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NEW CONGRUENCES FOR ℓ-REGULAR OVERPARTITIONS

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Abstract

Recently, Shen (2016) and Alanazi et al. (2016) studied the arithmetic properties of the ℓ-regular overpartition function $\overline{A}_\ell(n)$, which counts the number of overpartitions of n into parts not divisible by ℓ. In this note, we will present some new congruences modulo 5 when ℓ is a power of 5.

1. Introduction

A *partition* of a natural number n is a nonincreasing sequence of positive integers whose sum is n. For example, $6 = 3 + 2 + 1$ is a partition of 6. Let $p(n)$ denote the number of partitions of n. We also agree that $p(0) = 1$. It is well-known that the generating function of $p(n)$ is given by

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where we adopt the standard notation

$$(a;q)_\infty = \prod_{n\geq 0}(1-aq^n).$$

For any positive integer ℓ, a partition is called *ℓ-regular* if none of its parts are divisible by ℓ. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n. We know that its generating function is

$$\sum_{n\geq 0} b_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty}.$$

On the other hand, an *overpartition* of n is a partition of n in which the first occurrence of each distinct part can be overlined. Let $\overline{p}(n)$ be the number of overpartitions of n. We also know that the generating function of $\overline{p}(n)$ is

$$\sum_{n\geq 0} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}.$$

Many authors have studied the arithmetic properties of $b_\ell(n)$ and $\overline{p}(n)$. We refer the interested readers to the "Introduction" part of [6] and references therein for detailed description.

In [5], Lovejoy introduced a function $\overline{A}_\ell(n)$, which counts the number of overpartitions of n into parts not divisible by ℓ. According to Shen [6], this type of partition can be named as *ℓ-regular overpartition*. He also obtained the generating function of $\overline{A}_\ell(n)$, that is,

$$\sum_{n\geq 0}\overline{A}_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty^2(q^2;q^2)_\infty}{(q;q)_\infty^2(q^{2\ell};q^{2\ell})_\infty}. \tag{1}$$

Meanwhile, he presented several congruences for $\overline{A}_3(n)$ and $\overline{A}_4(n)$. For $\overline{A}_3(n)$, he got

$$\begin{aligned}
\overline{A}_3(4n+1) &\equiv 0 \pmod{2},\\
\overline{A}_3(4n+3) &\equiv 0 \pmod{6},\\
\overline{A}_3(9n+3) &\equiv 0 \pmod{6},\\
\overline{A}_3(9n+6) &\equiv 0 \pmod{24}.
\end{aligned}$$

More recently, Alanazi et al. [1] further studied the arithmetic properties of $\overline{A}_\ell(n)$ under modulus 3 when ℓ is a power of 3. They also gave some congruences satisfied by $\overline{A}_\ell(n)$ modulo 2 and 4.

In this note, our main purpose is to study the arithmetic properties of $\overline{A}_\ell(n)$ when ℓ is a power of 5. When $\ell = 5$ and 25, we connect $\overline{A}_\ell(n)$ with $r_4(n)$ and $r_8(n)$ respectively, where $r_k(n)$ denotes the number of representations of n by k squares. The method for $\ell = 5$ also applies to other prime ℓ. When $\ell = 125$, we show that

$$\overline{A}_{125}(25n) \equiv \overline{A}_{125}(625n) \pmod{5}.$$

This can be regarded as an analogous result of the following congruence for overpartition function $\overline{p}(n)$ proved by Chen et al. (see [4, Theorem 1.5])

$$\overline{p}(25n) \equiv \overline{p}(625n) \pmod{5}.$$

When $\ell = 5^\alpha$ with $\alpha \geq 4$, we obtain new congruences similar to a result of Alanazi et al. (see [1, Theorem 3]), which states that $\overline{A}_{3^\alpha}(27n+18) \equiv 0 \pmod{3}$ holds for all $n \geq 0$ and $\alpha \geq 3$.

