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CONGRUENCES ASSOCIATED WITH THE MODULAR j-INVARIANT

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Abstract

In this short note, we present proofs of congruences satisfied by the Fourier coefficients of the modular j-invariant. Our proofs are slightly different from the proof discovered by J. Lehner.

1. Introduction

The well-known modular j-invariant is defined by

$$j(\tau) = 1728\frac{E_4^3(q)}{E_4^3(q) - E_6^2(q)} = \frac{1}{q} + \sum_{j=0}^{\infty} c(j)q^j, \tag{1}$$

where $q = e^{2\pi i\tau}, \operatorname{Im}\tau > 0$,

$$E_4(q) = 1 + 240\sum_{j=1}^{\infty}\frac{j^3q^j}{1-q^j}$$

$$E_6(q) = 1 - 504\sum_{j=1}^{\infty}\frac{j^5q^j}{1-q^j}.$$

In 1949, J. Lehner [7] discovered that for any positive integer n, the following congruences hold:

$$c(2n) \equiv 0 \pmod{2^{11}} \tag{2}$$

$$c(3n) \equiv 0 \pmod{3^5} \tag{3}$$

$$c(5n) \equiv 0 \pmod{5^2} \tag{4}$$

$$c(7n) \equiv 0 \pmod{7}. \tag{5}$$

Lehner's proofs of these congruences were clearly explained in [2]. In Section 2, we will revisit Lehner's proofs and in Section 3, we provide proofs which are slightly different from Lehner's.

2. Lehner's Proofs of (2)-(5)

Let

$$\mathrm{SL}_2(\mathbf{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbf{Z}, ad - bc = 1 \right\}$$

and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) | c \equiv 0 \pmod{N} \right\}.$$

Definition 1. We say that a function $f(\tau)$ is automorphic under $\Gamma_0(N)$ if the following three conditions are satisfied [4, p. 125]:

(a) The function f is meromorphic in the upper half-plane

$$\mathbf{H} := \{z \in \mathbf{C} | \mathrm{Im}\, z > 0\}.$$

(b) The function

$$f(A\tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau)$$

for every $A \in G$.

(c) For any $\gamma \in \mathrm{SL}_2(\mathbf{Z})$, $f(\gamma\tau)$ has the form

$$\sum_{n=-m}^{\infty} a_n e^{2\pi i n\tau/N},$$

where $m \in \mathbf{Z}$.

For any function $f(\tau)$ and any prime number p, define $U_p(f(\tau))$ by

$$U_p(f)(\tau) = \frac{1}{p} \sum_{\lambda=0}^{p-1} f\left(\frac{\tau + \lambda}{p}\right).$$

Note that if

$$f(\tau) = \sum_{n=-m}^{\infty} a(n) q^n,$$

then

$$U_p(f)(\tau) = \sum_{n=[-m/p]}^{\infty} a(np) q^n.$$

It is known that [2, Section 4.5] that

Theorem 2. *If $f(\tau)$ is automorphic under $SL_2(\mathbf{Z})$ then $U_p(f)(\tau)$ is automorphic under $\Gamma_0(p)$.*

We observe that if we replace $\mathrm{SL}_2(\mathbf{Z})$ by $\Gamma_0(p)$, the same conclusion holds since the key in the proof is that $f(W\tau) = f(\tau)$ for $W \in \Gamma_0(p^2)$. In other words we have the following:

Theorem 3. *Let p be a prime number. If $f(\tau)$ is automorphic under $\Gamma_0(p)$ then $U_p(f)(\tau)$ is automorphic under $\Gamma_0(p)$.*

Lehner also showed that [2, Theorem 4.6]

Theorem 4. *If $f(\tau)$ is automorphic under $SL_2(\mathbf{Z})$ and if p is a prime, then*

$$U_p(f)(\tau) = U_p(f)\left(-\frac{1}{\tau}\right) - \frac{1}{p}f\left(-\frac{1}{p\tau}\right) + \frac{1}{p}f\left(\frac{\tau}{p}\right). \tag{6}$$

