AN ORDERED PARTITION EXPANSION OF THE DETERMINANT

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Abstract

From a transfer formula in multivariate finite operator calculus, we prove an expansion for the determinant similar to Ryser's formula for the permanent. However, the terms in this expansion are indexed by ordered partitions, and hence it contains many more terms than the usual determinant formula. To prove it, we consider the poset of ordered partitions, properties of the permutahedron, and some fundamental combinatorial techniques.

1. Introduction

One of the foundational concepts of linear algebra is the determinant. At the most basic level, this matrix parameter is celebrated for its intricate ties to the set of eigenvalues and as a similarity invariant. However, the determinant still surprises us as the solution to a varying array of problems.

In addition to solving systems of linear equations and performing a change of variables in calculus, the determinant can help us count! The determinant will calculate the number of nonintersecting n-paths in certain nonpermutable digraphs, where an n-path is a set of n paths from n distinct source vertices to n distinct sink vertices [1, 2, 3, 6]. In fact, the permanent will count the number of all n-paths.

The determinant of a matrix can be found recursively, as an alternating sum of minors. Often the determinant of an $n \times n$ matrix A is defined compactly using the

Leibniz formula, precisely

$$\det(A) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}.$$

Similarly, the permanent can be defined as a sum over subsets of $[n] := \{1, 2, \ldots, n\}$ using Ryser's formula [5]

$$\operatorname{perm}(A) = \sum_{S \subseteq [n]} (-1)^{n-|S|} \prod_{i=1}^{n} \sum_{j \in S} a_{i,j}.$$

In this paper, we prove a much messier expansion of the determinant by instead indexing our terms using the set of ordered partitions of $[n]$. Aptly, we call this the *ordered partition expansion of the determinant.* This expansion is analogous to Ryser's formula for the permanent. Section 5 explains the origins of this expansion as it relates to multivariate finite operator calculus, a branch of mathematics that has proven useful in enumerating ballot (generalized Dyck) paths containing certain patterns [7, 10].

Before stating the formula for our expansion of the determinant, we introduce it with two fundamental examples. When $n = 2$ the ordered partition expansion is as follows:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}(a_{12} + a_{22}) + a_{22}(a_{11} + a_{21}) - (a_{11} + a_{21})(a_{12} + a_{22}). \quad (1)$$

Likewise when $n = 3$ we get the following expansion:

$$\begin{aligned} |A| = \; & a_{11}(a_{12} + a_{22})(a_{13} + a_{23} + a_{33}) + a_{11}(a_{13} + a_{33})(a_{12} + a_{22} + a_{32}) \\ & + a_{22}(a_{11} + a_{21})(a_{13} + a_{23} + a_{33}) + a_{22}(a_{23} + a_{33})(a_{11} + a_{21} + a_{31}) \\ & + a_{33}(a_{11} + a_{31})(a_{12} + a_{22} + a_{32}) + a_{33}(a_{22} + a_{32})(a_{11} + a_{21} + a_{31}) \\ & - a_{11}(a_{12} + a_{22} + a_{32})(a_{13} + a_{23} + a_{33}) - a_{22}(a_{11} + a_{21} + a_{31})(a_{13} + a_{23} + a_{33}) \\ & - a_{33}(a_{11} + a_{21} + a_{31})(a_{12} + a_{22} + a_{32}) - (a_{11} + a_{21})(a_{12} + a_{22})(a_{13} + a_{23} + a_{33}) \\ & - (a_{11} + a_{31})(a_{13} + a_{33})(a_{12} + a_{22} + a_{32}) - (a_{22} + a_{32})(a_{23} + a_{33})(a_{11} + a_{21} + a_{31}) \\ & + (a_{11} + a_{21} + a_{31})(a_{12} + a_{22} + a_{32})(a_{13} + a_{23} + a_{33}). \end{aligned}$$

Our main theorem gives a general description of the ordered partition expansion of the determinant.

