NEW CONGRUENCES FOR $\ell$-REGULAR OVERPARTITIONS

Shane Chern
Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania
shanechern@psu.edu

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Abstract
Recently, Shen (2016) and Alanazi et al. (2016) studied the arithmetic properties of the $\ell$-regular overpartition function $A_\ell(n)$, which counts the number of overpartitions of $n$ into parts not divisible by $\ell$. In this note, we will present some new congruences modulo 5 when $\ell$ is a power of 5.

1. Introduction
A partition of a natural number $n$ is a nonincreasing sequence of positive integers whose sum is $n$. For example, $6 = 3 + 2 + 1$ is a partition of 6. Let $p(n)$ denote the number of partitions of $n$. We also agree that $p(0) = 1$. It is well-known that the generating function of $p(n)$ is given by

$$\sum_{n \geq 0} p(n)q^n = \frac{1}{(q; q)_{\infty}},$$

where we adopt the standard notation

$$(a; q)_{\infty} = \prod_{n \geq 0} (1 - aq^n).$$

For any positive integer $\ell$, a partition is called $\ell$-regular if none of its parts are divisible by $\ell$. Let $b_\ell(n)$ denote the number of $\ell$-regular partitions of $n$. We know that its generating function is

$$\sum_{n \geq 0} b_\ell(n)q^n = \frac{(q^\ell; q^\ell)_{\infty}}{(q; q)_{\infty}}.$$

On the other hand, an overpartition of $n$ is a partition of $n$ in which the first occurrence of each distinct part can be overlined. Let $\overline{p}(n)$ be the number of overpartitions of $n$. We also know that the generating function of $\overline{p}(n)$ is

$$\sum_{n \geq 0} \overline{p}(n)q^n = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}.$$
Many authors have studied the arithmetic properties of $b_\ell(n)$ and $p(n)$. We refer the interested readers to the “Introduction” part of [6] and references therein for detailed description.

In [5], Lovejoy introduced a function $\mathcal{A}_\ell(n)$, which counts the number of over-partitions of $n$ into parts not divisible by $\ell$. According to Shen [6], this type of partition can be named as $\ell$-regular overpartition. He also obtained the generating function of $\mathcal{A}_\ell(n)$, that is,

$$\sum_{n \geq 0} \mathcal{A}_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty^2(q^2;q^2)_\infty}{(q;q)_\infty^2(q^{2\ell};q^{2\ell})_\infty}. \quad (1)$$

Meanwhile, he presented several congruences for $\mathcal{A}_3(n)$ and $\mathcal{A}_4(n)$. For $\mathcal{A}_3(n)$, he got

$$\mathcal{A}_3(4n+1) \equiv 0 \pmod{2},$$
$$\mathcal{A}_3(4n+3) \equiv 0 \pmod{6},$$
$$\mathcal{A}_3(9n+3) \equiv 0 \pmod{6},$$
$$\mathcal{A}_3(9n+6) \equiv 0 \pmod{24}.$$

More recently, Alanazi et al. [1] further studied the arithmetic properties of $\mathcal{A}_\ell(n)$ under modulus 3 when $\ell$ is a power of 3. They also gave some congruences satisfied by $\mathcal{A}_\ell(n)$ modulo 2 and 4.

In this note, our main purpose is to study the arithmetic properties of $\mathcal{A}_\ell(n)$ when $\ell$ is a power of 5. When $\ell = 5$ and 25, we connect $\mathcal{A}_\ell(n)$ with $r_4(n)$ and $r_8(n)$ respectively, where $r_k(n)$ denotes the number of representations of $n$ by $k$ squares. The method for $\ell = 5$ also applies to other prime $\ell$. When $\ell = 125$, we show that

$$\mathcal{A}_{125}(25n) \equiv \mathcal{A}_{125}(625n) \pmod{5}.$$ 

This can be regarded as an analogous result of the following congruence for over-partition function $p(n)$ proved by Chen et al. (see [4, Theorem 1.5])

$$p(25n) \equiv p(625n) \pmod{5}.$$ 

When $\ell = 5^\alpha$ with $\alpha \geq 4$, we obtain new congruences similar to a result of Alanazi et al. (see [1, Theorem 3]), which states that $\mathcal{A}_{5^\alpha}(27n+18) \equiv 0 \pmod{3}$ holds for all $n \geq 0$ and $\alpha \geq 3$.