The above theorem is also true with slight modifications if $\mathrm{SL}_2(\mathbf{Z})$ is replaced by $\Gamma_0(p)$. The result is as follows:

Theorem 5. *Let p be a prime number. If $f(\tau)$ is automorphic under $\Gamma_0(p)$, then*

$$U_p(f)(p\tau) = U_p(f)\left(-\frac{1}{p\tau}\right) - \frac{1}{p}f\left(-\frac{1}{p^2\tau}\right) + \frac{1}{p}f(\tau). \tag{7}$$

We are now ready to describe Lehner's proofs. Recall that when $p = 2, 3, 5, 7, 13$ the function

$$\Phi_p(\tau) = \left(\frac{\eta(\tau)}{\eta(p\tau)}\right)^{24/(p-1)} \tag{8}$$

is automorphic under $\Gamma_0(p)$ with a simple pole at $i\infty$ [2, Theorem 4.9]. It is known that $j(\tau)$ is automorphic under $\mathrm{SL}_2(\mathbf{Z})$ and hence by Theorem 2, $U_p(j)(\tau)$ is automorphic under $\Gamma_0(p)$. Using Theorem 4 (see [2, p. 89]), Lehner showed that

$$U_p(j)(\tau) = p^{12/(p-1)-1}\left(a_1\Phi_p^{-1}(\tau) + a_2\Phi_p^{-2}(\tau) + \cdots + a_{p^2}\Phi_p^{-p^2}(\tau)\right) + c(0), \tag{9}$$

for some constants $a_1, a_2, \cdots, a_{p^2}$. Identity (9) immediately implies (2)–(5).

3. Alternative Approach to (2)–(5)

The functions $\Phi_p(\tau)$ for $p = 2, 3, 5, 7, 13$ are clearly analogues of $j(\tau)$ and it is natural to ask if congruences similar to (2)–(5) exist for the coefficients of $\Phi_p(\tau)$.

In our attempt to answer this question, we are led to an alternative approach to (2)–(5). We first recall that the index

$$[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(p)] = p + 1.$$

Since $j(\tau)$ is automorphic for $\Gamma_0(p)$, we conclude that $j(\tau)$ must be a rational function of $\Phi_p(\tau)$. In particular, $j(\tau) = P(\Phi_p(\tau))/Q(\Phi_p(\tau)))$, where $P(x), Q(x) \in \mathbf{Z}[x]$, and $P(x)$ is a polynomial of degree $p+1$. This is because there are $p+1$ poles of $j(\tau)$ in the fundamental region associated with $\Gamma_0(p)$. We list these identities as follow:

