FINITE RECIPROCAL SUMS OF PRODUCTS INVOLVING SQUARES OF SINES OR COSINES WITH ARGUMENTS IN ARITHMETIC PROGRESSION

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Received: 9/6/16, Revised: 4/19/17, Accepted: 7/21/17, Published: 8/7/17

Abstract

In this paper, we present closed forms for certain finite sums. In each case, the denominator of the summand is a product of sine or cosine functions, where at least one of these functions is squared. Furthermore, in each case, the arguments of the trigonometric functions in the denominator of the summand increase in arithmetic progression.

1. Introduction

In [1], we present closed forms for finite sums in which the denominator of the summand is a product of sine or cosine functions. In each case, the arguments of the trigonometric functions in the denominator of the summand increase in arithmetic progression. For instance, in [1, Sec. 4] we find

$$\sum_{i=1}^{n-1} \frac{\cos(i+1)}{\sin i \sin(i+1)\sin(i+2)} = \frac{(2\cos 2+1)\,(\cos 3-\cos(2n+1))}{4\sin 1\sin 2\sin 3\sin n\sin(n+1)}.$$

In the present paper, we continue this theme. Here, however, at least one of the trigonometric functions in the denominator of the summand is squared.

At the outset, we set the constraints on the parameters d and n that occur in this paper. Throughout, $d \geq 1$ and $n \geq 2$ are assumed to be integers.

In Section 3, we give a closed form for the finite sum

$$S_2(n,d) = \sum_{i=1}^{n-1} \frac{\sin(2i+d)}{\sin^2 i \sin^2(i+d)},$$

with an accompanying *dual* result. In the course of our investigation, we have found a dual result for each finite sum whose closed form we have been able to write down.

We categorize each finite sum that we consider according to the number of distinct factors in the denominator of its summand. For instance, S_2 has two distinct factors in the denominator of its summand.

We limit the scope of this paper so that each finite sum that we consider has at most five distinct factors in the denominator of its summand. Furthermore, in order to keep the presentation to a reasonable length, we give the closed forms for only a selection of the sums that we define. Our purpose is to give the reader an appreciation of the kinds of results that we have managed to discover.

In Section 2, we define four finite sums, Φ_s, Φ_c, Ψ_s, and Ψ_c, in terms of which we express all our results. We present our main results in Sections 3, 4, 5, and 6, and demonstrate the method of proof that can be used to prove each of these results in Section 5.

2. The Finite Sums Φ_s, Φ_c, Ψ_s, and Ψ_c

There are four finite sums that we use to express the closed forms for all the sums that occur in this paper. Let $0 \le l_1 < l_2$ be integers. Then, for the sine function, the finite sums in question are

$$\Phi_s(n, l_1, l_2) = \sum_{i=l_1}^{l_2-1} \frac{\sin(i+n-1)\sin(i+2) + \sin(i+n)\sin(i+1)}{\sin^2(i+2)\sin^2(i+n)},$$

$$\Psi_s(n, l_1, l_2) = \sum_{i=l_1}^{l_2-1} \frac{1}{\sin(i+2)\sin(i+n)}.$$

For the cosine function, the analogous finite sums are

$$\Phi_c(n, l_1, l_2) = \sum_{i=l_1}^{l_2-1} \frac{\cos(i+n-1)\cos(i+2) + \cos(i+n)\cos(i+1)}{\cos^2(i+2)\cos^2(i+n)},$$

$$\Psi_c(n, l_1, l_2) = \sum_{i=l_1}^{l_2-1} \frac{1}{\cos(i+2)\cos(i+n)}.$$

In the lemma that follows, we state identities for Ψ_s and Ψ_c that are required for the proofs of our theorems. These identities can be found in [1].

