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EQUALITY OF DEDEKIND SUMS MODULO $24\mathbb{Z}$

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Abstract

Let $S(a, b) = 12s(a, b)$, where $s(a, b)$ denotes the classical Dedekind sum. In a recent note E. Tsukerman gave a necessary and sufficient condition for $S(a_1, b) - S(a_2, b) \in 8\mathbb{Z}$. In the present paper we show that this condition is equivalent to $S(a_1, b) - S(a_2, b) \in 24\mathbb{Z}$, provided that $9 \nmid b$. Tsukerman also obtained a congruence mod 8 for $bT(a, b)$, where $T(a, b)$ is the alternating sum of the partial quotients of the continued fraction expansion of a/b. We show that the respective congruence holds mod 24 if $3 \nmid b$ and mod 72 if $3 \,|\, b$.

1. Introduction and Results

Let a be an integer, b a natural number, and $(a, b) = 1$. The classical Dedekind sum $s(a, b)$ is defined by

$$s(a, b) = \sum_{k=1}^{b} ((k/b))((ak/b)).$$

Here

$$((x)) = \begin{cases} x - \lfloor x \rfloor - 1/2 & \text{if } x \in \mathbb{R} \smallsetminus \mathbb{Z}; \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

(see [2, p. 1]). It is often more convenient to work with

$$S(a, b) = 12s(a, b)$$

instead (see, for instance, formula (7) below). Since $S(a + b, b) = S(a, b)$, we obtain all Dedekind sums if a is restricted to the range $0 \le a \le b - 1$, $(a, b) = 1$.

In the recent note [4], E. Tsukerman gave a necessary and sufficient condition for the equality of $S(a_1, b)$ and $S(a_2, b)$ modulo $8\mathbb{Z}$. This condition involves the function μ, which is defined, for a, b as above, as follows:

$$\mu(a, b) = \begin{cases} 2 - 2\left(\frac{a}{b}\right), & \text{if } b \text{ is odd;} \\ (a-1)(a+b-1), & \text{if } b \text{ is even.} \end{cases}$$

Here $\left(\frac{a}{b}\right)$ is the Jacobi symbol. Tsukerman's condition is phrased by means of the residue class

$$b(a_2\mu(b,a_1) - a_1\mu(b,a_2)) \mod 8b.$$

We observe, however, that this residue class depends only of the residue classes of $\mu(b,a_1)$ and $\mu(b,a_2)$ modulo 8, not of the values of $\mu(b,a_1)$ and $\mu(b,a_2)$ themselves. Therefore, we may replace the function μ by the following simpler function, which we henceforth also call μ.

$$\mu(a,b) = \begin{cases} 2 - 2\left(\frac{a}{b}\right), & \text{if } b \text{ is odd;} \\ 4, & \text{if } b \equiv 0 \mod 4 \text{ and } a \equiv 3 \mod 4; \\ 0, & \text{otherwise.} \end{cases}$$

In this paper we show

Theorem 1. *Let $a_1, a_2 \in \mathbb{N}$ be relatively prime to $b \in \mathbb{N}$. Suppose, further, that $9 \nmid b$. Then*

$$S(a_1,b) - S(a_2,b) \in 24\mathbb{Z}$$

if, and only if,

$$b(a_2\mu(b,a_1) - a_1\mu(b,a_2)) \equiv (a_1 - a_2)(b-1)(a_1a_2 + b - 1) \mod 8b. \tag{1}$$

This equivalence cannot be extended to the case $9 \,|\, b$ in an obvious way, as we show in Section 3. Tsukerman showed that (1) is equivalent to $S(a_1,b) - S(a_2,b) \in 8\mathbb{Z}$ for arbitrary natural numbers b, i.e., he needed not assume $9 \nmid b$.

For $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, let

$$\frac{a}{b} = [a_0, a_1, \ldots, a_n]$$

denote the regular continued fraction expansion of a/b. The partial quotients $a_1, \ldots, a_n$ are natural numbers. We do not assume $a_n \geq 2$, but require n to be odd, instead. Define

$$T(a,b) = \sum_{k=0}^{n} (-1)^{k-1} a_k. \tag{2}$$

In the said paper, Tsukerman showed, for $a, b \in \mathbb{N}$, $(a,b) = 1$,

$$bT(a,b) \equiv -\mu(a,b) + b^2 + 2 - a - a^* \mod 8, \tag{3}$$

with $a^* \in \{1, \ldots, b-1\}$, $aa^* \equiv 1 \mod b$. Our new definition of μ suggests a more explicit form of (3), which we use in the following Theorem. To this end we define $\varepsilon \in \{\pm 1\}$ by the congruence

$$a \equiv \varepsilon \mod 3 \tag{4}$$

for each $a \in \mathbb{Z}$, $3 \nmid a$.

Theorem 2. *Let $a \in \mathbb{Z}$ be relatively prime to $b \in \mathbb{N}$.*
(a) *Let b be odd. If $3 \nmid b$, then*

$$bT(a,b) \equiv 9 + 18\left(\frac{a}{b}\right) - a - a^* \mod 24.$$

If $3 \mid b$, then

$$bT(a,b) \equiv 9 + 18\left(\frac{a}{b}\right) - 16\varepsilon - a - a^* \mod 72.$$

(b) *Let $b \equiv 2 \bmod 4$ or let both $b \equiv 0 \bmod 4$ and $a \equiv 3 \bmod 4$ hold. If $3 \nmid b$, then*

$$bT(a,b) \equiv 6 - a - a^* \mod 24.$$

If $3 \mid b$, then

$$bT(a,b) \equiv 54 - 16\varepsilon - a - a^* \mod 72.$$

(c) *Let $b \equiv 0 \bmod 4$ and $a \equiv 1 \bmod 4$. If $3 \nmid b$, then*

$$bT(a,b) \equiv 18 - a - a^* \mod 24.$$

If $3 \mid b$, then

$$bT(a,b) \equiv 18 - 16\varepsilon - a - a^* \mod 72.$$

In Section 3 we exhibit many examples that illustrate both Theorem 1 and the fact that this theorem does not hold if $9 \mid b$.

