cdot Z \cdot W$ is a subsequence of $S$, where $A = 3^{[2]} \cdot (-2)^{[3]}$ and $X, Y, Z, W \in \{-C_1, C_1, -C_2, C_2\}$. By inspecting the sequence $S'$ for all possible choices of $X, Y, Z$, and $W$, we see that $S'$ admits z.s.s of lengths 6 and 11. In this case, $|S| \neq t + d$ for all $d \in [6, 11]$. Thus, we may assume that $S_L = 5^{[n_5]} \cdot 4^{[3]} \cdot 1^{[n_1]}$, where $n_5 \geq 2$ and $n_1 \leq 1$.

Now, it remains to show that $|S| \neq t + a$ for $a \in \{6, 11\}$. If $|S| = t + a$, then

$$5n_5 + 4(3) + n_1 = \sigma(S_L) = |S| = t + a,$$

which implies that $5n_5 = t + a - 12 - n_1$. This is a contradiction since $5 \mid t$ (by hypothesis) and $5 \nmid (a - 12 - n_1)$ for $a \in \{6, 11\}$ and $n_1 \in \{0, 1\}$. Thus, $|S| \neq t + d$ for all $d \in [6, 11]$. 


By Claim 1–Claim 3, we may assume $S$ also satisfies the following property:

(v) $S_L = 5^{[n_1]} \cdot 4^{[n_2]} \cdot 3^{[n_3]} \cdot 2^{[n_4]} \cdot 1^{[n_5]}$, where $0 \leq n_\ell \leq t/\ell - 1$ for all $\ell \in [1, 5]$;

$n_1, n_2, n_3 \leq 3$; $n_4 \leq 2$; $(n_4, n_3) \neq (2, 1)$; and $n_5 \geq 1$.

We will use this assumption in the remaining claims.

**Claim 4:** The statement $|S| \neq t + 6$ holds.

Assume that $|S| = t + 6$. If $n_1 \geq 1$, then $5 \cdot 1$ is a subsequence of $S_L$, which implies that $S$ contains a z.s.s. of length $5 + 1 = 6$ whose complementary sequence in $S$ is a z.s.s. of length $t$. Thus, $n_1 = 0$. By a similar reasoning, we infer that $n_2 \leq 2$, $n_3 \leq 1$, and $n_4n_2 = 0$. Moreover, the condition (v) implies that $n_4 \leq 2$ and $(n_4, n_3) \neq (2, 1)$. Thus, $|S| = \sigma(S_L) \leq 5n_5 + 4 \cdot 2 \leq 5(t/5) + 8 < t + 6$, which is a contradiction. Thus, $|S| \neq t + 6$.

**Claim 5:** The statement $|S| \neq t + 7$ holds.

Assume that $|S| = t + 7$. If $n_2 \geq 1$, then $5 \cdot 2$ is a subsequence of $S_L$, which implies that $S$ contains a z.s.s. of length $5 + 2 = 7$ whose complementary sequence in $S$ is a z.s.s. of length $t$. Thus, $n_2 = 0$. By a similar reasoning, we infer that $n_1 \leq 1$, $n_4n_3 = 0$, and $n_1 = 0$ if $n_3 \geq 2$. Moreover, the condition (v) implies that $n_3 \leq 3$ and $n_2 \leq 2$. Thus, $|S| = \sigma(S_L) \leq 5n_5 + 3 \cdot 3 \leq 5(t/5) + 9 < t + 7$, which is a contradiction. Thus, $|S| \neq t + 7$.

**Claim 6:** The statement $|S| \neq t + 8$ holds.

Assume that $|S| = t + 8$. If $n_3 \geq 1$, then $5 \cdot 3$ is a subsequence of $S_L$, which implies that $S$ contains a z.s.s. of length $5 + 3 = 8$ whose complementary sequence in $S$ is a z.s.s. of length $t$. Thus, $n_3 = 0$. By a similar reasoning, we infer that $n_4 \leq 1$, $n_2 \leq 3$, $n_1 \leq 2$, $n_1n_2 = 0$, and $n_4 \geq 1$ implies that $n_2 \leq 1$. Thus,

$$|S| = \sigma(S_L) \leq 5n_5 + 2 \cdot 3 \leq 5(t/5) + 6 < t + 8,$$

which is a contradiction. Thus, $|S| \neq t + 8$.