Theorem 1. *Let $A = (a_{ij})_{n \times n}$. The following formula is an expansion for the determinant of A:*

$$\det(A) = \sum_{B \vdash [n]} (-1)^{n-|B|} \prod_{\beta_k \in B} \prod_{j \in \beta_k} \sum_{i \in \beta'_k} a_{ij}, \quad (2)$$

where the outer summation runs over all ordered partitions $B = (\beta_1, \beta_2, \ldots, \beta_r)$ of the set $[n]$ and the inner summation runs over all integers i in the union of the first k parts $\beta'_k = \bigcup_{j=1}^{k} \beta_j$ of the partition B.

The following example serves to clarify the notation in Theorem 1.

Example 1. If

$$B = (\beta_1, \beta_2, \beta_3) = (\{2\}, \{1, 3\}, \{4, 5\}),$$

then

$$B' = (\beta'_1, \beta'_2, \beta'_3) = (\{2\}, \{1, 2, 3\}, \{1, 2, 3, 4, 5\}),$$

and the corresponding expression in Equation (2) for the ordered partition B is

$$a_{22}(a_{11}+a_{21}+a_{31})(a_{13}+a_{23}+a_{33})(a_{14}+a_{24}+a_{34}+a_{44}+a_{54})(a_{15}+a_{25}+a_{35}+a_{45}+a_{55}).$$

Notice that the first index runs through B', while the second index runs through the partition B.

The rest of this paper proceeds as follows: In Section 2, we analyze the functions $f : [n] \to [n]$ indexing the terms in the ordered partition expansion. After setting notation and proving a few fundamental lemmas, we end that section with Corollary 1, which proves that when $f : [n] \to [n]$ is a bijective function, or a permutation, then the coefficient $c_f = sgn(f)$. In other words, these coefficients are precisely the ones appearing in the determinant. In Section 3, we study the poset of ordered partitions and identify the importance of singleton partitions so that we can formulate our problem in more geometric terms as Euler characteristics of convex polytopes relating to the permutahedron. In Section 4, we prove that $c_f = 0$ for all non-bijective functions $f : [n] \to [n]$ by analyzing Euler characteristics of subsets of the permutahedron. This proves that the ordered partition expansion does indeed give a formula for the determinant. In Section 5, we give an extremely brief introduction to multivariate finite operator calculus, and state a more general open conjecture that motivated this paper.

2. Flattening Functions

Upon expanding the expression in Theorem 1, many terms will cancel. In this section, we set up the groundwork to keep track of each of the terms and show how they cancel. Now consider any function $f : [n] \to [n]$ and define the monomial

$$a_f := \prod_{j=1}^{n} a_{f(j),j}.$$

Expanding the terms in Equation (2) results in a sum of the form

$$\sum_{B \vdash [n]} (-1)^{n-|B|} \prod_{\beta_k \in B} \prod_{j \in \beta_k} \sum_{i \in \beta'_k} a_{ij} = \sum_f c_f a_f, \tag{3}$$

where each term $c_f a_f$ corresponds to a function from the set $[n]$ to itself. For instance, in Equation (1) the four functions $f_i : [2] \to [2]$ are

$$f_1(1) = 1, \quad f_1(2) = 1; \quad f_2(1) = 1, \quad f_2(2) = 2;$$
$$f_3(1) = 2, \quad f_3(2) = 2; \quad f_4(1) = 2, \quad f_4(2) = 1.$$

The terms corresponding to each function are labeled below:

$$\begin{aligned}
a_{11}(a_{12} + a_{22}) + a_{22}(a_{11} + a_{21}) &- (a_{11} + a_{21})(a_{12} + a_{22}) \\
&= (a_{11}a_{12} - a_{11}a_{12}) + (a_{11}a_{22} + a_{11}a_{22} - a_{11}a_{22}) \\
&\quad + (a_{21}a_{22} - a_{21}a_{22}) + (-a_{12}a_{21}) \\
&= 0 + a_{11}a_{22} + 0 - a_{12}a_{21} \\
&= c_{f_1}a_{f_1} + c_{f_2}a_{f_2} + c_{f_3}a_{f_3} + c_{f_4}a_{f_4},
\end{aligned}$$

where first we expand the terms and then we simplify. Hence we see that $c_{f_1} = 0$, $c_{f_2} = 1$, $c_{f_3} = 0$, and $c_{f_4} = -1$.