2. New Congruence Results

2.1. $\ell = 5$

One readily sees from the binomial theorem that for any prime p,

$$(q;q)_\infty^p \equiv (q^p;q^p)_\infty \pmod{p}. \tag{2}$$

Setting $p = 5$ and applying it to (1), we have

$$\sum_{n\geq 0} \overline{A}_5(n)q^n \equiv \left(\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}\right)^4 \pmod{5}. \tag{3}$$

Now let

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}.$$

It is well-known that (see [2, p. 37, Eq. (22.4)])

$$\varphi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}.$$

We therefore have

Theorem 1. *For any positive integer n, we have*

$$\overline{A}_5(n) \equiv \begin{cases} r_4(n) & \text{if } n \text{ is even} \\ -r_4(n) & \text{if } n \text{ is odd} \end{cases} \pmod{5}. \tag{4}$$

We know from [3, Theorem 3.3.1] that $r_4(n) = 8d^*(n)$ where

$$d^*(n) = \sum_{d|n,\, 4\nmid d} d.$$

Let $p \neq 5$ be an odd prime and k be a nonnegative integer. One readily verifies that

$$\sum_{i=0}^{4k+3} p^i \equiv (k+1)\sum_{i=1}^{4} i \equiv 0 \pmod{5}.$$

Note also that $d^*(n)$ is multiplicative. Thus we conclude

Theorem 2. *Let $p \neq 5$ be an odd prime and k be a nonnegative integer. Let n be a nonnegative integer. We have*

$$\overline{A}_5(p^{4k+3}(pn+i)) \equiv 0 \pmod{5}, \tag{5}$$

where $i \in \{1, 2, \ldots, p-1\}$.

Example 1. If we take $p = 3$, $k = 0$, and $i = 1$, then

$$\overline{A}_5(81n + 27) \equiv 0 \pmod{5} \tag{6}$$

holds for all $n \geq 0$.

We also note that if an odd prime p is congruent to 9 modulo 10, then $1+p \equiv 0 \pmod 5$. We therefore have $\overline{A}_5(p(pn+i)) \equiv 0 \pmod 5$ for $i \in \{1,2,\ldots,p-1\}$. On the other hand, if we require $1+p+p^2 \equiv 0 \pmod 5$, then $(2p+1)^2 \equiv -3 \pmod 5$. However, since $(-3|5) = -1$ (here $(*|*)$ denotes the Legendre symbol), such p does not exist. The above observation yields

Theorem 3. *Let $p \equiv 9 \pmod{10}$ be a prime and n be a nonnegative integer. We have*

$$\overline{A}_5(p(pn+i)) \equiv 0 \pmod 5, \tag{7}$$

where $i \in \{1,2,\ldots,p-1\}$.

Example 2. If we take $p = 19$ and $i = 1$, then

$$\overline{A}_5(361n+19) \equiv 0 \pmod 5 \tag{8}$$

holds for all $n \geq 0$.

We should mention that this method also applies to other primes ℓ. In fact, if we set $p = \ell$ in (2) and apply it to (1), then

$$\overline{A}_\ell(n) \equiv \begin{cases} r_{\ell-1}(n) & \text{if } n \text{ is even} \\ -r_{\ell-1}(n) & \text{if } n \text{ is odd} \end{cases} \pmod{\ell}. \tag{9}$$

Recall that the explicit formulas of $r_2(n)$ and $r_6(n)$ are also known. From [3, Theorems 3.2.1 and 3.4.1], we have

$$r_2(n) = 4\sum_{d|n} \chi(d),$$

and

$$r_6(n) = 16\sum_{d|n} \chi(n/d)d^2 - 4\sum_{d|n} \chi(d)d^2,$$

where

$$\chi(n) = \begin{cases} 1 & n \equiv 1 \pmod 4, \\ -1 & n \equiv 3 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Through a similar argument, we conclude that

Theorem 4. *For any nonnegative integers n, k, odd prime p, and $i \in \{1,2,\ldots,p-1\}$, we have*

$$\begin{aligned} &\overline{A}_3(p^{2k+1}(pn+i)) \equiv 0 \pmod 3 \quad \textit{where } p \equiv 3 \pmod 4, \\ &\overline{A}_3(p^{3k+2}(pn+i)) \equiv 0 \pmod 3 \quad \textit{where } p \equiv 1 \pmod 4, \\ &\overline{A}_7(p^{6k+5}(pn+i)) \equiv 0 \pmod 7 \quad \textit{where } p \neq 7. \end{aligned}$$

2.2. $\ell = 25$

Analogous to the congruences under modulus 3 for $\overline{A}_9(n)$ obtained by Alanazi et al. [1], we will present some arithmetic properties of $\overline{A}_{25}(n)$ modulo 5. Rather than using the technique of dissection identities, we build a connection between $\overline{A}_{25}(5n)$ and $r_8(n)$ and then apply the explicit formula of $r_8(n)$. It is necessary to mention that this method also applies to the results of Alanazi et al. as the following relation holds

$$\overline{A}_9(n) \equiv (-1)^n r_8(n) \pmod 3.$$

Note that

$$\begin{aligned}\sum_{n\geq 0} \overline{A}_{25}(n)q^n &= \frac{(q^{25};q^{25})_\infty^2}{(q^{50};q^{50})_\infty}\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}\\ &\equiv \left(\frac{(q^5;q^5)_\infty^2}{(q^{10};q^{10})_\infty}\right)^5 \left(\sum_{n\geq 0}\overline{p}(n)q^n\right) \pmod 5.\end{aligned}$$

