#A32 INTEGERS 17 (2017)

AN INFINITE FAMILY OF QUARTIC POLYNOMIALS WHOSE PRODUCTS OF CONSECUTIVE VALUES ARE FINITELY OFTEN PERFECT SQUARES

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Received: 10/4/16, Revised: 3/5/17, Accepted: 7/10/17, Published: 7/17/17

Abstract

Using an elementary identity, we prove that for infinitely many polynomials $P(x) \in \mathbb{Z}[X]$ of fourth degree, the equation $\prod_{k=1}^{n} P(k) = y^2$ has finitely many solutions in $\mathbb{Z}$. We also give an example of a quartic polynomial for which the product of its consecutive values is infinitely often a perfect square.

1. Introduction

Over the last few years, there has been a growing interest in identifying whether certain product sequences contain perfect squares. In 2008 Javier Cilleruelo [1] proved a conjecture of Amdeberhan [2], that the product $(1^2+1)(2^2+1)\cdots(n^2+1)$ is a square only for $n = 3$. Soon after, Jin-Hui Fang [3] achieved to prove that both products $\prod_{k=1}^{n}(4k^2+1)$ and $\prod_{k=1}^{n}\big(2k(k-1)+1\big)$ are never squares. There are not many similar results for quadratic polynomials. However, in a recent paper [4] two certain cases of quartic polynomials have been investigated in a similar vain. In this paper we will prove, using elementary arguments, that there is actually an infinite list of quartic polynomials $P(x)$ such that the product $\prod_{k=1}^{n} P(k)$ is a square finitely often. At the end of the paper we discuss some special examples that can be handled by this method. We begin with a polynomial identity which is the key ingredient throughout this article.

Lemma 1. *Let $f(x) = x^2 + ax + b$ be a quadratic polynomial. For every $x \in \mathbb{R}$, the following formula is valid:*

$$f\big(f(x)+x\big) = f(x)f(x+1).$$

Proof. We can verify this just by doing elementary manipulations but we will prove the lemma using a clever observation. Since $f(x)$ is a polynomial of second degree, Taylor's formula gives $f\big(f(x)+x\big) = f(x) + \frac{f'(x)f(x)}{1!} + f^2(x)$. This is equal to $f(x)\big(1 + f'(x) + f(x)\big)$. But $1 + f'(x) + f(x) = 1 + 2x + a + x^2 + ax + b = (x+1)^2 + a(x+1) + b = f(x+1)$. Hence we have: $f\big(f(x)+x\big) = f(x)f(x+1)$. $\square$

This simple formula will play a key role in the proof of the main theorem. For convenience of notation we set $P(k) = f\big(f(k)+k\big) = f(k)f(k+1)$. It can be seen that $P(k) = k^4 + 2(a+1)k^3 + \big((a+1)^2 + 2b + a\big)k^2 + (a+1)(2b+a)k + b^2 + ab + b$. In the proof of the main theorem, we require a and b to obey a certain restriction. Under this restriction, we are able to prove that equation (1) shown below, has finitely many solutions.

2. Main Results

Theorem. *Let $a, b, m \in \mathbb{Z}$ and $a + b + 1 = m^2$. Then the diophantine equation*

$$\prod_{k=1}^{n} P(k) = y^2 \tag{1}$$

has finitely many solutions, where $P(k)$ is as above.

Proof. Using Lemma 1 we can rewrite equation (1) as

$$f(1)f(2)f(2)f(3)\cdots f(n)f(n+1) = y^2$$

which reduces to $f(1)f(n+1)\prod_{k=2}^{n}(f(k))^2 = y^2$. Since $f(1) = a + b + 1 = m^2$ we conclude that $f(n+1) = \frac{y^2}{m^2 \prod_{k=2}^{n}(f(k))^2}$. It becomes clear that equation (1) is satisfied whenever $f(n+1)$ is a perfect square. It remains to prove that among the values of $f(k)$ occur finitely many squares. Write

$$k^2 + ak + b = z^2 \tag{2}$$

for some $z \in \mathbb{Z}$. This means that for sufficiently large k, $k^2 < z^2 < (k+2a)^2$ if $a > 0$ or, $(k+2a)^2 < z^2 < k^2$ if $a < 0$. (If $a = 0$ then equation (2) transposes to $(z-k)(z+k) = b$ which clearly has finitely many solutions). Both of the inequalities yield $z = k + c$ for some $c \in \mathbb{Z}$ with $|c| < |2a|$. So, (2) becomes $k^2 + ak + b = (k+c)^2$ which has finitely many solutions as the reader may easily verify. $\square$

It suffices to choose some nice values for a and b in order to demonstrate the theorem. Choosing $(a, b) = (-1, 1)$ we have $f(k) = k^2 - k + 1$ hence the following:

Corollary. $\prod_{k=1}^{n}(k^4+k^2+1)$ *is a square only for* $n=1$.

Proof. If $(a,b)=(-1,1)$ then $f(1)=1^2$. Repeating the previous arguments, it suffices to show that $k^2-k+1=y^2$ has one solution. Indeed, if $k^2-k+1=y^2$ then we must have $k^2 \le y^2 < (k+1)^2$ which yields $y=k$ and so $k=1$. The claim follows. $\square$

Remark. Arguing as in the previous section, we may present an example which shows that equation (1) has infinitely many solutions. Choosing $(a,b)=(-4,2)$ we have $f(k)=k^2-4k+2$ and $P(k)=\big(k(k-3)\big)^2-2$. We can prove that the product $\prod_{k=4}^{n}\Big(\big(k(k-3)\big)^2-2\Big)$ is a square infinitely often. Here we start with $k=4$ to omit any trivial case in which the product has negative factors. The product is a square if $f(4)f(n+1)=2(n-1)^2-4=y^2$. It is a routine matter to prove that both y and $n-1$ must be even. The last equation can be written as $(\frac{y}{2})^2-2(\frac{n-1}{2})^2=-1$ which is a special case of the negative Pell equation $X^2-2Y^2=-1$. This equation has the fundamental solution $(1,1)$ and all it's positive solutions can be found by taking odd powers of $1+\sqrt{2}$. The positive solutions are (X_n,Y_n) where $X_n+Y_n\sqrt{2}=(1+\sqrt{2})^{2n-1}$. The next solution is $(X_2,Y_2)=(7,5)$ which gives $n=11$. As an example we can verify that $\prod_{k=4}^{11}\Big(\big(k(k-3)\big)^2-2\Big)=246988938224^2$.

References

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