$$\begin{aligned}
j(\tau) &= \Phi_2(\tau) + 2^8 \cdot 3 + \frac{2^{16} \cdot 3}{\Phi_2(\tau)} + \frac{2^{24}}{\Phi_2^2(\tau)} \\
&= \Phi_3(\tau) + 2^2 \cdot 3^3 \cdot 7 + \frac{2 \cdot 3^9 \cdot 5}{\Phi_3(\tau)} + \frac{2^2 \cdot 3^{14}}{\Phi_3^2(\tau)} + \frac{3^{18}}{\Phi_3^3(\tau)} \\
&= \Phi_5(\tau) + 2 \cdot 3 \cdot 5^3 + \frac{3^2 \cdot 5^5 \cdot 7}{\Phi_5(\tau)} + \frac{2^2 \cdot 5^8 \cdot 13}{\Phi_5^2(\tau)} + \frac{3^2 \cdot 5^{10} \cdot 7}{\Phi_5^3(\tau)} + \frac{2 \cdot 3 \cdot 5^{13}}{\Phi_5^4(\tau)} + \frac{5^{15}}{\Phi_5^5(\tau)} \\
&= \Phi_7(\tau) + 2^2 \cdot 11 \cdot 17 + \frac{2 \cdot 7^4 \cdot 41}{\Phi_7(\tau)} + \frac{2^4 \cdot 7^6 \cdot 11}{\Phi_7^2(\tau)} + \frac{5 \cdot 7^7 \cdot 13^2}{\Phi_7^3(\tau)} + \frac{2^4 \cdot 7^9 \cdot 17}{\Phi_7^4(\tau)} \\
&\quad + \frac{2 \cdot 7^{11} \cdot 23}{\Phi_7^5(\tau)} + \frac{2^2 \cdot 7^{13}}{\Phi_7^6(\tau)} + \frac{7^{14}}{\Phi_7^7(\tau)} \\
&= \Phi_{13}(\tau) + 2 \cdot 373 + \frac{5 \cdot 13^2 \cdot 233}{\Phi_{13}(\tau)} + \frac{2^2 \cdot 7^4 \cdot 13^3}{\Phi_{13}^2(\tau)} + \frac{2^3 \cdot 7 \cdot 13^4 \cdot 487}{\Phi_{13}^3(\tau)} \\
&\quad + \frac{2^2 \cdot 5 \cdot 11^2 \cdot 13^5 \cdot 17}{\Phi_{13}^4(\tau)} + \frac{2 \cdot 11 \cdot 13^7 \cdot 137}{\Phi_{13}^5(\tau)} + \frac{2^2 \cdot 5 \cdot 13^7 \cdot 1283}{\Phi_{13}^6(\tau)} \\
&\quad + \frac{2 \cdot 13^8 \cdot 6043}{\Phi_{13}^7(\tau)} + \frac{2^2 \cdot 5 \cdot 11 \cdot 13^9}{\Phi_{13}^8(\tau)} + \frac{2^3 \cdot 7 \cdot 13^{10} \cdot 19}{\Phi_{13}^9(\tau)} + \frac{2^2 \cdot 7^2 \cdot 13^{11}}{\Phi_{13}^{10}(\tau)} \\
&\quad + \frac{5^2 \cdot 13^{12}}{\Phi_{13}^{11}(\tau)} + \frac{2 \cdot 13^{13}}{\Phi_{13}^{12}(\tau)} + \frac{13^{13}}{\Phi_{13}^{13}(\tau)}.
\end{aligned}$$

From the above identities, we deduce that

Theorem 6. *Let $\Phi_p(\tau)$ be defined as in* (8) *for $p = 2, 3, 5, 7, 13$ and let $j(\tau)$ be defined as in* (1)*. Then*

$$\begin{aligned}
j(\tau) &\equiv \Phi_2(\tau) + 2^8 \cdot 3 \pmod{2^{16}} \\
&\equiv \Phi_3(\tau) + 2^2 \cdot 3^3 \cdot 7 \pmod{3^9} \\
&\equiv \Phi_5(\tau) + 2 \cdot 3 \cdot 5^3 \pmod{5^5} \\
&\equiv \Phi_7(\tau) + 2^2 \cdot 11 \cdot 17 \pmod{7^4} \\
&\equiv \Phi_{13}(\tau) + 2 \cdot 373 \pmod{13^2}.
\end{aligned}$$

From Theorem 6, we observed the following:

Corollary 1. *Let $p = 2, 3, 5, 7, 13$ and*

$$\Phi_p(\tau) = \frac{1}{q} + \sum_{k=0}^{\infty} d_p(k) q^k.$$

Then for $n \geq 1$,

$$\begin{aligned} c(n) &\equiv d_2(n) \pmod{2^{16}} \\ &\equiv d_3(n) \pmod{3^9} \\ &\equiv d_5(n) \pmod{5^5} \\ &\equiv d_7(n) \pmod{7^4} \\ &\equiv d_{13}(n) \pmod{13^2} \end{aligned}$$

Corollary 1 implies that the coefficients $d_p(n)$ satisfy congruences similar to (2)–(5).

We now use Theorem 6 to prove (2)–(5). From Theorem 3, we note that $U_p(\Phi_p(\tau))$ is automorphic for $\Gamma_0(p)$ and hence, it is a polynomial in $1/\Phi_p(\tau)$. In fact, it is a polynomial of degree 1 in $1/\Phi_p(\tau)$, as shown in the following theorem.