Lemma 1. *With Ψ_s and Ψ_c as defined above,*

$$\sin(n-1)\Psi_s(n+1, l_1, l_2) - \sin(n-2)\Psi_s(n, l_1, l_2) = \frac{\sin(l_2 - l_1)}{\sin(n+l_1)\sin(n+l_2)}, \quad (1)$$

$$\sin(n-1)\Psi_c(n+1, l_1, l_2) - \sin(n-2)\Psi_c(n, l_1, l_2) = \frac{\sin(l_2 - l_1)}{\cos(n+l_1)\cos(n+l_2)}. \quad (2)$$

We also require identities, analogous to (1) and (2), for Φ_s and Φ_c. We show only how to prove the identity for Φ_s, since the identity for Φ_c can be proved in a similar fashion.

It follows from the well known identities

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}, \tag{3}$$

$$\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right), \tag{4}$$

that

$$\sin 1\sin(\alpha-\beta) = \sin(\alpha+\gamma-1)\sin(\beta+\gamma) - \sin(\alpha+\gamma)\sin(\beta+\gamma-1). \tag{5}$$

Specifically, upon applying (3) to each product on the right side of (5), and then applying (4) to the result, one obtains (5).

Next, in (5), replace α by n, β by 2, and γ by i, to obtain

$$\sin 1\sin(n-2) = \sin(i+n-1)\sin(i+2) - \sin(i+n)\sin(i+1).$$

It then follows that

$$\begin{aligned}\sin(n-2)\Phi_s(n,l_1,l_2) &= \sum_{i=l_1}^{l_2-1}\frac{\sin^2(i+n-1)\sin^2(i+2)-\sin^2(i+n)\sin^2(i+1)}{\sin 1\sin^2(i+2)\sin^2(i+n)}\\ &= \frac{1}{\sin 1}\sum_{i=l_1}^{l_2-1}\left(\frac{\sin^2(i+n-1)}{\sin^2(i+n)}-\frac{\sin^2(i+1)}{\sin^2(i+2)}\right).\end{aligned} \tag{6}$$

We can now prove the lemma that follows.

Lemma 2. *With Φ_s as defined at the beginning of this section,*

$$\begin{aligned}&\sin(n-1)\Phi_s(n+1,l_1,l_2)-\sin(n-2)\Phi_s(n,l_1,l_2)\\ &= \frac{\sin(l_2-l_1)\left(\cos 1\cos(l_2-l_1)-\cos(2n+l_1+l_2-1)\right)}{\sin^2(n+l_1)\sin^2(n+l_2)}.\end{aligned} \tag{7}$$

Proof. We have

$$\begin{aligned}&\sin(n-1)\Phi_s(n+1,l_1,l_2)-\sin(n-2)\Phi_s(n,l_1,l_2)\\ &= \frac{1}{\sin 1}\left(\frac{\sin^2(n+l_2-1)}{\sin^2(n+l_2)}-\frac{\sin^2(n+l_1-1)}{\sin^2(n+l_1)}\right) \text{ from (6)}\\ &= \frac{\sin(l_2-l_1)\left(\sin(n+l_2-1)\sin(n+l_1)+\sin(n+l_2)\sin(n+l_1-1)\right)}{\sin^2(n+l_1)\sin^2(n+l_2)} \qquad (8)\\ &= \frac{\sin(l_2-l_1)\left(\cos 1\cos(l_2-l_1)-\cos(2n+l_1+l_2-1)\right)}{\sin^2(n+l_1)\sin^2(n+l_2)}.\end{aligned}$$

The third line in (8) follows from (5). We obtain the fourth line in (8) by applying (3) twice, and then observing that $\cos(l_2-l_1-1)+\cos(l_2-l_1+1)$ can be expressed as a product with the use of a well known identity that is analogous to (4). □

An identity for Φ_c, analogous to (7), is given in the following lemma.

