#A23 INTEGERS 17 (2017)

WEAKENED RAMSEY NUMBERS AND THEIR HYPERGRAPH ANALOGUES

Mark Budden
Department of Mathematics and Computer Science, Western Carolina University, Cullowhee, North Carolina
mrbudden@email.wcu.edu

Matthew Stender
College of Engineering and Technology, Western Carolina University, Cullowhee, North Carolina
mgstender1@catamount.wcu.edu

Yanbo Zhang
College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, P.R. China
ybzhang@163.com

Received: 2/15/16, Revised: 4/19/17, Accepted: 5/29/17, Published: 6/23/17

Abstract

Weakened Ramsey numbers are a natural generalization of the definition of a t-colored Ramsey number $R^t(G)$, defined to be the least natural number p such that every t-coloring of the edges in the complete graph K_p has a monochromatic subgraph isomorphic to G. For $s < t$, one can define $R^t_s(G)$ to be the least natural number p such that every t-coloring of the edges in K_p contains a subgraph isomorphic to G that is spanned by edges in at most s colors. It follows that $R^t_1(G) = R^t(G)$, but few explicit values are known for values of $s > 1$. The goals of this article are two-fold. First, we show how the work of Su, Li, Luo, and Li done in 2002 can be used to derive new lower bounds for fifteen weakened Ramsey numbers of the form $R^3_2(K_n)$. Then we turn our attention to the analogue of weakened Ramsey numbers in the setting of r-uniform hypergraphs, proving some explicit and general bounds for such numbers.

1. Introduction

A common generalization of classical Ramsey numbers is the addition of multiple colors. In particular, one can define the Ramsey number $R^t(G)$ to be the least natural number n such that every coloring of the edges of the complete graph K_n

on n vertices using t colors results in a monochromatic subgraph isomorphic to the graph G. In 1977, Chung, Chung, and Liu [3] further generalized this definition by allowing for subgraphs isomorphic to G using at most s of the t colors ($1 \leq s < t$). Define $R_s^t(G)$ to be the least natural number n such that every coloring of the edges of K_n using t colors results in a subgraph isomorphic to G that is spanned by edges using at most s colors. The existence of these "weakened" Ramsey numbers follows from the observation that

$$R_s^t(G) \leq R^t(G).$$

Several authors have considered $R_s^t(G)$ (eg., see [3], [4], [6], and [7]) and many explicit values are now known. In Section 2, we review known bounds for certain small weakened Ramsey numbers and show how the work of Su, Li, Luo, and Li [9] on cubic residue graphs allows for the determination of new lower bounds for $R_2^3(K_n)$ for fifteen values of n.

Next, we turn our attention to generalizing the definition of $R_s^t(G)$ to hypergraphs and proving some preliminary bounds. Recall that an r-uniform hypergraph $H = (V, E)$ consists of a nonempty finite set V of vertices and a set E of unordered r-tuples of distinct elements in V, called hyperedges (or r-edges). We denote by $K_n^{(r)}$ the complete r-uniform hypergraph on n vertices, where $n = 1$ or $n \geq r$.

Analogous to $R_s^t(G)$, the weakened hypergraph Ramsey number $R_s^t(H; r)$ is the least natural number n such that every coloring of the r-edges in $K_n^{(r)}$ using t colors results in a subhypergraph isomorphic to H that is spanned by hyperedges using at most s colors. Anytime we refer to an s-colored hypergraph (or graph), we mean one that is spanned by hyperedges (or edges) using at most s colors. In other words, an s-colored hypergraph may use fewer than s colors.

In Section 3, we offer explicit lower and upper bounds for certain small weakened hypergraph Ramsey numbers. In Section 4, a couple of general results are given that are independent of the specific choices of s and t. Finally, we conclude with some directions for future inquiry involving weakened Ramsey numbers for both graphs and r-uniform hypergraphs.

2. Bounds for $R_2^3(K_n)$ when $n \geq 4$

As a simple first example, consider the weakened Ramsey number $R_2^3(K_3)$ (first considered in [4]). The 3-coloring of the edges of K_4 in Figure 1 lacks 2-colored triangles, from which it follows that $R_2^3(K_3) \geq 5$. Also, for every 3-coloring of the edges in K_5, a vertex x is incident with at least two edges that are the same color resulting in a 2-colored triangle, implying that $R_2^3(K_3) \leq 5$. In addition to $R_2^3(K_3) = 5$, it is known that $R_2^3(K_4) = 10$ [4], but little is known about $R_2^3(K_n)$ when $n \geq 5$. These weakened Ramsey numbers will be the focus of this section.