2. New Congruence Results

2.1. $\ell = 5$

One readily sees from the binomial theorem that for any prime $p$,

$$(q;q)^p_\infty \equiv (q^p;q^p)_\infty \pmod{p}. \quad (2)$$
Setting \( p = 5 \) and applying it to (1), we have
\[
\sum_{n \geq 0} \overline{A}_5(n)q^n \equiv \left( \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty} \right)^4 \pmod{5}.
\] (3)

Now let
\[
\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}.
\]
It is well-known that (see [2, p. 37, Eq. (22.4)])
\[
\varphi(-q) = \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty}.
\]

We therefore have

**Theorem 1.** For any positive integer \( n \), we have
\[
\overline{A}_5(n) \equiv \begin{cases} 
  r_4(n) & \text{if } n \text{ is even} \\
  -r_4(n) & \text{if } n \text{ is odd} 
\end{cases} \pmod{5}.
\] (4)

We know from [3, Theorem 3.3.1] that \( r_4(n) = 8d^*(n) \) where
\[
d^*(n) = \sum_{d|n, 4|d} d.
\]

Let \( p \neq 5 \) be an odd prime and \( k \) be a nonnegative integer. One readily verifies that
\[
\sum_{i=0}^{4k+3} p^i \equiv (k+1) \sum_{i=1}^{4} i \equiv 0 \pmod{5}.
\]

Note also that \( d^*(n) \) is multiplicative. Thus we conclude

**Theorem 2.** Let \( p \neq 5 \) be an odd prime and \( k \) be a nonnegative integer. Let \( n \) be a nonnegative integer. We have
\[
\overline{A}_5(p^{4k+3}(pn + i)) \equiv 0 \pmod{5},
\] (5)

where \( i \in \{1, 2, \ldots, p-1\} \).

**Example 1.** If we take \( p = 3, k = 0, \) and \( i = 1, \) then
\[
\overline{A}_5(81n + 27) \equiv 0 \pmod{5}
\] (6)
holds for all \( n \geq 0 \).
We also note that if an odd prime \( p \) is congruent to 9 modulo 10, then \( 1 + p \equiv 0 \) (mod 5). We therefore have \( \overline{A}_5(p(pm+i)) \equiv 0 \) (mod 5) for \( i \in \{1, 2, \ldots, p-1\} \). On the other hand, if we require \( 1 + p + p^2 \equiv 0 \) (mod 5), then \( (2p+1)^2 \equiv -3 \) (mod 5). However, since \(-3/5 = -1\) (here \((\ast|\ast)\) denotes the Legendre symbol), such \( p \) does not exist. The above observation yields

**Theorem 3.** Let \( p \equiv 9 \) (mod 10) be a prime and \( n \) be a nonnegative integer. We have

\[
\overline{A}_5(p(pm+i)) \equiv 0 \pmod{5},
\]

where \( i \in \{1, 2, \ldots, p-1\} \).

**Example 2.** If we take \( p = 19 \) and \( i = 1 \), then

\[
\overline{A}_5(361n+19) \equiv 0 \pmod{5}
\]

holds for all \( n \geq 0 \).

We should mention that this method also applies to other primes \( \ell \). In fact, if we set \( p = \ell \) in (2) and apply it to (1), then

\[
\overline{A}_\ell(n) \equiv \begin{cases} r_{\ell-1}(n) & \text{if } n \text{ is even} \\ -r_{\ell-1}(n) & \text{if } n \text{ is odd} \end{cases} \pmod{\ell}.
\]

Recall that the explicit formulas of \( r_2(n) \) and \( r_6(n) \) are also known. From [3, Theorems 3.2.1 and 3.4.1], we have

\[
r_2(n) = 4 \sum_{d \mid n} \chi(d),
\]

and

\[
r_6(n) = 16 \sum_{d \mid n} \chi(n/d)d^2 - 4 \sum_{d \mid n} \chi(d)d^2,
\]

where

\[
\chi(n) = \begin{cases} 1 & n \equiv 1 \pmod{4}, \\ -1 & n \equiv 3 \pmod{4}, \\ 0 & \text{otherwise}. \end{cases}
\]

Through a similar argument, we conclude that

**Theorem 4.** For any nonnegative integers \( n, k \), odd prime \( p \), and \( i \in \{1, 2, \ldots, p-1\} \), we have

\[
\begin{align*}
\overline{A}_3(p^{2k+1}(pm+i)) & \equiv 0 \pmod{3} \quad \text{where } p \equiv 3 \pmod{4}, \\
\overline{A}_3(p^{3k+2}(pm+i)) & \equiv 0 \pmod{3} \quad \text{where } p \equiv 1 \pmod{4}, \\
\overline{A}_7(p^{6k+5}(pm+i)) & \equiv 0 \pmod{7} \quad \text{where } p \neq 7.
\end{align*}
\]
2.2. \( \ell = 25 \)