Lemma 3. *With Φ_c as defined at the beginning of this section,*

$$\begin{aligned}&\sin(n-1)\Phi_c(n+1,l_1,l_2)-\sin(n-2)\Phi_c(n,l_1,l_2)\\&=\frac{\sin(l_2-l_1)\left(\cos 1\cos(l_2-l_1)+\cos(2n+l_1+l_2-1)\right)}{\cos^2(n+l_1)\cos^2(n+l_2)}.\end{aligned}$$

3. The Closed Form for S_2

In the introduction, we defined the sum

$$S_2(n,d)=\sum_{i=1}^{n-1}\frac{\sin(2i+d)}{\sin^2 i\sin^2(i+d)}.$$

In this section, we give the closed form for S_2, together with its dual result. Recall that in the introduction we stated that, throughout this paper, $d\geq 1$ and $n\geq 2$ are assumed to be integers. Also, throughout this paper, we take Φ_s, Φ_c, Ψ_s, and Ψ_c to be as defined in Section 2.

Theorem 1. *With S_2 as defined above,*

$$\sin 1\sin d\left(S_2(n,d)-S_2(2,d)\right)=\sin(n-2)\left(-\Phi_s(n,0,d)+2\cos 1\Psi_s(n,0,d)\right). \quad (9)$$

The dual result for (9) is as follows: In the denominator of the summand of S_2, replace each occurrence of sin by cos. In (9), replace Φ_s and Ψ_s by Φ_c and Ψ_c, respectively. Then (9) remains valid if the right side is multiplied by -1.

For $d=1$, after simplification, formula (9) and its dual become, respectively,

$$\sum_{i=1}^{n-1}\frac{\sin(2i+1)}{\sin^2 i\sin^2(i+1)}=\frac{1}{\sin 1}\left(\frac{1}{\sin^2 1}-\frac{1}{\sin^2 n}\right), \quad (10)$$

$$\sum_{i=1}^{n-1}\frac{\sin(2i+1)}{\cos^2 i\cos^2(i+1)}=\frac{1}{\sin 1}\left(\frac{1}{\cos^2 n}-\frac{1}{\cos^2 1}\right). \quad (11)$$

Again, in (9) let $d=2$. Then (9) and its dual become, respectively,

$$\sum_{i=1}^{n-1}\frac{\sin(2i+2)}{\sin^2 i\sin^2(i+2)}=\frac{1}{\sin 2}\left(\frac{3+2\cos 2}{\sin^2 2}-\frac{1}{\sin^2 n}-\frac{1}{\sin^2(n+1)}\right),$$

$$\sum_{i=1}^{n-1}\frac{\sin(2i+2)}{\cos^2 i\cos^2(i+2)}=\frac{1}{\sin 2}\left(\frac{1}{\cos^2 n}+\frac{1}{\cos^2(n+1)}-\frac{1}{\cos^2 1}-\frac{1}{\cos^2 2}\right).$$

4. The Summand Has Three Distinct Factors in the Denominator

In this section, we present closed forms for the three finite sums that we define below.

Define

$$S_3^0(n,d) = \sum_{i=1}^{n-1} \frac{\cos(i+d)}{\sin^2 i \sin(i+d)\sin^2(i+2d)},$$
$$S_3^1(n,d) = \sum_{i=1}^{n-1} \frac{\cos^3(i+d)}{\sin^2 i \sin(i+d)\sin^2(i+2d)},$$
$$S_3^2(n,d) = \sum_{i=1}^{n-1} \frac{\cos(i+d)\cos(2(i+d))}{\sin^2 i \sin(i+d)\sin^2(i+2d)}.$$

In order to succinctly present the closed form for S_3^0, we make use of certain coefficients that are functions of the parameter d. Specifically, set

$$\begin{aligned} a_0 &= a_0(d) = 2\sin 1 \sin^3 d \sin(2d), \\ a_1 &= a_1(d) = -\sin d, \\ b_1 &= b_1(d) = 2\sin(d-1) \\ b_2 &= b_2(d) = 2\sin(d+1). \end{aligned} \tag{12}$$

We now give the closed form for S_3^0 in the following theorem.