Theorem 7. *Let* $\Phi_p(\tau)$ *be defined as in* (8) *for* $p = 2, 3, 5, 7, 13$. *Then*

$$U_p(\Phi_p)(\tau) = -p^{-1+12/(p-1)} \left(\frac{1}{\Phi_p(\tau)}\right)^{24/(p-1)} - \frac{24}{p-1}. \tag{10}$$

Proof. By making use of the following formula [2, Theorem 3.1]

$$\eta\left(\frac{-1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau), \tag{11}$$

we note that

$$\Phi_p\left(\frac{-1}{p\tau}\right) = \left(\frac{\eta\left(\frac{-1}{p\tau}\right)}{\eta\left(\frac{-1}{\tau}\right)}\right)^{24/(p-1)} = \frac{p^{12/(p-1)}}{\Phi_p(\tau)}. \tag{12}$$

Moreover, $U_p(\Phi_p)$ has no pole at $i\infty$. We thus focus on the pole at 0 by applying the transformation $\tau \to -1/p\tau$. From Theorem 5, we find that

$$U_p(\Phi_p)\left(\frac{-1}{p\tau}\right) + \frac{1}{p}\Phi_p(\tau) = \frac{1}{p}\Phi_p\left(\frac{-1}{p^2\tau}\right) + U_p(\Phi_p)(p\tau). \tag{13}$$

The terms on the right-hand side have no pole at $i\infty$ as a result of (12). Thus, replacing $-1/(p\tau)$ by τ, we deduce together with (12) that

$$F(\tau) = U_p(\Phi_p)(\tau) + \frac{1}{p}\Phi_p\left(\frac{-1}{p\tau}\right) = U_p(\Phi_p)(\tau) + \frac{p^{-1+12/(p-1)}}{\Phi_p(\tau)}$$

has no poles at either 0 and $i\infty$ and so, $F(\tau)$ must be a constant, and in particular, must be equal to the constant term $d_p(0)$ of $\Phi_p(\tau)$. We also see that $d_p(0)$ is the coefficient of q in $(1-q)^{24/(p-1)}$, since all other non-constant terms in the expression for $\Phi_p(\tau)$ involve higher powers of q. Thus, we have $d_p(0) = -24/(p-1)$. $\square$

This result was previously established by P. Jenkins and N. Andersen in [1]. We discovered this proof without the knowledge of their work.

Applying U_p to both sides of the congruences stated in Theorem 6, we deduce that

Theorem 8. *Let* $\Phi_p(\tau)$ *be defined as in* (8) *for* $p = 2, 3, 5, 7, 13$ *and let* $j(\tau)$ *be defined as in* (1)*. Then*

$$\begin{aligned}
U_2(j(\tau)) - 744 &\equiv -\frac{2^{11}}{\Phi_2(\tau)} \pmod{2^{16}} \\
U_3(j(\tau)) - 744 &\equiv -\frac{3^5}{\Phi_3(\tau)} \pmod{3^9} \\
U_5(j(\tau)) - 744 &\equiv -\frac{5^2}{\Phi_5(\tau)} \pmod{5^5} \\
U_7(j(\tau)) - 744 &\equiv -\frac{7}{\Phi_7(\tau)} \pmod{7^4} \\
U_{13}(j(\tau)) - 744 &\equiv -\frac{1}{\Phi_{13}(\tau)} \pmod{13^2}.
\end{aligned}$$

From the first four congruence relations in the above Theorem, it is now clear that (2)–(5) hold. Moreover, by observing that since $1-q^{13k} \equiv (1-q^k)^{13} \pmod{13}$,

$$\frac{1}{\Phi_p(\tau)} = q \prod_{k=1}^{\infty} \frac{(1-q^{13k})^2}{(1-q^k)^2} \equiv q \prod_{k=1}^{\infty} \left(1-q^k\right)^{24} \pmod{13},$$

hence the last congruence gives, for $n \geq 1$, Newman's congruence [8, (13)]

$$c(13n) \equiv -\tau(n) \pmod{13}, \tag{14}$$

where $\tau(n)$ is given by

$$q \prod_{k=1}^{\infty} \left(1-q^k\right)^{24} = \sum_{j=1}^{\infty} \tau(j) q^j.$$

We emphasize that our proof of (14) is different from Newman's proof given in [8].

