CONGRUENCES FOR 1-SHELL TOTALLY SYMMETRIC PLANE PARTITIONS

Shane Chern
Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania
and
School of Mathematical Sciences, Zhejiang University, Hangzhou, China
shanechern@psu.edu

Received: 4/7/16, Accepted: 5/8/17, Published: 5/24/17

Abstract

Let $f(n)$ denote the number of 1-shell totally symmetric plane partitions of weight n. Recently, Hirschhorn and Sellers, Yao, and Xia established a number of congruences modulo 2 and 5, 4 and 8, and 25 for $f(n)$, respectively. In this note, we shall prove several new congruences modulo 125 and 11 by using some results of modular forms. For example, for all $n \geq 0$, we have

$$\begin{aligned} f(1250n+125) &\equiv 0 \pmod{125}, \\ f(1250n+1125) &\equiv 0 \pmod{125}, \\ f(2750n+825) &\equiv 0 \pmod{11}, \\ f(2750n+1925) &\equiv 0 \pmod{11}. \end{aligned}$$

1. Introduction

A plane partition is a two-dimensional array of integers $\pi_{i,j}$ that are weakly decreasing in both indices and that add up to the given number n, namely, $\pi_{i,j} \geq \pi_{i+1,j}$, $\pi_{i,j} \geq \pi_{i,j+1}$, and $\sum \pi_{i,j} = n$. If a plane partition is invariant under any permutation of the three axes, we call it a totally symmetric plane partition (see, e.g., Andrews *et al.* [1] and Stembridge [7] for more details). In 2012, Blecher [3] studied a special class of totally symmetric plane partitions which he called 1-shell totally symmetric plane partitions. A 1-shell totally symmetric plane partition has a self-conjugate first row/column (as an ordinary partition) and all other entries are 1.

For example,

$$\begin{array}{cccc} 4 & 4 & 2 & 2 \\ 4 & 1 & 1 & 1 \\ 2 & 1 & & \\ 2 & 1 & & \end{array}$$

is a totally symmetric plane partition.

Let $f(n)$ denote the number of 1-shell totally symmetric plane partitions of weight n, namely, the parts of the totally symmetric plane partition sum to n. In [3], Blecher found the generating function of $f(n)$,

$$\sum_{n\geq 0} f(n)q^n = 1 + \sum_{n\geq 1} q^{3n-2} \prod_{i=0}^{n-2} \left(1 + q^{6i+3}\right).$$

Recently, Hirschhorn and Sellers [4], Yao [9], and Xia [8] established a number of congruences for $f(n)$, respectively. For example, for all $n \geq 0$, Hirschhorn and Sellers proved that

$$f(10n + 5) \equiv 0 \pmod{5}, \tag{1}$$

while Xia proved that

$$f(250n + 125) \equiv 0 \pmod{25}. \tag{2}$$

Moreover, Yao showed that, for all $n \geq 0$,

$$f(8n + 3) \equiv 0 \pmod{4}. \tag{3}$$

In this note, we shall prove several new congruences modulo 125 and 11 for $f(n)$. Here our methods are based on some results of modular forms, which are quite different from the proofs of the previous congruences. In fact, Radu and Sellers gave a strategy in [6] to prove these Ramanujan-like congruences, and their methods can be tracked back to [5]. Our results are stated as follows.