Theorem 2. *With the a_i and b_i as defined in (12),*

$$\begin{aligned} a_0\left(S_3^0(n,d) - S_3^0(2,d)\right) &= a_1 \sin(n-2)\left(\Phi_s(n,0,d) + \Phi_s(n,d,2d)\right) \\ &\quad + \sin(n-2)\left(b_1\Psi_s(n,0,d) + b_2\Psi_s(n,d,2d)\right). \end{aligned} \tag{13}$$

Next, we present the closed form for S_3^1. In order to keep the notation simple, we merely redefine the coefficients a_0, a_1, b_1, and b_2. Accordingly, define

$$\begin{aligned} a_0 &= a_0(d) = 4\sin 1 \sin^3 d \sin(2d), \\ a_1 &= a_1(d) = -\cos d \sin(2d), \\ b_1 &= b_1(d) = 2\cos d\left(\cos 1 \sin(2d) - 2\sin 1\right), \\ b_2 &= b_2(d) = 2\cos d\left(\cos 1 \sin(2d) + 2\sin 1\right). \end{aligned} \tag{14}$$

We then have the following theorem.

Theorem 3. *With the a_i and b_i as defined in (14),*

$$\begin{aligned} a_0\left(S_3^1(n,d) - S_3^1(2,d)\right) &= a_1 \sin(n-2)\left(\Phi_s(n,0,d) + \Phi_s(n,d,2d)\right) \\ &\quad + \sin(n-2)\left(b_1\Psi_s(n,0,d) + b_2\Psi_s(n,d,2d)\right). \end{aligned} \tag{15}$$

To present the closed form for S_3^2, we redefine the a_i and b_i as

$$\begin{aligned}
a_0 &= a_0(d) = \sin 1 \sin^3 d \sin(2d), \\
a_1 &= a_1(d) = -\frac{1}{2} \sin d \cos(2d), \\
b_1 &= b_1(d) = \cos 1 \sin d \cos(2d) - \sin 1 \cos d, \\
b_2 &= b_2(d) = \cos 1 \sin d \cos(2d) + \sin 1 \cos d.
\end{aligned} \tag{16}$$

We then have the theorem that follows.

Theorem 4. *With the a_i and b_i as defined in (16),*

$$\begin{aligned}
a_0 \left(S_3^2(n,d) - S_3^2(2,d)\right) &= a_1 \sin(n-2) \left(\Phi_s(n,0,d) + \Phi_s(n,d,2d)\right) \\
&\quad + \sin(n-2) \left(b_1 \Psi_s(n,0,d) + b_2 \Psi_s(n,d,2d)\right).
\end{aligned} \tag{17}$$

The dual results for (13) and (15) are as follows: In the summands of S_3^0 and S_3^1, replace each occurrence of sin by cos, and replace each occurrence cos by sin. In (13) and (15), replace each occurrence of Φ_s and Ψ_s by Φ_c and Ψ_c, respectively. Then (13) and (15) remain valid if the right side of each is multiplied by -1.

The only dual result for (17) that we could find is (9). Specifically, in the summand of S_3^2, replace each occurrence of cos by sin. Then the summand that results is the summand of S_2 with d replaced by $2d$.

To conclude this section, we set $d = 1$ in (13). Then upon simplification, (13) and its dual become, respectively,

$$\sum_{i=1}^{n-1} \frac{\cos(i+1)}{\sin^2 i \sin(i+1) \sin^2(i+2)} = \frac{\sin(n-1)\sin(n+2)\left(\cos 1 - \cos(n-1)\cos(n+2)\right)}{16 \sin^5 1 \cos^3 1 \sin^2 n \sin^2(n+1)},$$

$$\sum_{i=1}^{n-1} \frac{\sin(i+1)}{\cos^2 i \cos(i+1) \cos^2(i+2)} = \frac{\sin(n-1)\sin(n+2)\left(\cos 1 + \cos(n-1)\cos(n+2)\right)}{4 \sin 1 \cos^3 1 \cos^2 2 \cos^2 n \cos^2(n+1)}.$$