The goal of this paper is to combinatorially identify the coefficients c_f for each such function, and show that they agree with the coefficients of a_f in the determinant. To do so, we must identify the set

$$S_f = \{B \vdash [n] : a_f \text{ appears as a summand of the product indexed by } B\}.$$

The following definition and lemma describe criteria for when an ordered partition $B \vdash [n]$ appears in S_f.

Definition 1. Let $B = (\beta_1, \beta_2, \ldots, \beta_r)$ be an ordered partition of $[n]$. For each $i \in [n]$, define $\beta(i) = k$ if $i \in \beta_k$. We say *i precedes or is in the same part as j in B*, denoted $i \preceq_B j$ (or simply $i \preceq j$ if the partition B is clearly specified), if

$$\beta(i) \leq \beta(j).$$

Lemma 1. *Let $B \vdash [n]$ be an ordered partition of $[n]$. The term a_f appears in the product $\prod_{\beta_k \in B} \prod_{j \in \beta_k} \sum_{i \in \beta'_k} a_{ij}$ if and only if $f(j) \preceq j$ in B for all $1 \leq j \leq n$.*

Proof. Let $B \vdash [n]$ be an ordered partition of $[n]$. Since $\beta'_k = \cup_{j=1}^{k} \beta_j$, we see that a_f appears as a term in the product $\prod_{\beta_k \in B} \prod_{j \in \beta_k} \sum_{i \in \beta'_k} a_{ij}$ precisely when $i = f(j) \preceq j$ in the ordered partition B for all $1 \leq j \leq n$. □

Now that Lemma 1 has established that an ordered partition B appears in the set S_f if and only if $f(j)$ precedes or is in the same part as j in B for all integers $1 \leq j \leq n$, we are ready to analyze which functions $f : [n] \to [n]$ correspond to nonzero terms in the ordered partition expansion. To do so, we start by introducing some notation regarding the structure of such functions.

Definition 2. Let $f : [n] \to [n]$ be any function. We say that the function f is *acyclic* if $f^k(i) = i$ implies $k = 1$ for all $i \in [n]$. Otherwise, we say f contains a *cycle*.

Henceforth we will prefer to work with acyclic functions. The next definition describes how each function $f : [n] \to [n]$ which contains a cycle can be simplified or *flattened* to an acyclic function $\overline{f}$ on a different set of elements.

Definition 3. Let $f : [n] \to [n]$ be any function, $C_f = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ represent the cycles of f, $N_f \subseteq [n]$ be the elements not in a cycle, and $D_f = C_f \cup N_f$. Define the *flattened* function $\overline{f} : D_f \to D_f$ as follows

$$\overline{f}(i) = \begin{cases} f(i) & \text{if } f(i) \text{ is not in a cycle of } f \\ \sigma_j & \text{if } f(i) \text{ belongs to the cycle } \sigma_j \\ i & \text{if } i \in C_f \end{cases}.$$

Intuitively, $\overline{f}$ acts just like f, but shrinks each cycle of f to a fixed point (or cycle of length one), and thus it is an acyclic function. The following example illustrates Definitions 2 and 3.

Example 2. Let $f : [6] \to [6]$ be defined by $f(1) = 1, f(2) = 3, f(3) = 2,$ $f(4) = 3,$ $f(5) = 6$, and $f(6) = 5$. Then $C_f = \{\sigma_1, \sigma_2\}$ where $\sigma_1 = (23)$ and $\sigma_2 = (56)$. The sets $N_f = \{1, 4\}$ and $D_f = N_f \cup C_f = \{\sigma_1, \sigma_2, 1, 4\}$. The function $\overline{f} : D_f \to D_f$ is defined by $\overline{f}(\sigma_1) = \sigma_1$, $\overline{f}(\sigma_2) = \sigma_2$, $\overline{f}(1) = 1$, and $\overline{f}(4) = \sigma_1$.

The following lemma shows that we can reduce the problem of calculating the coefficients c_f in the ordered partition expansion, to that of calculating the coefficients $c_{\bar{f}}$ corresponding to acyclic functions.