Extracting powers of the form q^{5n} from both sides and replacing q^5 by q, we have

$$\begin{aligned}\sum_{n\geq 0} \overline{A}_{25}(5n)q^n &\equiv \left(\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}\right)^5 \left(\sum_{n\geq 0}\overline{p}(5n)q^n\right)\\ &\equiv \varphi(-q)^8 \pmod 5.\end{aligned}$$

Here we use the following celebrated result due to Treneer [7]

$$\sum_{n\geq 0}\overline{p}(5n)q^n \equiv \varphi(-q)^3 \pmod 5.$$

It is known that the explicit formula of $r_8(n)$ (see [3, Theorem 3.5.4]) is given by

$$r_8(n) = 16(-1)^n \sigma_3^-(n),$$

where

$$\sigma_3^-(n) = \sum_{d|n}(-1)^d d^3.$$

We therefore conclude that

Theorem 5. *For any positive integer n, we have*

$$\overline{A}_{25}(5n) \equiv \sigma_3^-(n) \pmod 5. \tag{10}$$

One also readily deduces several congruences from Theorem 5.

Theorem 6. *Let $p \neq 5$ be an odd prime and k be a nonnegative integer. Let n be a nonnegative integer. We have*

$$\overline{A}_{25}(5p^{4k+3}(pn+i)) \equiv 0 \pmod 5, \tag{11}$$

where $i \in \{1, 2, \ldots, p-1\}$.

Proof. It is easy to see that

$$\sigma_3^-(p^{4k+3}) = -\sum_{i=0}^{4k+3} p^{3i} \equiv -(k+1)\sum_{i=1}^{4} i \equiv 0 \pmod 5.$$

Note also that $\sigma_3^-(n)$ is multiplicative. The theorem therefore follows. □

Furthermore, if $p \equiv 9 \pmod{10}$ is a prime, then $1+p^3 \equiv 0 \pmod 5$. This yields

Theorem 7. *Let $p \equiv 9 \pmod{10}$ be a prime and n be a nonnegative integer. We have*

$$\overline{A}_{25}(5p(pn+i)) \equiv 0 \pmod 5, \tag{12}$$

where $i \in \{1, 2, \ldots, p-1\}$.

2.3. $\ell = 125$

From the generating function (1), we have

$$\sum_{n\geq 0} \overline{A}_{125}(n)q^n = \frac{(q^{125};q^{125})_\infty^2 (q^2;q^2)_\infty}{(q;q)_\infty^2 (q^{250};q^{250})_\infty} = \varphi(-q^{125}) \sum_{n\geq 0} \overline{p}(n)q^n.$$

Extracting terms of the form q^{125n} and replacing q^{125} by q, we have

$$\sum_{n\geq 0} \overline{A}_{125}(125n)q^n = \varphi(-q) \sum_{n\geq 0} \overline{p}(125n)q^n.$$

According to [4, Eq. (5.3)], we know that $\overline{p}(125(5n \pm 1)) \equiv 0 \pmod 5$. We also know from [2, p. 49, Corollary (i)] that

$$\varphi(-q) = \varphi(-q^{25}) - 2qM_1(-q^5) + 2q^4 M_2(-q^5),$$

where $M_1(q) = f(q^3, q^7)$ and $M_2(q) = f(q, q^9)$. Here $f(a, b)$ is the Ramanujan's theta function defined as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty.$$

Now we extract terms involving q^{5n} from $\sum_{n\geq 0}\overline{A}_{125}(125n)q^n$ and replace q^5 by q, then

$$\sum_{n\geq 0}\overline{A}_{125}(625n)q^n \equiv \varphi(-q^5)\sum_{n\geq 0}\overline{p}(625n)q^n \pmod 5.$$