5. The Summand Has Four Distinct Factors in the Denominator

We have managed to find a closed form for the finite sum

$$S_4(n,d) = \sum_{i=1}^{n-1} \frac{\sin^3(2i+3d)}{\sin^2 i \sin^2(i+d) \sin^2(i+2d) \sin^2(i+3d)}.$$

The other finite sums in this section for which we have managed to find closed forms fall into several categories. One category consists of the three finite sums

$$S_4^0(n,d,k) = \sum_{i=1}^{n-1} \frac{\sin(k(2i+3d))}{\sin^2 i \sin^2(i+d) \sin^2(i+2d) \sin^2(i+3d)}, \ k = 1,2,3.$$

A second category consists of the two finite sums

$$S_4^1(n,d,k) = \sum_{i=1}^{n-1} \frac{\sin(k(2i+3d))}{\sin^2 i \sin(i+d)\sin(i+2d)\sin^2(i+3d)}, \ k=1,2.$$

Finally, the third category consists of the two finite sums

$$S_4^2(n,d,k) = \sum_{i=1}^{n-1} \frac{\sin(k(2i+3d))}{\sin i \sin^2(i+d)\sin^2(i+2d)\sin(i+3d)}, \ k=1,2.$$

Next, we present the closed form for $S_4^0(n,d,1)$, together with a proof. Our method of proof can be used to prove all the theorems in this paper. During the course of our proof, we make use of the *Mathematica* command FullSimplify. To quote, "FullSimplify[*expr*] tries a wide range of transformations on *expr* involving elementary and special functions, and returns the simplest form it finds". For instance, FullSimplify$\left[\sin(3\theta) - 3\sin\theta + 4\sin^3\theta\right] = 0$. In an age where computer algorithms are used in fields as diverse as primality testing, and the proof of binomial identities, we feel that our use of *Mathematica's* FullSimplify to fast track the simplification of lengthy trigonometric identities is entirely appropriate.

Define $a_i = a_i(d)$, $0 \le i \le 3$, and $b_i = b_i(d)$, $1 \le i \le 3$, by

$$\begin{aligned}
a_0 &= \sin 1 \sin^2 d \sin^3(2d)\sin(3d), \\
a_1 &= a_3 = -\sin(2d), \\
a_2 &= -4\sin(2d)\cos^2 d, \\
b_1 &= 2\cos 1 \sin(2d) - 2\sin 1\left(2\cos(2d)+1\right), \\
b_2 &= 8\cos 1 \sin(2d)\cos^2 d, \\
b_3 &= 2\cos 1 \sin(2d) + 2\sin 1\left(2\cos(2d)+1\right).
\end{aligned} \tag{18}$$

We now give the closed form for $S_4^0(n,d,1)$ in the following theorem.

Theorem 5. *With the a_i and b_i as defined in (18),*

$$\begin{aligned}
a_0\left(S_4^0(n,d,1) - S_4^0(2,d,1)\right) \quad = \quad & \sin(n-2)\left(\sum_{i=1}^{3} a_i \Phi_s(n,(i-1)d,id)\right. \\
& \left. + \sum_{i=1}^{3} b_i \Psi_s(n,(i-1)d,id)\right).
\end{aligned} \tag{19}$$

Proof. Denote the quantities on the left and right sides of (19) by $L(n,d)$ and $R(n,d)$, respectively. Then

$$L(n+1,d) - L(n,d) = \frac{\sin 1 \sin^2 d \sin^3(2d)\sin(3d)\sin(2n+3d)}{\sin^2 n \sin^2(n+d)\sin^2(n+2d)\sin^2(n+3d)}.$$

Also, with the use of (1) and (7), we have

$$R(n+1,d)-R(n,d) = \sin d\left(\sum_{i=1}^{3} a_i \frac{\cos 1\cos d-\cos(2n+(2i-1)d-1)}{\sin^2(n+(i-1)d)\sin^2(n+id)} + \sum_{i=1}^{3}\frac{b_i}{\sin(n+(i-1)d)\sin(n+id)}\right). \tag{20}$$