2. Proofs

Our main tools are two congruences modulo 3 for Dedekind sums. First we observe that $bS(a,b)$ is an integer; moreover, if 3 does not divide b, then

$$bS(a,b) \equiv 0 \mod 3. \tag{5}$$

These assertions follow from [2, p. 27, Th. 2]). On the other hand, if $3 \mid b$,

$$bS(a,b) \equiv 2\varepsilon \mod 9, \tag{6}$$

where ε is defined as in (4) (see [3, formula (70)]).

Proof of Theorem 1. Suppose, first, that $3 \nmid b$. Because of (5), we may write

$$S(a_1,b) = \frac{3k_1}{b},\ \ S(a_2,b) = \frac{3k_2}{b}$$

with integers k_1, k_2. By [4, Th. 3.1], the congruence (1) is equivalent to $S(a_1,b) - S(a_2,b) \in 8\mathbb{Z}$. Accordingly, (1) is also equivalent to

$$\frac{3(k_1 - k_2)}{b} = 8r,\ \ r \in \mathbb{Z}.$$

However, $3 \nmid b$, and so this means $3 \,|\, r$. This proves Theorem 1 in the case $3 \nmid b$. Suppose now that $3 \,|\, b$. Then the congruence (1) implies $(a_1 - a_2)(a_1 a_2 - 1) \equiv 0 \mathrm{mod} 3$. Hence we obtain, from (6)

$$S(a_1, b) = \frac{2\varepsilon + 9k_1}{b}, \quad S(a_2, b) = \frac{2\varepsilon + 9k_2}{b}$$

with a common value $\varepsilon \equiv a_1 \equiv a_2 \mod 3$ and $k_1, k_2 \in \mathbb{Z}$. Accordingly, (1) is equivalent to

$$\frac{9(k_1 - k_2)}{b} = 8r, \quad r \in \mathbb{Z}.$$

If $9 \,|\, b$, this simply means $S(a_1, b) - S(a_2, b) \in 8\mathbb{Z}$, so this is just Tsukerman's result. However, if $9 \nmid b$, we obtain $3 \,|\, r$, which yields the theorem in the case $3 \,|\, b, 9 \nmid b$. □

Proof of Theorem 2. The Barkan-Hickerson-Knuth formula says

$$S(a, b) = T(a, b) + \frac{a + a^*}{b} - 3 \tag{7}$$

(see, for instance, [1]). Note that this formula is often enunciated only for the case $0 \leq a < b$, but it is, in fact, valid for arbitrary integers a relatively prime to b, provided that $T(a, b)$ is defined as in (2). Hence we obtain, by (5) and (7),

$$bT(a, b) \equiv -a - a^* \mod 3 \tag{8}$$

if $3 \nmid b$. In the case $3 \,|\, b$, (6) and (7) give

$$bT(a, b) \equiv 2\varepsilon - a - a^* \mod 9 \tag{9}$$

instead. Further, Tsukerman's congruence (3) is also valid for arbitrary integers a relatively prime to b, as we easily check. We combine (3) with the congruences (8) and (9) by means of the Chinese remainder theorem. This readily gives Theorem 2. □

3. A Proposition Yielding Examples

Our examples arise from the following proposition.

Proposition 1. *Let c, d be odd natural numbers, $d \geq 3$. Put $b = cd^2$ and $a = cd+1$. Then*

$$S(1, b) - S(a, b) = c(d^2 - 1).$$

Proof. We apply the reciprocity law for Dedekind sums (see [2, p. 5]), which gives

$$S(a,b) = -S(b,a) + \frac{a}{b} + \frac{b}{a} + \frac{1}{ab} - 3.$$

Now $b \equiv -d \mod a$, hence the reciprocity law says

$$S(b,a) = S(-d,a) = S(a,d) - \frac{a}{d} - \frac{d}{a} - \frac{1}{ad} + 3.$$

However, $a \equiv 1 \mod d$, and so $S(a,d) = S(1,d) = d - 3 + 2/d$. Inserting the values $b = cd^2$ and $a = cd + 1$ gives

$$S(a,b) = c - 3 + \frac{2}{b}.$$

Since $S(1,b) = b - 3 + 2/b$, we obtain the desired result. $\square$

In the setting of the proposition, let $3 \nmid d$. Then $d^2 - 1 \equiv 0 \mod 24$, so the proposition yields many examples with $S(1,b) - S(a,b) > 0$ and $S(1,b) - S(a,b) \equiv 0$ mod 24. On the other hand, if $3 \mid d$, then $d^2 - 1 \equiv 0$ mod 8, but $d^2 - 1 \not\equiv 0$ mod 24. If, therefore, $3 \nmid c$, we obtain many examples with $S(1,b) - S(a,b) \equiv 0$ mod 8, but $S(1,b) - S(a,b) \not\equiv 0$ mod 24.

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