Finally, we use the following congruence from [4, Theorem 1.5] for $\overline{p}(n)$

$$\overline{p}(25n) \equiv \overline{p}(625n) \pmod 5,$$

and obtain

$$\sum_{n\geq 0}\overline{A}_{125}(625n)q^n \equiv \varphi(-q^5)\sum_{n\geq 0}\overline{p}(25n)q^n \pmod 5,$$

which coincides with

$$\sum_{n\geq 0}\overline{A}_{125}(25n)q^n \equiv \varphi(-q^5)\sum_{n\geq 0}\overline{p}(25n)q^n \pmod 5.$$

We therefore conclude

Theorem 8. *For any nonnegative integer n, we have*

$$\overline{A}_{125}(25n) \equiv \overline{A}_{125}(625n) \pmod 5. \tag{13}$$

2.4. $\ell = 5^\alpha$ with $\alpha \geq 4$

We know from (1) that in this case

$$\sum_{n\geq 0}\overline{A}_{5^\alpha}(n)q^n = \varphi\left(-q^{5^\alpha}\right)\sum_{n\geq 0}\overline{p}(n)q^n.$$

Note that for $\alpha \geq 4$, $\varphi\left(-q^{5^\alpha}\right)$ is a function of q^{625}. Hence $\overline{A}_{5^\alpha}(625n+125)$ (resp. $\overline{A}_{5^\alpha}(625n+500)$) is a linear combination of values of $\overline{p}(625n+125)$ (resp. $\overline{p}(625n+500)$). Thanks to [4, Eq. (5.3)], we know that $\overline{p}(625n+125) \equiv 0 \pmod 5$ and $\overline{p}(625n+500) \equiv 0 \pmod 5$ hold for all $n \geq 0$. Hence

Theorem 9. *For any nonnegative integer n and positive integer $\alpha \geq 4$, we have*

$$\overline{A}_{5^\alpha}(625n+i) \equiv 0 \pmod 5, \tag{14}$$

where $i = 125$ and 500.

Note also that for $\alpha \geq 2$, extracting terms of the form q^{25n} and replacing q^{25} by q, we obtain

$$\sum_{n\geq 0}\overline{A}_{5^\alpha}(25n)q^n = \varphi\left(-q^{5^{\alpha-2}}\right)\sum_{n\geq 0}\overline{p}(25n)q^n.$$

On the other hand, we extract terms involving q^{625n} from $\sum_{n\geq 0}\overline{A}_{5^{\alpha+2}}(n)q^n$ and replace q^{625} by q, then

$$\sum_{n\geq 0}\overline{A}_{5^{\alpha+2}}(625n)q^n = \varphi\left(-q^{5^{\alpha-2}}\right)\sum_{n\geq 0}\overline{p}(625n)q^n.$$

Thanks again to [4, Theorem 1.5], which states that $\overline{p}(25n) \equiv \overline{p}(625n) \pmod 5$ for all $n \geq 0$, we conclude

Theorem 10. *For any nonnegative integer n, we have*

$$\overline{A}_{5^\alpha}(25n) \equiv \overline{A}_{5^{\alpha+2}}(625n) \pmod 5 \tag{15}$$

for all $\alpha \geq 2$.

It follows from Theorems 9 and 10 that

Theorem 11. *For any nonnegative integer n and positive integer $\alpha \geq 4$, we have*

$$\overline{A}_{5^\alpha}(5^{2j}(625n+i)) \equiv 0 \pmod 5 \tag{16}$$

for all integers $0 \leq j \leq (\alpha-4)/2$, where $i = 125$ and 500.

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References

[1] A. M. Alanazi, A. O. Munagi, and J. A. Sellers, An infinite family of congruences for ℓ-regular overpartitions, *Integers* **16** (2016), Paper No. A37, 8 pp.

[2] B. C. Berndt, *Ramanujan's Notebooks. Part III*, Springer-Verlag, New York, 1991.

[3] B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, Student Mathematical Library, **34**. American Mathematical Society, Providence, RI, 2006.

[4] W. Y. C. Chen, L. H. Sun, R. H. Wang, L. Zhang, Ramanujan-type congruences for overpartitions modulo 5, *J. Number Theory* **148** (2015), 62–72.

[5] J. Lovejoy, Gordon's theorem for overpartitions, *J. Combin. Theory Ser. A* **103** (2003), no. 2, 393–401.

[6] E. Y. Y. Shen, Arithmetic properties of l-regular overpartitions, *Int. J. Number Theory* **12** (2016), no. 3, 841–852.

[7] S. Treneer, Congruences for the coefficients of weakly holomorphic modular forms, *Proc. London Math. Soc. (3)* **93** (2006), no. 2, 304–324.