Now, with the use of *Mathematica 8*, we apply the command FullSimplify to the right side of (20) to obtain

$$\frac{8\sin 1\cos^3 d\sin^6 d\,(1+2\cos(2d))\sin(2n+3d)}{\sin^2 n\sin^2(n+d)\sin^2(n+2d)\sin^2(n+3d)}.$$

Furthermore, an application of FullSimplify to

$$\sin 1\sin^2 d\sin^3(2d)\sin(3d)-8\sin 1\cos^3 d\sin^6 d\,(1+2\cos(2d))$$

returns zero, so that

$$L(n+1,d)-L(n,d)=R(n+1,d)-R(n,d). \tag{21}$$

Also

$$L(2,d)=R(2,d)=0. \tag{22}$$

Together, (21) and (22) show that (19) is true. This completes the proof of Theorem 5. □

Since the closed form for $S_4^2(n,d,1)$ is relatively succinct, we present this closed form in our next theorem. To this end, define

$$\begin{aligned} a_0 &= a_0(d)=\sin 1\sin d\sin^3(2d),\\ a_1 &= a_1(d)=2\cos d\sin(2d),\\ b_1 &= b_1(d)=2\sin 1\cos d,\\ b_2 &= b_2(d)=4\cos 1\cos d\sin(2d). \end{aligned} \tag{23}$$

We now give the closed form for $S_4^2(n,d,1)$ in the theorem that follows.

Theorem 6. *With the a_i and b_i as defined in (23),*

$$a_0\left(S_4^2(n,d,1)-S_4^2(2,d,1)\right) = \sin(n-2)\left(a_1\Phi_s(n,d,2d)+b_1\Psi_s(n,0,d) - b_2\Psi_s(n,d,2d)-b_1\Psi_s(n,2d,3d)\right). \tag{24}$$

The dual result for (24) is obtained as follows: In the denominator of the summand of $S_4^2(n,d,1)$, replace each occurrence of sin by cos. In (24), replace each occurrence of Φ_s and Ψ_s by Φ_c and Ψ_c, respectively. Then (24) remains valid if the right side is multiplied by -1. The dual result for (19) is obtained in precisely the same manner.

6. The Summand Has Five Distinct Factors in the Denominator

As in the previous section, the finite sums in this section for which we have been able to find closed forms fall into several categories.

One category consists of the four sums

$$S_5^0(n,d,k) = \sum_{i=1}^{n-1} \frac{\cos^k(i+2d)}{\sin^2 i \sin^2(i+d)\sin(i+2d)\sin^2(i+3d)\sin^2(i+4d)}, \; k = 1,3,5,7.$$

A second category consists of the three finite sums

$$S_5^1(n,d,k) = \sum_{i=1}^{n-1} \frac{\cos(k(i+2d))}{\sin^2 i \sin^2(i+d)\sin(i+2d)\sin^2(i+3d)\sin^2(i+4d)}, \; k = 3,5,7.$$

The next category consists of the three finite sums

$$\begin{aligned}
S_5^2(n,d) &= \sum_{i=1}^{n-1} \frac{\cos(i+d)\cos(i+2d)\cos(i+3d)}{\sin^2 i \sin^2(i+d)\sin(i+2d)\sin^2(i+3d)\sin^2(i+4d)},\\
S_5^3(n,d) &= \sum_{i=1}^{n-1} \frac{\cos(i+d)\cos(i+2d)\cos(i+3d)}{\sin^2 i \sin(i+d)\sin(i+2d)\sin(i+3d)\sin^2(i+4d)},\\
S_5^4(n,d) &= \sum_{i=1}^{n-1} \frac{\cos(i+d)\cos(i+2d)\cos(i+3d)}{\sin i \sin^2(i+d)\sin(i+2d)\sin^2(i+3d)\sin(i+4d)}.
\end{aligned}$$

Finally, in the summand of each of S_5^2, S_5^3, and S_5^4, keep the same denominator, but replace the numerator by $\cos(i+d)\cos(i+2d)\cos(i+4d)$. Then we have managed to find closed forms for the corresponding finite sums.