Analogous to the congruences under modulus 3 for \( \mathcal{A}_6(n) \) obtained by Alanazi et al. [1], we will present some arithmetic properties of \( \mathcal{A}_{25}(n) \) modulo 5. Rather than using the technique of dissection identities, we build a connection between \( \mathcal{A}_{25}(5n) \) and \( r_8(n) \) and then apply the explicit formula of \( r_8(n) \). It is necessary to mention that this method also applies to the results of Alanazi et al., as the following relation holds

\[
\mathcal{A}_6(n) \equiv (-1)^n r_8(n) \pmod{3}.
\]

Note that

\[
\sum_{n \geq 0} \mathcal{A}_{25}(n)q^n = \frac{(q^{25}; q^{25})_{\infty}^2 (q^2; q^2)_{\infty}}{(q^{50}; q^{50})_{\infty}^2 (q; q^2)_{\infty}}
\]

\[
\equiv \left( \frac{(q^5; q^5)_{\infty}^2}{(q^{10}; q^{10})_{\infty}} \right)^5 \left( \sum_{n \geq 0} \overline{p}(n)q^n \right) \pmod{5}.
\]

Extracting powers of the form \( q^{5n} \) from both sides and replacing \( q^5 \) by \( q \), we have

\[
\sum_{n \geq 0} \mathcal{A}_{25}(5n)q^n = \frac{(q; q)_{\infty}^2 (q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \left( \sum_{n \geq 0} \overline{p}(5n)q^n \right)
\]

\[
\equiv \varphi(-q)^8 \pmod{5}.
\]

Here we use the following celebrated result due to Treneer [7]

\[
\sum_{n \geq 0} \overline{p}(5n)q^n \equiv \varphi(-q)^3 \pmod{5}.
\]

It is known that the explicit formula of \( r_8(n) \) (see [3, Theorem 3.5.4]) is given by

\[
r_8(n) = 16(-1)^n \sigma_3^-(n),
\]

where

\[
\sigma_3^-(n) = \sum_{d|n} (-1)^d d^3.
\]

We therefore conclude that

**Theorem 5.** For any positive integer \( n \), we have

\[
\mathcal{A}_{25}(5n) \equiv \sigma_3^-(n) \pmod{5}. \tag{10}
\]

One also readily deduces several congruences from Theorem 5.
\textbf{Theorem 6.} Let \( p \neq 5 \) be an odd prime and \( k \) be a nonnegative integer. Let \( n \) be a nonnegative integer. We have
\[
\overline{A}_{25}(5^{4k+3}(pn+i)) \equiv 0 \pmod{5},
\]
where \( i \in \{1, 2, \ldots, p-1\} \).

\textit{Proof.} It is easy to see that
\[
\sigma_3(5^{4k+3}) = -\sum_{i=0}^{4k+3} p^{3i} \equiv -(k+1) \sum_{i=1}^{4} i \equiv 0 \pmod{5}.
\]
Note also that \( \sigma_3(n) \) is multiplicative. The theorem therefore follows. \( \square \)

Furthermore, if \( p \equiv 9 \pmod{10} \) is a prime, then \( 1 + p^3 \equiv 0 \pmod{5} \). This yields
\textbf{Theorem 7.} Let \( p \equiv 9 \pmod{10} \) be a prime and \( n \) be a nonnegative integer. We have
\[
\overline{A}_{25}(5p(pm+i)) \equiv 0 \pmod{5},
\]
where \( i \in \{1, 2, \ldots, p-1\} \).

