Abstract
Let \( S(a, b) = 12s(a, b) \), where \( s(a, b) \) denotes the classical Dedekind sum. In a recent note E. Tsukerman gave a necessary and sufficient condition for \( S(a_1, b) - S(a_2, b) \in 8\mathbb{Z} \). In the present paper we show that this condition is equivalent to \( S(a_1, b) - S(a_2, b) \in 24\mathbb{Z} \), provided that \( 9 \nmid b \). Tsukerman also obtained a congruence mod 8 for \( bT(a, b) \), where \( T(a, b) \) is the alternating sum of the partial quotients of the continued fraction expansion of \( a/b \). We show that the respective congruence holds mod 24 if \( 3 \nmid b \) and mod 72 if \( 3|b \).

1. Introduction and Results
Let \( a \) be an integer, \( b \) a natural number, and \( (a, b) = 1 \). The classical Dedekind sum \( s(a, b) \) is defined by
\[
s(a, b) = \sum_{k=1}^{b} ((k/b))(ak/b).
\]
Here
\[
((x)) = \begin{cases} 
  x - [x] - 1/2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}; \\
  0 & \text{if } x \in \mathbb{Z}
\end{cases}
\]
(see [2, p. 1]). It is often more convenient to work with
\[
S(a, b) = 12s(a, b)
\]
instead (see, for instance, formula (7) below). Since \( S(a + b, b) = S(a, b) \), we obtain all Dedekind sums if \( a \) is restricted to the range \( 0 \leq a < b - 1 \), \( (a, b) = 1 \).

In the recent note [4], E. Tsukerman gave a necessary and sufficient condition for the equality of \( S(a_1, b) \) and \( S(a_2, b) \) modulo \( 8\mathbb{Z} \). This condition involves the function \( \mu \), which is defined, for \( a, b \) as above, as follows:
\[
\mu(a, b) = \begin{cases} 
  2 - 2 \left( \frac{a}{b} \right) & \text{if } b \text{ is odd;} \\
  (a - 1)(a + b - 1) & \text{if } b \text{ is even.}
\end{cases}
\]
Here \( \left( \frac{a}{b} \right) \) is the Jacobi symbol. Tsukerman’s condition is phrased by means of the residue class
\[
b(a_2 \mu(b, a_1) - a_1 \mu(b, a_2)) \mod 8b.
\]
We observe, however, that this residue class depends only of the residue classes of \( \mu(b, a_1) \) and \( \mu(b, a_2) \) modulo 8, not of the values of \( \mu(b, a_1) \) and \( \mu(b, a_2) \) themselves. Therefore, we may replace the function \( \mu \) by the following simpler function, which we henceforth also call \( \mu \).

\[
\mu(a, b) = \begin{cases} 
2 - 2 \left( \frac{a}{b} \right), & \text{if } b \text{ is odd;} \\
4, & \text{if } b \equiv 0 \mod 4 \text{ and } a \equiv 3 \mod 4; \\
0, & \text{otherwise.}
\end{cases}
\]

In this paper we show

**Theorem 1.** Let \( a_1, a_2 \in \mathbb{N} \) be relatively prime to \( b \in \mathbb{N} \). Suppose, further, that \( 9 \nmid b \). Then
\[
S(a_1, b) - S(a_2, b) \in 24\mathbb{Z}
\]
if, and only if,
\[
b(a_2 \mu(b, a_1) - a_1 \mu(b, a_2)) \equiv (a_1 - a_2)(b - 1)(a_1a_2 + b - 1) \mod 8b. \tag{1}
\]

This equivalence cannot be extended to the case \( 9 \mid b \) in an obvious way, as we show in Section 3. Tsukerman showed that \( 1 \) is equivalent to \( S(a_1, b) - S(a_2, b) \in 8\mathbb{Z} \) for arbitrary natural numbers \( b \), i.e., he needed not assume \( 9 \nmid b \).

For \( a \in \mathbb{Z} \) and \( b \in \mathbb{N} \), let
\[
\frac{a}{b} = [a_0, a_1, \ldots, a_n]
\]
denote the regular continued fraction expansion of \( a/b \). The partial quotients \( a_1, \ldots, a_n \) are natural numbers. We do not assume \( a_n \geq 2 \), but require \( n \) to be odd, instead. Define
\[
T(a, b) = \sum_{k=0}^{n} (-1)^{k-1} a_k. \tag{2}
\]
In the said paper, Tsukerman showed, for \( a, b \in \mathbb{N} \), \( (a, b) = 1 \),
\[
bT(a, b) \equiv -\mu(a, b) + b^2 + 2 - a - a^* \mod 8, \tag{3}
\]
with \( a^* \in \{1, \ldots, b-1\} \), \( aa^* \equiv 1 \mod b \). Our new definition of \( \mu \) suggests a more explicit form of \( 3 \), which we use in the following Theorem. To this end we define \( \varepsilon \in \{\pm 1\} \) by the congruence
\[
a \equiv \varepsilon \mod 3 \tag{4}
\]
for each \( a \in \mathbb{Z} \), \( 3 \nmid a \).
**Theorem 2.** Let $a \in \mathbb{Z}$ be relatively prime to $b \in \mathbb{N}$.