Since the pattern of the closed forms that we present in this paper is now clear, in this section we present only the closed form for $S_5^0(n,d,1)$. Accordingly, define $a_i = a_i(d)$, $0 \le i \le 4$, and $b_i = b_i(d)$, $1 \le i \le 4$, by

$$\begin{aligned}
a_0 &= 4\sin 1 \sin d \cos(2d)\sin^4(2d)\sin^3(3d),\\
a_1 &= a_4 = -2\cos d \sin(3d),\\
a_2 &= a_3 = -2\cos d \sin(3d)\left(1+2\cos(2d)\right)^2,\\
b_1 &= -2\sin 1 \cos(2d)\left(5+6\cos(2d)\right) + 4\cos 1 \cos d \sin(3d),\\
b_2 &= 2\sin(2d-1)\left(1+2\cos(2d)\right)^3,\\
b_3 &= 2\sin(2d+1)\left(1+2\cos(2d)\right)^3,\\
b_4 &= 2\sin 1 \cos(2d)\left(5+6\cos(2d)\right) + 4\cos 1 \cos d \sin(3d).
\end{aligned} \tag{25}$$

We give the closed form for $S_5^0(n,d,1)$ in the following theorem.

Theorem 7. *With the a_i and b_i as defined in (25),*

$$\begin{aligned} a_0\left(S_5^0(n,d,1)-S_5^0(2,d,1)\right) &= \sin(n-2)\left(\sum_{i=1}^{4} a_i\Phi_s(n,(i-1)d,id)\right. \\ &\qquad \left. +\sum_{i=1}^{4} b_i\Psi_s(n,(i-1)d,id)\right). \end{aligned} \tag{26}$$

The dual result for (26) is obtained as follows: In the summand of $S_5^0(n,d,1)$, replace each occurrence of sin by cos, and replace each occurrence cos by sin. In (26), replace each occurrence of Φ_s and Ψ_s by Φ_c and Ψ_c, respectively. Then (26) remains valid if the right side is multiplied by -1.

7. Our Results Generalized

As in [1], in this paper we present all our results in abbreviated form. To write down our results in their most general form is easy. Let θ be any real number that is not a rational multiple of π. This condition on θ excludes the possibility of vanishing denominators. Then this entire paper can be generalized in the following manner: take *every* occurrence of sin and cos, and multiply the argument by θ.

For example, redefine Φ_s and Ψ_s as

$$\Phi_s(n,l_1,l_2,\theta)=\sum_{i=l_1}^{l_2-1}\frac{\sin((i+n-1)\theta)\sin((i+2)\theta)+\sin((i+n)\theta)\sin((i+1)\theta)}{\sin^2((i+2)\theta)\sin^2((i+n)\theta)},$$
$$\Psi_s(n,l_1,l_2,\theta)=\sum_{i=l_1}^{l_2-1}\frac{1}{\sin((i+2)\theta)\sin((i+n)\theta)}.$$

Also, redefine S_3^0 as

$$S_3^0(n,d,\theta)=\sum_{i=1}^{n-1}\frac{\cos((i+d)\theta)}{\sin^2(i\theta)\sin((i+d)\theta)\sin^2((i+2d)\theta)}.$$

We next include θ in the arguments of the a_i and b_i in (12). Accordingly, define

$$\begin{aligned} a_0 &= a_0(d,\theta)=2\sin\theta\sin^3(d\theta)\sin(2d\theta),\\ a_1 &= a_1(d,\theta)=-\sin(d\theta),\\ b_1 &= b_1(d,\theta)=2\sin((d-1)\theta)\\ b_2 &= b_2(d,\theta)=2\sin((d+1)\theta). \end{aligned} \tag{27}$$

Then, with the a_i and b_i as in (27), (13) generalizes to

$$a_0\left(S_3^0(n,d,\theta)-S_3^0(2,d,\theta)\right)=a_1\sin((n-2)\theta)\left(\Phi_s(n,0,d,\theta)+\Phi_s(n,d,2d,\theta)\right)\\+\sin((n-2)\theta)\left(b_1\Psi_s(n,0,d,\theta)+b_2\Psi_s(n,d,2d,\theta)\right).$$

As we state at the end of the introduction, our purpose in this paper is to give the reader an appreciation of the kinds of results that we have managed to discover. We trust that our presentation achieves this purpose.