Lemma 2. *If $f : [n] \to [n]$ is any function, then the coefficients c_f and $c_{\overline{f}}$ are related by the equation $c_f = (-1)^{n-|D_f|} c_{\overline{f}}$.*

Proof. Once again, let

$$S_f := \{B \vdash [n] : a_f \text{ appears as a summand of the product indexed by } B\},$$

and similarly let

$$S_{\overline{f}} := \{A \vdash D_f : a_{\overline{f}} \text{ appears as a summand of the product indexed by } A\}.$$

If B is an ordered partition for which a_f appears as a summand, then Lemma 1 implies that for each cycle

$$\sigma_j = (i, f(i), f^2(i), \ldots, f^k(i))$$

of f, the following precedence relation

$$i \preceq f^k(i) \preceq f^{k-1}(i) \preceq \cdots \preceq f(i) \preceq i$$

must hold in the ordered partition B. Therefore c_j must be contained in the same part β of the ordered partition B. From here, it is clear to see that S_f is equivalent to the set of ordered partitions of D_f, and so each term in Equation (3) has the form

$$\begin{aligned} c_f a_f &= \sum_{B \in S_f} (-1)^{n-|B|} a_f \\ &= (-1)^{n-|D_f|} \sum_{A \in D_f} (-1)^{|D_f|-|A|} a_f \\ &= (-1)^{n-|D_f|} c_{\overline{f}} a_f, \end{aligned}$$

and the lemma follows. □

The number of ordered partitions of $[n]$ with k parts is well known to be $k!S(n,k)$, where $S(n,k)$ are the Stirling numbers of the second kind. We obtain an important corollary to Lemma 2 from the following well-known result about Stirling numbers of the second kind.

Lemma 3. *The following identity holds:*

$$\sum_{k=0}^{n} (-1)^{n-k} k! S(n,k) = 1.$$

Proof. The result follows immediately upon setting $x = -1$ in the following identity on Stirling numbers of the second kind[1] [9, p. 35]:

$$\sum_{k=0}^{n} S(n,k)(x)_k = x^n.$$ □

Corollary 1. *If $f : [n] \to [n]$ is bijective, i.e., $f = \pi$ for some $\pi \in \mathfrak{S}_n$, then $c_f = \operatorname{sgn}(\pi)$.*

Proof. Since f is bijective, it consists only of cycles. Thus $D_f = C_f = \{\sigma_1, \ldots, \sigma_r\}$ and $|D_f| = r$. By Lemma 2 and 3 we see

$$\begin{aligned} c_f &= (-1)^{n-|D_f|} \sum_{A \in \mathcal{D}} (-1)^{|D_f|-|A|} \\ &= (-1)^{n-r} \sum_{B \vdash [r]} (-1)^{r-|B|} \\ &= (-1)^{n-r} \sum_{k=0}^{n} (-1)^{r-k} k! S(r,k) \\ &= (-1)^{n-r}. \end{aligned}$$

We leave it to the reader to verify that $\operatorname{sgn}(\pi) = (-1)^{n-r}$. □

[1] This identity was originally proved by Stirling himself in 1730 with, of course, very different notation.

With Corollary 1, we have that the ordered partition expansion contains every term of the determinant with the correct coefficient. It remains to show that whenever f is not bijective, $c_f = 0$. By Lemma 2, it suffices to show this for acyclic functions.

3. The Poset of Ordered Partitions

We next consider the poset of ordered partitions in order to show that the set S_f has a nice structure when f is acyclic. This will allow us to eventually switch to a more geometric viewpoint.

Let $\mathcal{P}_n$ denote *the poset of ordered partitions of the set* $[n]$ ordered by refinement, i.e., for any A, B in $\mathcal{P}_n$ the relation $A \leq B$ holds if and only if for each part $\alpha \in A$ there is a part $\beta \in B$ such that $\alpha \subseteq \beta$. In this case, A is said to be a *refinement* of B. Note that $\mathcal{P}_n$ is a graded poset of rank n, where the rank of an ordered partition $A \in \mathcal{P}_n$ is $n - |A|$.