\subsection*{2.3. \( \ell = 125 \)}

From the generating function (1), we have
\[
\sum_{n \geq 0} \overline{A}_{125}(n)q^n = \frac{(q^{125}; q^{125})_\infty^2}{(q; q^2)_\infty^2(q^{250}; q^{250})_\infty} = \varphi(-q^{125}) \sum_{n \geq 0} \overline{p}(n)q^n.
\]
Extracting terms of the form \( q^{125n} \) and replacing \( q^{125} \) by \( q \), we have
\[
\sum_{n \geq 0} \overline{A}_{125}(125n)q^n = \varphi(-q) \sum_{n \geq 0} \overline{p}(125n)q^n.
\]

According to [4, Eq. (5.3)], we know that \( \overline{p}(125(5n \pm 1)) \equiv 0 \pmod{5} \). We also know from [2, p. 49, Corollary (i)] that
\[
\varphi(-q) = \varphi(-q^{25}) - 2qM_1(-q^5) + 2q^4M_2(-q^5),
\]
where \( M_1(q) = f(q^3, q^7) \) and \( M_2(q) = f(q, q^9) \). Here \( f(a, b) \) is the Ramanujan’s theta function defined as
\[
f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2} = (-a; ab)_\infty(-b; ab)_\infty(ab; ab)_\infty.
\]
Now we extract terms involving $q^{5n}$ from $\sum_{n \geq 0} A_{125}(125n)q^n$ and replace $q^5$ by $q$, then
\[
\sum_{n \geq 0} A_{125}(625n)q^n \equiv \varphi(-q^5) \sum_{n \geq 0} p(625n)q^n \pmod{5}.
\]
Finally, we use the following congruence from [4, Theorem 1.5] for $p(n)$
\[
p(25n) \equiv p(625n) \pmod{5},
\]
and obtain
\[
\sum_{n \geq 0} A_{125}(625n)q^n \equiv \varphi(-q^5) \sum_{n \geq 0} p(25n)q^n \pmod{5},
\]
which coincides with
\[
\sum_{n \geq 0} A_{125}(25n)q^n \equiv \varphi(-q^5) \sum_{n \geq 0} p(25n)q^n \pmod{5}.
\]
We therefore conclude
\begin{align*}
\textbf{Theorem 8.} & \quad \text{For any nonnegative integer } n, \text{ we have} \\
A_{125}(25n) & \equiv A_{125}(625n) \pmod{5}. \tag{13}
\end{align*}

\subsection*{2.4. $t = 5^\alpha$ with $\alpha \geq 4$}

We know from (1) that in this case
\[
\sum_{n \geq 0} A_{5^\alpha}(n)q^n = \varphi\left(-q^{5^\alpha}\right) \sum_{n \geq 0} p(n)q^n.
\]
Note that for $\alpha \geq 4$, $\varphi\left(-q^{5^\alpha}\right)$ is a function of $q^{625}$. Hence $A_{5^\alpha}(625n + 125)$ (resp. $A_{5^\alpha}(625n + 500)$) is a linear combination of values of $p(625n + 125)$ (resp. $p(625n + 500)$). Thanks to [4, Eq. (5.3)], we know that $p(625n + 125) \equiv 0 \pmod{5}$ and $p(625n + 500) \equiv 0 \pmod{5}$ hold for all $n \geq 0$. Hence
\begin{align*}
\textbf{Theorem 9.} & \quad \text{For any nonnegative integer } n \text{ and positive integer } \alpha \geq 4, \text{ we have} \\
A_{5^\alpha}(25n + i) & \equiv 0 \pmod{5}, \tag{14}
\end{align*}
where $i = 125$ and 500.

Note also that for $\alpha \geq 2$, extracting terms of the form $q^{25n}$ and replacing $q^{25}$ by $q$, we obtain
\[
\sum_{n \geq 0} A_{5^\alpha}(25n)q^n = \varphi\left(-q^{5^{\alpha-2}}\right) \sum_{n \geq 0} p(25n)q^n.
\]
On the other hand, we extract terms involving $q^{625n}$ from $\sum_{n \geq 0} A_{5^{n+2}}(n)q^n$ and replace $q^{625}$ by $q$, then

$$\sum_{n \geq 0} A_{5^{n+2}}(625n)q^n = \varphi \left( -q^{5^{n+2}} \right) \sum_{n \geq 0} p(625n)q^n.$$ 

Thanks again to [4, Theorem 1.5], which states that $p(25n) \equiv p(625n) \pmod{5}$ for all $n \geq 0$, we conclude

**Theorem 10.** For any nonnegative integer $n$, we have

$$A_{5^n}(25n) \equiv A_{5^{n+2}}(625n) \pmod{5} \quad (15)$$

for all $\alpha \geq 2$.

It follows from Theorems 9 and 10 that

**Theorem 11.** For any nonnegative integer $n$ and positive integer $\alpha \geq 4$, we have

$$A_{5^n}(5^{2j}(625n+i)) \equiv 0 \pmod{5} \quad (16)$$

for all integers $0 \leq j \leq (\alpha - 4)/2$, where $i = 125$ and 500.

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**References**