4. The Group $\Gamma_0(11)$ and $c(11n)$

Besides (2)–(5), Lehner [6, (1.4)] discovered that

$$c(11n) \equiv 0 \pmod{11}. \tag{15}$$

The proof of (15) is more complicated than the congruences we considered so far as there is no "Hauptmodul" such as $\Phi_p(\tau)$ $(p = 2, 3, 5, 7, 13)$ for us to work with.

Let

$$\beta(\tau) = \frac{11^2 E_4(11\tau) - E_4(\tau)}{120\eta^4(\tau)\eta^4(11\tau)}, \tag{16}$$

$$\alpha(\tau) = \frac{1}{\eta^2(11\tau)\eta^2(\tau)} \left(\sum_{m,n\in\mathbf{Z}} q^{m^2+mn+3n^2} \right) \tag{17}$$

and

$$\gamma(\tau) = \frac{1}{2\cdot 11^2}\left(\alpha^2(\tau) - 10\alpha(\tau) - 22 - \beta(\tau)\right). \tag{18}$$

Lehner [5, pp. 501–505] showed that the set of modular functions invariant under $\Gamma_0(11)$ is generated by $\alpha(\tau)$ and $\gamma(\tau)$. Since $j(\tau)$ is a modular function for $\Gamma_0(11)$, we find that

$$\begin{aligned} j(\tau) = &-2^3\cdot 3\cdot 5^3\cdot 8243 + \alpha(\tau) - 2\cdot 11^2\cdot 1531\cdot 2467\gamma(\tau)\\ &+ 2\cdot 3\cdot 11^2\cdot 23\cdot 1481\alpha(\tau)\gamma(\tau) - 3\cdot 11^4\cdot 37\cdot 79\cdot 103\gamma^2(\tau)\\ &+ 3\cdot 11^4\cdot 14621\gamma^2(\tau)\alpha(\tau) - 2^2\cdot 7\cdot 11^6\cdot 1879\gamma^3(\tau)\\ &+ 5\cdot 11^6\cdot 23\cdot 29\gamma^3(\tau)\alpha(\tau) - 11^8\cdot 1483\gamma^4(\tau) + 11^8\cdot 103\gamma^4(\tau)\alpha(\tau)\\ &- 2^4\cdot 11^{10}\gamma^5(\tau) + 11^{10}\gamma^5(\tau)\alpha(\tau). \end{aligned} \tag{19}$$

To continue with our proof of (15), we need to show that $\gamma(\tau)$ has a power series in q with integer coefficients. This is given by the following lemma of Lehner [5, Lemma 3]:

Lemma 1. *Let $\gamma(\tau)$ be defined as in* (18) *and*

$$\delta(\tau) = \frac{1}{2}\left(\alpha^2(\tau) - 10\alpha(\tau) - 22 + \beta(\tau)\right). \tag{20}$$

If

$$\gamma(\tau) = \sum_{k=1}^{\infty} c_k q^k$$

and

$$\delta(\tau) = \sum_{k=1}^{\infty} c'_k q^k$$

then $c_k, c'_k \in \mathbf{Z}$.

Proof. First, we observe that

$$\alpha^2(\tau) \pm \beta(\tau) = \frac{1}{\eta^4(\tau)\eta^4(11\tau)}\left(\left(\sum_{m,n\in\mathbb{Z}} q^{m^2+mn+3n^2} \right)^2 \pm \left(11^2 g(\tau) - g(\tau)\right) \right).$$

Note that the right-hand side can be written as

$$\left(1+2\sum_{m>0,n} q^{m^2+mn+3n^2}\right)^2 \pm 1 \pm 2\left(11^2\sum_{k=1}^{\infty}\frac{k^3q^{11k}}{1-q^{11k}} - \sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}\right)$$

and this shows that the coefficients in the series expansion of

$$\alpha^2(\tau) - 10\alpha(\tau) - 22 \pm \beta(\tau)$$

are even integers and that in particular, we have $c_k' \in \mathbf{Z}$.