(a) Let $b$ be odd. If $3 \nmid b$, then

$$bT(a,b) \equiv 9 + 18 \left(\frac{a}{b}\right) - a - a^* \mod 24.$$

If $3 \mid b$, then

$$bT(a,b) \equiv 9 + 18 \left(\frac{a}{b}\right) - 16\varepsilon - a - a^* \mod 72.$$

(b) Let $b \equiv 2 \mod 4$ or let both $b \equiv 0 \mod 4$ and $a \equiv 3 \mod 4$ hold. If $3 \nmid b$, then

$$bT(a,b) \equiv 6 - a - a^* \mod 24.$$

If $3 \mid b$, then

$$bT(a,b) \equiv 54 - 16\varepsilon - a - a^* \mod 72.$$

(c) Let $b \equiv 0 \mod 4$ and $a \equiv 1 \mod 4$. If $3 \nmid b$, then

$$bT(a,b) \equiv 18 - a - a^* \mod 24.$$

If $3 \mid b$, then

$$bT(a,b) \equiv 18 - 16\varepsilon - a - a^* \mod 72.$$

In Section 3 we exhibit many examples that illustrate both Theorem 1 and the fact that this theorem does not hold if $9 \mid b$.

## 2. Proofs

Our main tools are two congruences modulo 3 for Dedekind sums. First we observe that $bS(a,b)$ is an integer; moreover, if 3 does not divide $b$, then

$$bS(a,b) \equiv 0 \mod 3.$$  \tag{5}

These assertions follow from [2, p. 27, Th. 2]). On the other hand, if $3 \mid b$,

$$bS(a,b) \equiv 2\varepsilon \mod 9,$$  \tag{6}

where $\varepsilon$ is defined as in (4) (see [3, formula (70)]).

**Proof of Theorem 1.** Suppose, first, that $3 \nmid b$. Because of (5), we may write

$$S(a_1,b) = \frac{3k_1}{b}, \quad S(a_2,b) = \frac{3k_2}{b}$$

with integers $k_1, k_2$. By [4, Th. 3.1], the congruence (1) is equivalent to $S(a_1,b) - S(a_2,b) \in 8\mathbb{Z}$. Accordingly, (1) is also equivalent to

$$\frac{3(k_1 - k_2)}{b} = 8r, \quad r \in \mathbb{Z}.$$
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However, $3 \nmid b$, and so this means $3 \mid r$. This proves Theorem 1 in the case $3 \nmid b$. Suppose now that $3 \mid b$. Then the congruence (1) implies $(a_1 - a_2)(a_1 a_2 - 1) \equiv 0 \pmod{3}$. Hence we obtain, from (6)

$$S(a_1, b) = \frac{2\varepsilon + 9k_1}{b}, \quad S(a_2, b) = \frac{2\varepsilon + 9k_2}{b}$$

with a common value $\varepsilon \equiv a_1 \equiv a_2 \pmod{3}$ and $k_1, k_2 \in \mathbb{Z}$. Accordingly, (1) is equivalent to

$$\frac{9(k_1 - k_2)}{b} = 8r, \quad r \in \mathbb{Z}.$$ 

If $9 \mid b$, this simply means $S(a_1, b) - S(a_2, b) \in 8\mathbb{Z}$, so this is just Tsukerman’s result. However, if $9 \nmid b$, we obtain $3 \mid r$, which yields the theorem in the case $3 \mid b, 9 \nmid b$.

Proof of Theorem 2. The Barkan-Hickerson-Knuth formula says

$$S(a, b) = T(a, b) + \frac{a + a^*}{b} - 3 \quad (7)$$

(see, for instance, [1]). Note that this formula is often enunciated only for the case $0 \leq a < b$, but it is, in fact, valid for arbitrary integers $a$ relatively prime to $b$, provided that $T(a, b)$ is defined as in (2). Hence we obtain, by (5) and (7),

$$bT(a, b) \equiv -a - a^* \pmod{3} \quad (8)$$

if $3 \nmid b$. In the case $3 \mid b$, (6) and (7) give

$$bT(a, b) \equiv 2\varepsilon - a - a^* \pmod{9} \quad (9)$$

instead. Further, Tsukerman’s congruence (3) is also valid for arbitrary integers $a$ relatively prime to $b$, as we easily check. We combine (3) with the congruences (8) and (9) by means of the Chinese remainder theorem. This readily gives Theorem 2.

3. A Proposition Yielding Examples

Our examples arise from the following proposition.

Proposition 1. Let $c, d$ be odd natural numbers, $d \geq 3$. Put $b = cd^2$ and $a = cd + 1$. Then

$$S(1, b) - S(a, b) = c(d^2 - 1).$$
Proof. We apply the reciprocity law for Dedekind sums (see [2, p. 5]), which gives

$$S(a, b) = -S(b, a) + \frac{a}{b} + \frac{b}{a} + \frac{1}{ab} - 3.$$ 

Now $b \equiv -d \mod a$, hence the reciprocity law says

$$S(b, a) = S(-d, a) = S(a, d) - \frac{a}{d} - \frac{d}{a} + \frac{1}{ad} + 3.$$ 

However, $a \equiv 1 \mod d$, and so $S(a, d) = S(1, d) = d - 3 + 2/d$. Inserting the values $b = cd^2$ and $a = cd + 1$ gives

$$S(a, b) = c - 3 + \frac{2}{b}.$$ 

Since $S(1, b) = b - 3 + 2/b$, we obtain the desired result.

In the setting of the proposition, let $3 \nmid d$. Then $d^2 - 1 \equiv 0 \mod 24$, so the proposition yields many examples with $S(1, b) - S(a, b) > 0$ and $S(1, b) - S(a, b) \equiv 0 \mod 24$. On the other hand, if $3 \mid d$, then $d^2 - 1 \equiv 0 \mod 8$, but $d^2 - 1 \not\equiv 0 \mod 24$. If, therefore, $3 \nmid c$, we obtain many examples with $S(1, b) - S(a, b) \equiv 0 \mod 8$, but $S(1, b) - S(a, b) \not\equiv 0 \mod 24$.

References


