CLOSED FORM EXPRESSIONS FOR THE STIRLING NUMBERS OF THE FIRST KIND

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Abstract

In this note, we give a probabilistic representation for the Stirling numbers of the first kind in terms of moments of suitable random variables. As an application, two multiple integral representations of such numbers are easily obtained.

1. Introduction and main result

The Stirling numbers of the first kind $s(n,k)$ appear as ingredients in a great variety of formulas involving functions commonly used in analytic number theory. Such numbers may be defined or generated in various equivalent ways (see, for instance, Abramowitz and Stegun [1, pp. 824–825], Katriel [5], and Qi [6]). Here, we consider the definition of $s(n,k)$ by means of the generating function

$$\frac{\log^k(1+z)}{k!} = \sum_{n=k}^{\infty} \frac{s(n,k)}{n!} z^n, \quad z \in \mathbb{C}, \quad |z| < 1, \quad k = 1, 2, \ldots \tag{1}$$

Let $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, n$. This note is motivated by two recent papers by Qi [6] and Agoh and Dilcher [3], in which these authors obtain multiple integral

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representations for $s(n,k)$. Specifically, Qi [6] gives various representations, such as

$$s(n,k) = (-1)^{n-k}\binom{n}{k}\lim_{x\to 0}\frac{d^{n-k}}{dx^{n-k}}\left\{\left[\int_0^\infty\left(\int_{1/e}^1 t^{xu-1}\,dt\right)e^{-u}\,du\right]^k\right\}, \quad (2)$$

whereas Agoh and Dilcher [3] show that

$$\begin{aligned} s(n,k) =& (-1)^{n-k}\frac{(n-1)!}{k!} \\ &\times\int_0^1\int_0^{1-x_1}\cdots\int_0^{1-x_1-\ldots-x_{k-2}}\frac{S_n(x_1,\ldots,x_k)}{x_1\cdots x_k}\,dx_{k-1}\cdots dx_1, \end{aligned} \quad (3)$$

where $k\geq 2$, $x_1+\cdots+x_k=1$, and

$$S_n(x_1,\cdots,x_k) = 1+\sum_{r=1}^{k-1}(-1)^{k-r}\sum_{1\leq i_1<\cdots<i_r\leq k}(x_{i_1}+\cdots+x_{i_r})^n. \quad (4)$$

In the case $k=2$, the multiple integral in (3) is understood as the single integral from 0 to 1. A different integral representation for $s(n,k)$ can be found in Agoh and Dilcher [3, Theorem 2].

Let U and T be two independent random variables such that U is uniformly distributed on $(0,1)$ and T has the exponential density with unit mean

$$\rho(\theta) = e^{-\theta}, \quad \theta\geq 0. \quad (5)$$

Our probabilistic representation of $s(n,k)$ is based upon the following simple observation. Taking into account (5) and the fact that U and T are independent, the Fourier transform of UT is given by

$$\varphi(t) = Ee^{itUT} = E\frac{1}{1-itU} = \frac{\log(1-it)}{-it}, \quad t\in\mathbb{R}, \quad (6)$$

where E stands for mathematical expectation. Let $(U_j)_{j\geq 1}$ and $(T_j)_{j\geq 1}$ be two sequences of independent copies of U and T, respectively, both of them mutually independent. Denote by

$$S_k = U_1T_1+\cdots+U_kT_k, \quad k=1,2,\ldots \quad (S_0=0). \quad (7)$$

In the following result, we give a representation of $s(n,k)$ in terms of appropriate moments of the random variable S_k.

Theorem 1. *For any $n=1,2,\ldots$ and $k=1,2,\ldots,n$, we have*

$$s(n,k) = (-1)^{n-k}\binom{n}{k}ES_k^{n-k}.$$

Proof. Let $t \in \mathbb{R}$ with $|t| < 1$. Since the random variables involved are independent, we have from (6) and (7)

$$\left(\frac{\log(1-it)}{-it}\right)^k = Ee^{itS_k} = \sum_{m=0}^{\infty} \frac{ES_k^m}{m!}(it)^m, \tag{8}$$

thus implying that

$$\frac{\log^k(1-it)}{k!} = \sum_{m=0}^{\infty} \frac{(-1)^m ES_k^m}{k!\,m!}(-it)^{m+k} = \sum_{n=k}^{\infty} \frac{(-1)^{n-k} ES_k^{n-k}}{k!\,(n-k)!}(-it)^n. \tag{9}$$

The conclusion follows by equating coefficients in (1) and (9). □

We finally mention that the random variable S_k defined in (7) can be used as an important tool to obtain fast computations of the Stieltjes constants (c.f. [2]).

2. Applications

Thanks to Theorem 1, we can easily derive multiple integral representations for $s(n,k)$. The first one is the following.

Corollary 1. *For any $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, n$, we have*

$$s(n,k) = (-1)^{n-k}\binom{n}{k} \times \int_{(0,1)^k} du_1 \cdots du_k \int_{[0,\infty)^k} \left(\sum_{j=1}^{k} u_j \theta_j\right)^{n-k} e^{-(\theta_1+\cdots+\theta_k)}\, d\theta_1 \cdots d\theta_k.$$

Proof. It suffices to apply Theorem 1, by writing in integral form the moment ES_k^{n-k}. In this regard, recall that $U_1, \ldots, U_k$ are uniformly distributed, $T_1, \ldots, T_k$ are exponentially distributed, and that all of these random variables are mutually independent. □

The integral representation in Corollary 1 seems to be more tractable than those given in (2) and (3). On the other hand, Theorem 1 suggests that the Stirling numbers of the first kind could, in principle, be written in terms of one single integral, since

$$s(n,k) = (-1)^{n-k}\binom{n}{k} \int_0^{\infty} \theta^{n-k} f_k(\theta)\, d\theta, \quad n = 1, 2, \ldots, \quad k = 1, 2, \ldots, n, \tag{10}$$

where $f_k(\theta)$ is the probability density of the random variable S_k defined in (7). However, $f_k(\theta)$ is a very involved function and therefore it is doubtful that formula (10) represents a real advantage with respect to the multiple integral given in Corollary 1.

Despite these comments, we can give a representation of $s(n,k)$ as a double integral, whenever $n = 2, 3, \dots$ and $k = 2, 3, \dots, n$.

Corollary 2. *For any $n = 2, 3, \dots$ and $k = 2, 3, \dots, n$, we have*

$$s(n,k) = \frac{(-1)^n}{2\pi} \binom{n}{k} \int_0^\infty \theta^{n-k} \, d\theta \int_{-\infty}^{\infty} e^{-it\theta} \left(\frac{\log(1-it)}{it} \right)^k dt.$$

Proof. As follows from (8), the Fourier transform of S_k, i.e.,

$$\varphi_k(t) = Ee^{itS_k} = \left(\frac{\log(1-it)}{-it} \right)^k, \quad t \in \mathbb{R}, \tag{11}$$

is absolutely integrable for $k = 2, 3, \cdots$. Therefore, by the Fourier inversion formula (see, for instance, Billingsley [4, p. 347]), the probability density of S_k can be written as

$$f_k(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\theta} \varphi_k(t) \, dt, \quad \theta > 0, \quad k = 2, 3, \dots. \tag{12}$$

The conclusion follows from (10)–(12). □

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