Theorem 1. *For all $n \geq 0$, we have*

$$f(1250n + 125) \equiv 0 \pmod{125} \tag{4}$$

and

$$f(1250n + 1125) \equiv 0 \pmod{125}. \tag{5}$$

Theorem 2. *For all $n \geq 0$, we have*

$$f(2750n + 825) \equiv 0 \pmod{11} \tag{6}$$

and

$$f(2750n + 1925) \equiv 0 \pmod{11}. \tag{7}$$

By (1) and Theorem 2, we immediately get

Theorem 3. *For all $n \geq 0$, we have*

$$f(2750n+825) \equiv 0 \pmod{55} \tag{8}$$

and

$$f(2750n+1925) \equiv 0 \pmod{55}. \tag{9}$$

2. Preliminaries

We first introduce some notations of [6]. Let M be a positive integer. We denote by $R(M)$ the set of integer sequences $\{r : r = (r_{\delta_1}, \ldots, r_{\delta_k})\}$ indexed by the positive divisors $1 = \delta_1 < \cdots < \delta_k = M$ of M. For a positive integer m, let $[s]_m$ be the set of all elements congruent to s modulo m. We also write $\mathbb{Z}_m^*$ the set of all invertible elements in $\mathbb{Z}_m$, and $\mathbb{S}_m$ the set of all squares in $\mathbb{Z}_m^*$. For $t \in \{0, \ldots, m-1\}$, we define by $\overline{\odot}_r$ the map $\mathbb{S}_{24m} \times \{0, \ldots, m-1\} \rightarrow \{0, \ldots, m-1\}$ with

$$([s]_{24m}, t) \mapsto [s]_{24m} \overline{\odot}_r t \equiv ts + \frac{s-1}{24} \sum_{\delta | M} \delta r_\delta \pmod{m}.$$

Furthermore, we put $P_{m,r}(t) := \{[s]_{24m} \overline{\odot}_r t \mid [s]_{24m} \in \mathbb{S}_{24m}\}$.

Let $\Gamma := SL_2(\mathbb{Z})$ and $\Gamma_\infty := \left\{ \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \,\middle|\, h \in \mathbb{Z} \right\}$. For a positive integer N, we define the congruence subgroup of level N as

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, c \equiv 0 \pmod{N} \right\}.$$

We also know that

$$[\Gamma : \Gamma_0(N)] = N \prod_{p | N} (1 + p^{-1}),$$

where the product runs through the distinct prime numbers dividing N.

Now denote by Δ^* the set of tuples $(m, M, N, t, r = (r_\delta))$ which satisfy conditions given in [6, p. 2255][1]. Let $\kappa = \kappa(m) = \gcd(m^2-1, 24)$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $r \in R(M)$, and $r' \in R(N)$, we set

$$p_{m,r}(\gamma) = \min_{\lambda \in \{0, \ldots, m-1\}} \frac{1}{24} \sum_{\delta | M} r_\delta \frac{\gcd^2(\delta(a + \kappa\lambda c), mc)}{\delta m}$$

[1]According to a private communication between the author and S. Radu, the last condition of Δ^* should read: "for $(s, j) = \pi(M, (r_\delta))$, if $2 \mid m$, we have ($4 \mid \kappa N$ and $8 \mid Ns$) or ($2 \mid s$ and $8 \mid N(1-j)$)."

and

$$p_{r'}^{*}(\gamma) = \frac{1}{24}\sum_{\delta|N}\frac{r'_\delta \gcd^2(\delta,c)}{\delta}.$$

Finally, we write $(a;q)_\infty := \prod_{n\geq 0}(1-aq^n)$, and let

$$f_r(q) := \prod_{\delta|M}(q^\delta;q^\delta)_\infty^{r_\delta} = \sum_{n\geq 0} c_r(n)q^n$$

for some $r \in R(M)$. The following lemma (see [5, Lemma 4.5] or [6, Lemma 2.4]) is a key to our proof.

Lemma 1. *Let u be a positive integer, $(m, M, N, t, r = (r_\delta)) \in \Delta^*$, $r' = (r'_\delta) \in R(N)$, n be the number of double cosets in $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$ and $\{\gamma_1, \dots, \gamma_n\} \subset \Gamma$ be a complete set of representatives of the double coset $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Assume that $p_{m,r}(\gamma_i) + p_{r'}^*(\gamma_i) \geq 0$ for all $i = 1, \dots, n$. Let $t_{\min} := \min_{t'\in P_{m,r}(t)} t'$ and*

$$v := \frac{1}{24}\left(\left(\sum_{\delta|M} r_\delta + \sum_{\delta|N} r'_\delta\right)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N}\delta r'_\delta\right) - \frac{1}{24m}\sum_{\delta|M}\delta r_\delta - \frac{t_{\min}}{m}.$$

Then if

$$\sum_{n=0}^{\lfloor v\rfloor} c_r(mn+t')q^n \equiv 0 \pmod{u}$$

for all $t' \in P_{m,r}(t)$, then

$$\sum_{n\geq 0} c_r(mn+t')q^n \equiv 0 \pmod{u}$$

for all $t' \in P_{m,r}(t)$.