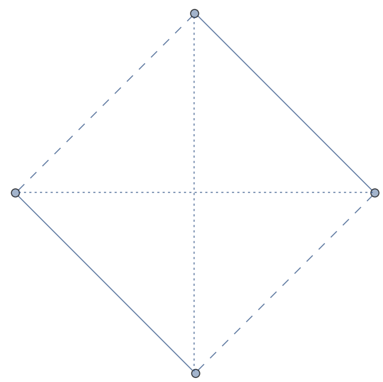

Figure 1: A 3-coloring of the edges of K_4 that lacks a 2-colored triangle.

Building on the ubiquitous role Paley graphs have played in the determination of lower bounds for certain diagonal Ramsey numbers, in 2002, Su, Li, Luo, and Li [9] developed an algorithm for determining the clique numbers of cubic residue graphs and their complements. Using their algorithm, they were able to produce 16 new lower bounds for non-diagonal Ramsey numbers. In this section, we show how their results on the complements of cubic residue graphs can be used to obtain new lower bounds for fifteen weakened Ramsey numbers of the form $R_2^3(K_n)$, when $n \geq 5$.

Let $p \equiv 1 \pmod 6$ be a prime and $\mathbb{F}_p$ the finite field of order p. The multiplicative group $\mathbb{F}_p^\times$ is cyclic, so we can write $\mathbb{F}_p^\times = \langle g \rangle$ for some $g \in \mathbb{F}_p^\times$. Let $\chi : \mathbb{F}_p^\times \longrightarrow \mathbb{C}^\times$ be a cubic character (ie., a character of order 3). That is, χ is a homomorphism, and if we let $\zeta_3 = e^{2\pi i/3}$, then $\chi(g)$ maps to either ζ_3 or ζ_3^2, and this choice uniquely determines χ. Without loss of generality, assume that g is a fixed generator that satisfies $\chi(g) = \zeta_3$. An arbitrary element in $\mathbb{F}_p^\times$ has the form g^k and the congruence class of k modulo 3 uniquely determines its image under χ.

The cubic residue graph G_p is defined to have vertex set $V(G_p) = \mathbb{F}_p$ and edge set

$$E(G_p) = \{ab \mid b - a \in Ker(\chi)\}.$$

The assumption $p \equiv 1 \pmod 6$ guarantees that $\chi(b - a) = \chi(a - b)$, making G_p well-defined. It should also be observed that the set of cubic residues

$$\mathbb{F}_p^{\times 3} = \{y \in \mathbb{F}_p^\times \mid y = x^3 \text{ for some } x \in \mathbb{F}_p^\times\}$$

is the kernel of χ. The algorithm developed by Su, Li, Luo, and Li [9] allowed them to determine the clique numbers $\omega(G_p)$ and $\omega(\overline{G_p})$ for primes $p \equiv 1 \pmod 6$ up to 1327. The following theorem will enable us to apply their results in [9] to weakened Ramsey numbers.

Theorem 1. *Let $p \equiv 1 \pmod 6$ be a prime and G_p be the corresponding cubic residue graph. If $n = \omega(\overline{G_p})$, then $R_2^3(K_{n+1}) > p$.*

Proof. Form a 3-coloring of the edges in K_p by identifying the vertices with $\mathbb{F}_p$ and coloring each edge ab according to which left coset of $Ker(\chi)$ the difference $b-a$ is in. If $b - a \in Ker(\chi)$, then color ab red. If $b-a \in gKer(\chi)$, then color ab blue. Finally, if $b-a \in g^2Ker(\chi)$, then color ab green. The graph spanned by the red edges is isomorphic to G_p. Let G_p' be the subgraph spanned by the blue edges and G_p'' be the subgraph spanned by the green edges. The maps $\sigma_1 : V(G_p) \longrightarrow G_p'$ given by $x \mapsto gx$ and $\sigma_2 : V(G_p) \longrightarrow G_p''$ given by $x \mapsto g^2x$ are easily confirmed to be isomorphisms. Of course, this implies that $\overline{G_p}$, $\overline{G_p'}$, and $\overline{G_p''}$ are isomorphic and these are precisely the subgraphs that include exactly two of the three colors. From this construction, it follows that $R_2^3(K_{\omega(\overline{G_p})+1}) > p$. $\square$

p	$\omega(\overline{G_p})$	Ramsey bound
19	4	$R_2^3(K_5) > 19$
31	6	$R_2^3(K_7) > 31$
43	7	$R_2^3(K_8) > 43$
97	8	$R_2^3(K_9) > 97$
109	9	$R_2^3(K_{10}) > 109$
163	10	$R_2^3(K_{11}) > 163$
229	11	$R_2^3(K_{12}) > 229$
349	13	$R_2^3(K_{14}) > 349$
439	14	$R_2^3(K_{15}) > 439$
487	15	$R_2^3(K_{16}) > 487$
673	16	$R_2^3(K_{17}) > 673$
769	17	$R_2^3(K_{18}) > 769$
1051	18	$R_2^3(K_{19}) > 1051$
1303	19	$R_2^3(K_{20}) > 1303$
1327	20	$R_2^3(K_{21}) > 1327$

Table 1: Lower bounds for some weakened Ramsey numbers $R_2^3(K_n)$.