8. Observations of an Anonymous Referee

The entire contents of this section are the observations of an anonymous referee. These observations were formulated with the assistance of a computer algebra system.

Regarding (10) and (11), set

$$X_n := \sum_{i=1}^{n-1} \frac{\sin(2i+1)}{\sin^2 i \sin^2(i+1)},$$
$$Y_n := \sum_{i=1}^{n-1} \frac{\sin(2i+1)}{\cos^2 i \cos^2(i+1)}.$$

Then we have

$$Y_n\left(1-X_n\sin^3 1\right)\left(1-\sin^2 1\right)-X_n\sin^2 1-(Y_n-X_n)\sin^2 1\left(1-\sin^2 1\right)=0.$$

Setting $Z := \sin 1$ and replacing n by $n+1$, we obtain a second polynomial equation. The two polynomials in question belong to $\mathbb{Q}\left(X_n, Y_n\right)[Z]$ and $\mathbb{Q}\left(X_{n+1}, Y_{n+1}\right)[Z]$, respectively. Now, we can remove $\sin 1$ from both polynomial equations. We do this by computing the resultant of the two polynomials with respect to the variable Z. This gives an integer polynomial in X_n, X_{n+1}, Y_n, Y_{n+1} consisting of forty-four monomials, and which is independent of n. In particular, for every integer $n \geq 1$, the four numbers X_n, X_{n+1}, Y_n, Y_{n+1} are algebraically dependent over $\mathbb{Q}$.

Next, set $s := \sin d$ and set

$$T(x,y) := \left((x-1)(2y-1)-s^2(2x-1)(2y-1)+\left(2s^2-1\right)(y-1)(2x-1)\right)^2 \\ -4s^2\left(1-s^2\right)y(y-1)(2x-1)^2.$$

Then for all integers $n, d \geq 1$, we have

$$T\left(\frac{S_3^1(n+1,d)-S_3^1(n,d)}{S_3^2(n+1,d)-S_3^2(n,d)}, \frac{S_3^1(n-d+1,d)-S_3^1(n-d,d)}{S_3^2(n-d+1,d)-S_3^2(n-d,d)}\right)=0.$$

Multiplying the left side by

$$\left(S_3^2(n+1,d)-S_3^2(n,d)\right)^2\left(S_3^2(n-d+1,d)-S_3^2(n-d,d)\right)^2,$$

we obtain a polynomial from $\mathbb{Q}(\sin d)\,[x_1,\ldots,x_8]\setminus\{0\}$ with two hundred and twenty-five monomials, which is independent of n, and which vanishes for

$$\begin{aligned}
x_1 &= S_3^1(n,d),\\
x_2 &= S_3^1(n+1,d),\\
x_3 &= S_3^2(n,d),\\
x_4 &= S_3^2(n+1,d),\\
x_5 &= S_3^1(n-d,d),\\
x_6 &= S_3^1(n-d+1,d),\\
x_7 &= S_3^2(n-d,d),\\
x_8 &= S_3^2(n-d+1,d).
\end{aligned}$$

In particular, $x_1,\ldots,x_8$ are algebraically dependent over $\mathbb{Q}(\sin d)$.

We express our gratitude to the referee for a careful reading of the manuscript, and for the interest shown in this work.

References

[1] R. S. Melham, Closed forms for finite sums in which the denominator of the summand is a product of trigonometric functions, *Fibonacci Quart.* **54.3** (2016), 196–203.