3. Proofs of the Theorems

3.1. The Upper Bound

In the first part of our proofs, we will compute the upper bound of $\lfloor v\rfloor$ in Lemma 1 for each theorem. Let $g(n)$ be given by

$$\sum_{n\geq 0} g(n)q^n := \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2}. \tag{10}$$

In [4], Hirschhorn and Sellers proved that

$$f(6n+1) = g(n). \tag{11}$$

Moreover, we write

$$\sum_{n\geq 0} g_{\alpha,p}(n)q^n := \frac{(q;q)_\infty^{p^\alpha-2}(q^2;q^2)_\infty^3}{(q^p;q^p)_\infty^{p^{\alpha-1}}}, \tag{12}$$

where α is a positive integer and p is prime. By [6, Lemma 1.2], we obtain

$$\sum_{n\geq 0} g_{\alpha,p}(n)q^n \equiv \sum_{n\geq 0} g(n)q^n \pmod{p^\alpha}. \tag{13}$$

Note that [4, Theorem 2.1] tells that $f(n)=0$ if $n\equiv 0,2 \pmod 3$ for all $n\geq 1$. We therefore have $f(1250\cdot 3n+125) = f(1250\cdot(3n+2)+125) = 0$. To prove (4), it suffices to prove $f(3750n+1375) = f(1250\cdot(3n+1)+125) \equiv 0 \pmod{125}$, which yields

$$g_{3,5}(625n+229) \equiv 0 \pmod{125}. \tag{14}$$

Similarly, to prove (5), (6), and (7), we only need to prove

$$g_{3,5}(625n+604) \equiv 0 \pmod{125}, \tag{15}$$

$$g_{1,11}(1375n+1054) \equiv 0 \pmod{11}, \tag{16}$$

and

$$g_{1,11}(1375n+779) \equiv 0 \pmod{11}, \tag{17}$$

respectively.

Let

$$r^{(\alpha,p)} := (r_1, r_2, r_p, r_{2p}) = (p^\alpha-2, 3, -p^{\alpha-1}, 0) \in R(2p).$$

By the definition of $P_{m,r}(t)$, we have

$$P_{m,r^{(\alpha,p)}}(t) = \left\{t' \mid t' \equiv ts + (s-1)/6 \pmod m, 0\leq t'\leq m-1, [s]_{24m}\in \mathbb{S}_{24m}\right\}.$$

One readily verifies $P_{625,r^{(3,5)}}(229) = \{229, 604\}$. Next we set

$$(m, M, N, t, r = (r_1, r_2, r_5, r_{10})) = (625, 10, 10, 229, (123, 3, -25, 0)) \in \Delta^*$$

and

$$r' = (r'_1, r'_2, r'_5, r'_{10}) = (13, 0, 0, 0).$$

Moreover, by [6, Lemma 2.6], $\{\gamma_\delta : \delta \mid N\}$ contains a complete set of representatives of the double coset $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$ where $\gamma_\delta = \begin{pmatrix}1 & 0\\ \delta & 1\end{pmatrix}$. One may see that all these constants satisfy the assumption of Lemma 1. We therefore obtain $\lfloor v\rfloor = 84$.

To get the upper bound $\lfloor v\rfloor$ for Theorem 2, we have $P_{1375,r^{(1,11)}}(1054) = \{779, 1054\}$. Similarly, we can compute other relevant constants of (16) and (17), which are listed in Table 3.1.