Using the results from Table 1 of [9] for the clique numbers of $\overline{G_p}$, we immediately obtain the lower bounds for $R_2^3(K_n)$ given in Table 1.

3. Weakened Hypergraph Ramsey Numbers

Now, we turn our attention to $R_s^t(H;r)$, the analogues of weakened Ramsey numbers in the setting of r-uniform hypergraphs. The first nontrivial case that we consider is $R_2^3(K_4^{(3)};3)$. Theorem 2 gives a lower bound, while Theorem 3 provides an upper bound.

Theorem 2. $R_2^3(K_4^{(3)};3) \geq 7$.

Proof. The hypergraph $K_6^{(3)}$ has 20 hyperedges. If we denote the vertices by a, b, c, d, e, and f, then we can color the hyperedges as follows.

$$\begin{aligned} Red &: abc, abd, cde, adf, bcf, bef \\ Blue &: bcd, ade, abe, acf, bdf, cef \\ Green &: acd, ace, bce, bde, abf, cdf, def \end{aligned}$$

One can check directly that removing any two vertices results in a $K_4^{(3)}$ that contains hyperedges in all three colors. Thus, we have constructed a 3-coloring of the hyperedges in $K_6^{(3)}$ that lacks a 2-colored $K_4^{(3)}$. □

As with classical Ramsey numbers, constructions provide lower bounds, while upper bounds are achieved by more theoretical means.

Theorem 3. $R_2^3(K_4^{(3)};3) \leq 8$.

Proof. We use $K_4^{(3)} - t$ to denote $K_4^{(3)}$ with one hyperedge deleted. From Page 52 in [8], we see that $R^2(K_4^{(3)} - t, K_4^{(3)};3) = 8$. That is, for any 3-coloring of $K_8^{(3)}$ with colors red, blue, and green, there is either a red $K_4^{(3)} - t$, or a $K_4^{(3)}$ with colors blue and green. In both cases, there is a 2-colored $K_4^{(3)}$. □

Theorem 4. *The weakened Ramsey number* $R_3^4(K_4^{(3)};3) = 5$.

Proof. Trivially, $R_3^4(K_4^{(3)};3) > 4$ since the four hyperedges in $K_4^{(3)}$ could each receive a different color. Assume that $R_3^4(K_4^{(3)};3) > 5$. Then there exists a 4-coloring of the hyperedges in $K_5^{(3)}$ such that each of the five subsets of four vertices can have their underlying subhypergraphs colored using all four colors. However, each edge among these four sets is shared with exactly one other set, so each edge is counted twice among the five subsets of four vertices and at most two distinct hyperedges of the $K_5^{(3)}$ can receive the same color. This leads to at most eight hyperedges, leaving two uncolored, giving a contradiction. Thus,

$$R_3^4(K_4^{(3)};3) \leq 5,$$

resulting in the equality in the statement of the theorem. □

4. General Constructive Bounds

Now, we shift our attention to proving a couple of general inequalities for weakened hypergraph Ramsey numbers. The following theorem applies to two of the weakened (hypergraph) Ramsey numbers that we have already considered ($R_2^3(K_3)$ and $R_3^4(K_4^{(3)};3)$). In the case of $R_2^3(K_3)$, it agrees with the upper bound noted at the beginning of Section 2, and in the case of $R_3^4(K_4^{(3)};3)$, it is weaker than the upper bound we found in Theorem 4.

Theorem 5. *For $r \geq 2$, we have $R_r^{r+1}(K_{r+1}^{(r)};r) \leq 2r+1$.*

Proof. Pick $v_1, v_2, \ldots, v_{r-1} \in V(K_{2r+1}^{(r)})$. There will be

$$2r + 1 - (r - 1) = r + 2$$

hyperedges that include $v_1, v_2, \ldots, v_{r-1}$, and one other vertex. Coloring these hyperedges with $r+1$ colors, there must exist two hyperedges colored the same color. Without loss of generality, assume that $av_1v_2 \cdots v_{r-1}$ and $bv_1v_2 \cdots v_{r-1}$ are red. The remaining $r-1$ hyperedges that include both a, b, and $r-2$ vertices from $\{v_1, v_2, \ldots, v_{r-1}\}$ can be colored with any arbitrary $r-1$ colors other than red. Then we are able to color a $K_{r+1}^{(r)}$ using at most r colors. □