To show that $c_k \in \mathbf{Z}$, we consider

$$E_2(\tau) = 1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}, q = e^{2\pi i\tau}$$

and define

$$\mathcal{E}(\tau) = \frac{1}{\eta^2(\tau)\eta^2(11\tau)}\left(11E_2(11\tau) - E_2(\tau)\right)^2.$$

Let

$$G_1(\tau) = 2\cdot 11^2\gamma(\tau) = \alpha^2(\tau) - 10\alpha - 22 - \beta(\tau)$$

and

$$G_2(\tau) = 2\delta(\tau) = \alpha^2(\tau) - 10\alpha - 22 + \beta(\tau).$$

Then we find that

$$\mathcal{E}(\tau) = \frac{170}{121}G_1(\tau)G_2(\tau) + 50G_1(\tau) + 50G_2(\tau) + 1776. \tag{21}$$

From (21), we deduce that $(G_1(\tau)G_2(\tau))/121$ has series expansion with integer coefficients. Moreover, $1/G_2(\tau)$ has integer series expansion with integer coefficients, because the leading term of $G_2(\tau)$ is q^{-2}. Hence,

$$2\gamma(\tau) = \frac{1}{11^2}G_1(\tau) = \frac{1}{11^2}\left(\alpha^2(\tau) - 10\alpha(\tau) - 22 - \beta(\tau)\right) \tag{22}$$

has series expansion with integer coefficients. We have already seen in the beginning of the proof that the right hand side of (22) has series expansion with even coefficients and hence, we conclude that $c_k \in \mathbf{Z}$.

□

Our proof of Lemma 1 uses (21) and is different from Lehner's proof.

We now continue with our proof of (15). From (19) and Lemma 1, we deduce that

$$j(\tau) \equiv \alpha(\tau) - 109 \pmod{11^2}. \tag{23}$$

Next, observe that

$$\begin{aligned} U_{11}(\alpha)(\tau) = &-2^3 \cdot 3 \cdot 283 \cdot 331 + 11^9\gamma^5(\tau)\alpha(\tau) - 2^4 \cdot 11^9\gamma^5(\tau) + 11^7 \cdot 103\gamma^4(\tau)\alpha(\tau) \\ &- 11^7 \cdot 1483\gamma^4(\tau) + 5 \cdot 11^5 \cdot 23 \cdot 29\gamma^3(\tau)\alpha(\tau) - 2^2 \cdot 7 \cdot 11^5 \cdot 1879\gamma^3(\tau) \\ &+ 3 \cdot 11^3 \cdot 14621\gamma^2(\tau)\alpha(\tau) - 3 \cdot 11^3 \cdot 37 \cdot 79 \cdot 103\gamma^2(\tau) \\ &+ 2 \cdot 3 \cdot 11 \cdot 23 \cdot 1481\gamma(\tau)\alpha(\tau) - 2 \cdot 11 \cdot 1531 \cdot 2467\gamma(\tau). \end{aligned}$$

Again, by Lemma 1, we conclude that

$$U_{11}(\alpha)(\tau) \equiv 6 \pmod{11},$$

which together with (23) yields (15).