$P_{1375,r^{(1,11)}}(1054) = \{779, 1054\}$
$(m, M, N, t, r = (r_1, r_2, r_{11}, r_{22})) = (1375, 22, 110, 1054, (9, 3, -1, 0))$
$r' = (r'_1, r'_2, r'_5, r'_{10}, r'_{11}, r'_{22}, r'_{55}, r'_{110}) = (6, 0, 0, 0, 0, 0, 0, 0)$
$\lfloor v \rfloor = 152$

Table 1: Relevant constants of (16) and (17)

3.2. Simplifying the Verification

We should notice that as n approaches the upper bound $\lfloor v \rfloor$ in both theorems, the verification will become difficult. Hence we provide a method that can simplify the calculation. First, we notice that

$$\sum_{n\geq 0} g(n)q^n = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2} = \frac{1}{(q^2;q^2)_\infty}\left(\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}\right)^2. \tag{18}$$

From [2, Chapter 16, Entry 22(ii)] we know that

$$\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} = \sum_{n\geq 0} q^{T_n},$$

where $T_n = n(n+1)/2$ is the triangular number. Now we write

$$\left(\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}\right)^2 = \sum_{n\geq 0} a(n)q^n.$$

Then

$$a(n) = \sharp\left\{(n_1, n_2) \in (\mathbb{N}\cup\{0\})^2 : n = T_{n_1} + T_{n_2}\right\}.$$

Notice that $n = T_{n_1} + T_{n_2}$ implies $8n+2 = (2n_1+1)^2 + (2n_2+1)^2$. Now if $8n+2$ is a sum of two squares, then both the squares are odd. This is because a square is congruent to $0, 1, 4$ modulo 8. We therefore have

$$4a(n) = r(8n+2),$$

where $r(n)$ denotes the number of representations of n by two squares. For example, $r(5) = 8$ since $5 = (\pm 1)^2 + (\pm 2)^2 = (\pm 2)^2 + (\pm 1)^2$.

Let $p(n)$ be the partition function given by

$$\sum_{n\geq 0} p(n)q^n := \frac{1}{(q;q)_\infty}.$$

It follows by (18) that

$$g(n) = \sum_{\substack{2i+j=n \\ i,j\geq 0}} p(i)a(j) = \frac{1}{4} \sum_{\substack{2i+j=n \\ i,j\geq 0}} p(i)r(8j+2).$$

Note that $p(n)$ and $r(n)$ are computable by *Mathematica* via functions `PartitionsP` and `SquaresR`, respectively. Thus we can complete our verification with much less time. In fact, with the help of *Mathematica*, we see that (14) and (15) hold up to the bound $\lfloor v \rfloor = 84$, and thus they hold for all $n \geq 0$ by Lemma 1. This completes our proof of Theorem 1. We also end our proof of Theorem 2 by a similar verification.

Acknowledgments The author thanks S. Radu for interpreting the definition of Δ^*.

References

[1] G. E. Andrews, P. Paule, and C. Schneider, Plane partitions. VI. Stembridge's TSPP theorem, *Adv. in Appl. Math.* **34** (2005), no. 4, 709–739.

[2] B. C. Berndt, *Ramanujan's Notebooks. Part III*, Springer-Verlag, New York, 1991. xiv+510 pp.

[3] A. Blecher, Geometry for totally symmetric plane partitions (TSPPs) with self-conjugate main diagonal, *Util. Math.* **88** (2012), 223–235.

[4] M. D. Hirschhorn and J. A. Sellers, Arithmetic properties of 1-shell totally symmetric plane partitions, *Bull. Aust. Math. Soc.* **89** (2014), no. 3, 473–478.

[5] S. Radu, An algorithmic approach to Ramanujan's congruences, *Ramanujan J.* **20** (2009), no. 2, 215–251.

[6] S. Radu and J. A. Sellers, Congruence properties modulo 5 and 7 for the pod function, *Int. J. Number Theory* **7** (2011), no. 8, 2249–2259.

[7] J. R. Stembridge, The enumeration of totally symmetric plane partitions, *Adv. Math.* **111** (1995), no. 2, 227–243.

[8] E. X. W. Xia, A new congruence modulo 25 for 1-shell totally symmetric plane partitions, *Bull. Aust. Math. Soc.* **91** (2015), no. 1, 41–46.

[9] O. X. M. Yao, New infinite families of congruences modulo 4 and 8 for 1-shell totally symmetric plane partitions, *Bull. Aust. Math. Soc.* **90** (2014), no. 1, 37–46.