**Claim 7:** The statement $|S| \neq t + 9$ holds.

Assume that $|S| = t + 9$. If $n_3 \geq 1$, then $S$ contain a z.s.s. $T$ of length 3. Thus, $S' = S \cdot T^{-1}$ is a z.s.s of length $|S| - 3 = t + 6$ which does not contain a z.s.s. of length $t$. This contradicts Claim 4. Thus, $n_3 = 0$. Similarly, $n_2 = 0$ (by Claim 5) and $n_1 = 0$ (by Claim 6). Moreover, the condition (v) implies that $n_4 \leq 2$. Thus,

$$|S| = \sigma(S_L) = 5n_5 + 4n_4 \leq 5(t/5) + 4 \cdot 2 < t + 9,$$

which is a contradiction. Thus, $|S| \neq t + 9$.

**Claim 8:** The statement $|S| \neq t + d$ holds for $d \in \{10, 11\}$.

Assume that $|S| = t + 10$. If $n_\ell \geq 1$ for some $\ell \in [1, 4]$, then $S$ contain a z.s.s. $T$ of length $\ell$. Thus, $S' = S \cdot T^{-1}$ is a z.s.s of length $|S| - \ell = t + 10 - \ell$ which does not
contain a z.s.s. of length $t$. Since $|S| - t = t + 9$, this contradicts one of the four previous claims (Claim 4–Claim 7). So we may assume that $n_\ell = 0$ for every $\ell \in [1, 4]$. Thus, $|S| = \sigma(S_L) = 5n_5 \leq 5(t/5 - 1) < t + 10$, which is a contradiction. Thus, $|S| \neq t + 10$.

Finally, assume that $|S| = t + 11$. Since $n_5 \geq 1$, $S$ contains a z.s.s. $T$ of length 5. Thus, $S' = S \cdot T^{-1}$ is a z.s.s of length $|S| - 5 = t + 6$ which does not contain a z.s.s. of length $t$. This contradicts Claim 1. Thus, $|S| \neq t + 11$.

In conclusion, we have shown that if $S$ is an arbitrary z.s.s over $I_3 = [-3, 3]$ which does not contain a z.s.s. of length $t$, then $|S| \neq t + d$ for $d \in [6, 11]$. Thus, $s'_5(I_3) = t + 6$.

\[\square\]

3. Appendix

In this section, we include Zhong’s proofs of Lemma 1 and Remark 1.

**Proof of Lemma 1.** (i) Since $s(G) \leq s'(G)$, it suffices to prove that $s'(G) \geq s(G)$. Let $S = \prod_{i=1}^{s(G)-1} g_i$ be a sequence in $\cal F(G)$ of length $|S| = s(G) - 1$ such that $S$ has no z.s.s. of length $\exp(G)$. Assume that $\sigma(S) = h$ is in $G$, and let $t \in \mathbb{N}$ be such that $(s(G) - 1)t \equiv 1 \pmod{\exp(G)}$. Thus, $(s(G) - 1)th = h$ in $G$. Define $S' = \prod_{i=1}^{s(G)-1} (g_i - th)$. Since $\sigma(S') = \sigma(S) - (s(G) - 1)th = 0$ and $S'$ does not contain a z.s.s. of length $\exp(G)$, it follows that $s'(G) \geq s(G)$.

(ii) Let $S \in \cal B(G)$ be such that $|S| = s(G) - 1$. We want to prove that $S$ contains a z.s.s. of length $n = \exp(G)$. If we assume to the contrary that $S$ does not contain a z.s.s. of length $n$, then Property D (defined on page 3) implies that there exists $T \in \cal F(G)$ such that $S = T^{[n-1]}$. Thus, $|T| = c$ and $\sigma(T) = 0$. Therefore $T^{[n/c]}$ is a z.s.s of length $n$, a contradiction. \[\square\]

**Proof of Remark 1.** (i) Let $n$ be odd and $G \cong \mathbb{Z}_n^2$. Since $s(G) = 4n - 3$, then $\gcd(s(G) - 1, n) = 1$. Thus, $s(G) = s'(G)$ by Lemma 1(i).

(ii) Let $h \geq 2$ be an integer and $G \cong \mathbb{Z}_{2h}^2$. Thus, $\exp(G) = 2^h$, $s(G) = 4(2^h - 1) + 1$, $\gcd(s(G) - 1, \exp(G)) = 4$, and $G$ has Property $D$ (by [12, Theorem 3.2]). Thus, Lemma 1(ii) yields $s'(G) < s(G)$. Since $\gcd(s(G) - 2, \exp(G)) = 1$, the proof of Lemma 1(i) yields $s'(G) > s(G) - 2$. Thus, $s'(G) = s(G) - 1$. \[\square\]

**Acknowledgement.** We thank Alfred Geroldinger for providing references and for his valuable comments which helped clarify the definitions and terminology. We also thank Qinghai Zhong for allowing us to include Lemma 1 and Remark 1. Finally, we thank the reviewer for mathematical corrections, and the editor for editorial corrections.
Note added to the paper: Aaron Berger [4] has recently announced a proof of Conjecture 1.

References