Theorem 6. *If $m > r \geq 2$, then*

$$R_s^t(K_{(m-1)^2+1}^{(r)};r) > (R_s^t(K_m^{(r)};r) - 1)^2.$$

Proof. Let $n = R_s^t(K_m^{(r)};r) - 1$ and consider n copies of $K_n^{(r)}$. Color the hyperedges within each copy of $K_n^{(r)}$ using t colors such that no s-color copy of $K_m^{(r)}$ exists. Identify the copies of $K_n^{(r)}$ with the vertices in $K_n^{(r)}$ and color the hyperedges whose vertices all come from distinct copies of $K_n^{(r)}$ using t colors, again avoiding an s-color copy of $K_m^{(r)}$. Color the remaining hyperedges arbitrarily using the t colors (these are the hyperedges that have at least two but not all vertices in some copy of $K_n^{(r)}$. Then the largest s-color complete hypergraph will include vertices from at most $m-1$ copies of $K_n^{(r)}$ and at most $m-1$ vertices from each included copy of $K_n^{(r)}$. Thus,

$$R_s^t(K_{(m-1)^2+1}^{(r)};r) > n^2,$$

completing the proof of the theorem. □

5. Future Inquiry

We conclude by listing some directions for future inquiry involving weakened (hypergraph) Ramsey numbers.

1. Su, Li, Luo, and Li's [9] algorithm was published in 2002, so current computing power may enable the computation of clique numbers for cubic residue graphs and their complements that was not possible previously. Such results would imply new lower bounds for $R_2^3(K_n)$ analogous to the bounds found in Section 2.

2. It is likely that Su, Li, Luo, and Li's [9] algorithm can be extended to character difference graphs (see [1]) for characters of order $t \geq 3$. Implementing such an algorithm to find the clique numbers for the complements of such graphs would result in lower bounds for $R_{t-1}^t(K_n)$.

3. In the most general form, Chung, Chung, and Liu ([3] and [4]) defined the weakened Ramsey number $R_2^3(G_1, G_2, G_3)$ to be the least natural number n such that every 3-coloring of the edges in K_n results in a red and blue copy of G_1, a blue and green copy of G_2, or a red and green copy of G_3. Extend weakened hypergraph Ramsey numbers to this non-diagonal case and even consider subhypergraphs other than those that are complete (eg., paths, cycles, trees, stars, etc...).

4. In [2], numerous inequalities were given for hypergraph Ramsey numbers building on previously known inequalities for graphs. Do any of the constructions considered in [2] have "weakened" analogues?

5. One generalization of classical Ramsey numbers comes when one restricts to Gallai colorings. A Gallai coloring is one that avoids rainbow (3-colored) triangles. The corresponding Ramsey numbers in this setting are called Gallai-Ramsey numbers (eg., see [5]). What can be said about weakened Ramsey numbers when only Gallai colorings are considered and can these definitions also be extended to hypergraphs?

References

[1] M. Budden, N. Calkins, W. Hack, J. Lambert, and K. Thompson, Dirichlet character difference graphs, *Acta Math. Univ. Comenian.* **82**, No. 1 (2013), 21-28.

[2] M. Budden, J. Hiller, and A. Rapp, Generalized Ramsey theorems for r-uniform hypergraphs, *Australas. J. Combin.* **63** (2015), 142-152.

[3] K. Chung, M. Chung, and C. Liu, A generalization of Ramsey theory for graphs - with stars and complete graphs as forbidden subgraphs, *Congr. Numer.* **19** (1977), 155-161.

[4] K. Chung and C. Liu, A generalization of Ramsey theory for graphs, *Discrete Math.* **21** (1978), 117-127.

[5] R. Faudree, R. Gould, M. Jacobson, and C. Magnant, Ramsey numbers in rainbow triangle free colorings, *Australas. J. Combin.* **46** (2010), 269-284.

[6] H. Harborth and M. Möller, Weakened Ramsey numbers, *Discrete Appl. Math.* **95** (1999), 279-284.

[7] M.S. Jacobson, On a generalization of Ramsey theory, *Discrete Math.* **38** (1982), 191-195.

[8] S. Radziszowski, Small Ramsey numbers - revision 15, *Electron. J. Combin.* **DS1.15** (2017), 1-104.

[9] W. Su, Q. Li, H. Luo, and G. Li, Lower bounds of Ramsey numbers based on cubic residues, *Discrete Math.* **250** (2002), 197-209.