We conclude this section by giving another approach to the derivation of (19). Since $j(\tau)$ is a modular function invariant under $\mathrm{SL}_2(\mathbf{Z})$, $j(11\tau)$ is a modular function invariant under $\Gamma_0(11)$. Therefore, $j(11\tau)$ can be expressed in terms of $\delta(\tau)$ (see (20)) and $\alpha(\tau)$. The resulting identity is

$$\begin{aligned} j(11\tau) = &\,\delta^5(\tau)\alpha(\tau) - 2^4\delta^5(\tau) + 103\delta^4(\tau)\alpha(\tau) - 1483\delta^4(\tau) \\ &+ 5 \cdot 23 \cdot 29\delta^3(\tau)\alpha(\tau) - 2^2 \cdot 7 \cdot 1879\delta^3(\tau) + 3 \cdot 14621\delta^2(\tau)\alpha(\tau) \\ &- 3 \cdot 37 \cdot 79 \cdot 103\delta^2(\tau) + 2 \cdot 3 \cdot 23 \cdot 1481\delta(\tau)\alpha(\tau) \\ &- 2 \cdot 1531 \cdot 2467\delta(\tau) + \alpha(\tau) - 2^3 \cdot 3 \cdot 5^3 \cdot 8243. \end{aligned} \tag{24}$$

It is known from [5, (4.53)] that

$$\alpha\left(-\frac{1}{11\tau}\right) = \alpha(\tau) \quad \text{and} \quad \beta\left(-\frac{1}{11\tau}\right) = -\beta(\tau).$$

Hence,

$$\delta\left(-\frac{1}{11\tau}\right) = 11^2\gamma(\tau). \tag{25}$$

Replacing τ by $-1/(11\tau)$ in (24) and using the fact that $j(-1/\tau) = j(\tau)$, we obtain (19). Note that the presence of 11^2 in most of the coefficients of (19) is a consequence of the transformation formula (25) for $\delta(\tau)$. We also observe that it is necessary to show that the coefficients of the q-series expansion of $\gamma(\tau)$ are integers (Lemma 1) so that the right-hand side of (25) is congruent to 0 modulo 11^2.

5. Conclusion

At the beginning of Section 3, we mentioned that this article is motivated by our attempt to understand if Fourier coefficients of other Hauptmoduls such as $\Phi_p(\tau), p = 2, 3, 5, 7, 13$, satisfy congruences similar to Fourier coefficients of $j(\tau)$.

We have shown that the answer is affirmative. The natural question to ask is if the Fourier coefficients of other Hauptmoduls associated to "genus 0" discrete subgroups of $\mathrm{SL}_2(\mathbf{Z})$ (see [3]) satisfy congruences modulo power of primes related to the level of the Hauptmoduls. It turns out that the answer is again positive. For example, if

$$F(\tau) = \left(\frac{\eta(\tau)\eta(2\tau)}{\eta(3\tau)\eta(6\tau)}\right)^4 = \frac{1}{q} + \sum_{j=0}^{\infty} \nu(j) q^j,$$

then for $n \geq 1$,

$$\nu(3n) \equiv 0 \pmod{3^3}. \tag{26}$$

One way to prove this congruence is to use the identity

$$U_3(F)(\tau) = -\frac{27}{F(\tau)} - 4.$$

Another way is to first write

$$F(\tau) = G(\tau) - 7 - \frac{8}{G(\tau)}$$

where

$$G(\tau) = \frac{\eta^3(2\tau)\eta^9(3\tau)}{\eta^3(\tau)\eta^9(6\tau)},$$

followed by applying U_3 to both sides. One then notes that since

$$U_3(G)(\tau) = -\frac{3}{1+G(\tau)} + 3$$

and

$$U_3\left(\frac{8}{G}\right)(\tau) = \frac{24}{G(\tau)-8},$$

we have whenever $|G(\tau)| < 1$,

$$U_3(F)(\tau) = -3(1 - G(\tau) + G^2(\tau) + \cdots) + 3 - 7 + 3(1 + G(\tau)/8 + (G(\tau)/8)^2 + \cdots).$$

If we simplify the right-hand side as a power series in $G(\tau)$, the numerator of the coefficient of $G^j(\tau)$ is

$$3((-1)^{j+1} \cdot 8^j + 1).$$

Since

$$(-1)^{j+1} \cdot 8^j + 1 \equiv 0 \pmod{9},$$

we deduce that

$$U_3(F)(\tau) \equiv -4 \pmod{3^3},$$

which implies (26). The second proof is obviously motivated by the proof we gave in Section 3.

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