In Figure 1, we see the poset $\mathcal{P}_3$ of ordered partitions on 3 elements. The rank 0 elements in $\mathcal{P}_n$ are the $n!$ ordered partitions consisting of singletons; these partitions correspond bijectively with the elements of $\mathfrak{S}_n$. Because of their importance later, we will refer to them as *singleton partitions.*

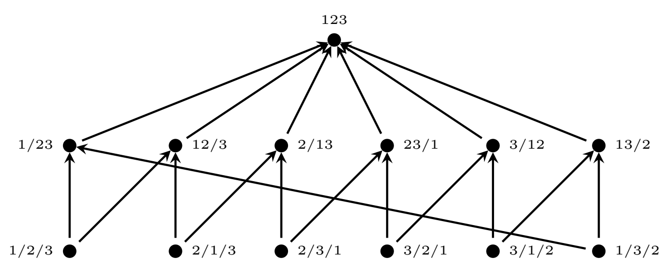

Figure 1: Hasse diagram of the poset of ordered partitions on 3 elements

An ordered partition B in $\mathcal{P}_n$ is covered by the ordered partitions formed by merging two consecutive parts in B. For example, the singleton partition 3/1/2/4 is covered by the ordered partitions 13/2/4, 3/12/4, and 3/1/24. Define $\mathfrak{B}$ to be the subposet of all ordered partitions which are formed by merging various parts of a singleton partition B. Notice that $\mathfrak{B}$ is isomorphic to the Boolean algebra B_{n-1}, which we will refer to as the $(n-1)$-*cube* associated with B. An example is given in Figure 2.

An acyclic function can be viewed as a rooted forest, where the fixed points are the roots. An example is given in Figure 3. Given an acyclic function f, if a path exists from p to q, with p closer to the root than q, then $f^k(q) = p$ for some k. Thus, $p \preceq q$, and so we say f has the *rule* $p \preceq q$. In this way, each function f stipulates a set of rules

$$R_f = \left\{ p \preceq q \mid f^k(q) = p \text{ for } p, q, k \in [n] \right\}.$$

The following lemma and corollary will show that the set of ordered partitions S_f has a nice structure in $\mathcal{P}_n$.

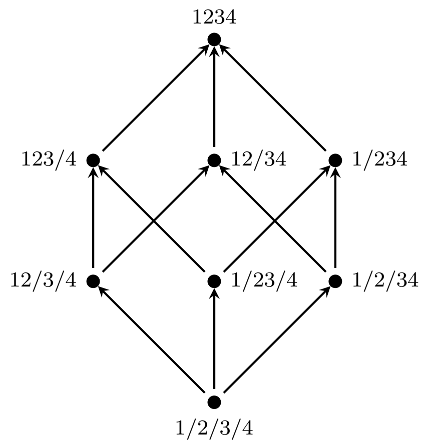

Figure 2: Hasse diagram of the cube for the singleton partition 1/2/3/4

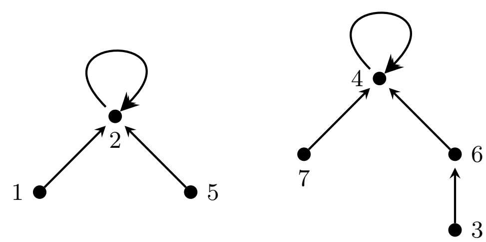

Figure 3: Acyclic function represented by a rooted forest

Lemma 4. *If a singleton partition A in $\mathcal{P}_n$ satisfies the rules R_f given by an acyclic function f then every ordered partition $B \geq A$ in $\mathcal{P}_n$ also satisfies the rules R_f. If an ordered partition B in $\mathcal{P}_n$ satisfies the rules R_f then there exists at least one singleton partition $A \leq B$ in $\mathcal{P}_n$ that satisfies the precedence rules R_f.*

Proof. The first statement is obvious. If A is a singleton partition with the rule $p \preceq q$ in A and $B \geq A$ is an ordered partition in $\mathcal{P}_n$, then the parts of B are unions of consecutive parts of A. Hence $p \preceq q$ in B as well.

The second statement is slightly less obvious. Let f be an acyclic function, and suppose that B in $\mathcal{P}_n$ satisfies the rules R_f and is not a singleton partition. We must show there is a singleton partition $A \leq B$ above it in $\mathcal{P}_n$ that also satisfies R_f. Consider a part β of B that is not a singleton. If no pair of elements in β has a rule associated with it, then the elements of β can be ordered arbitrarily. Otherwise, there are elements of β that have rules imposed on them. Consider the elements in the intersection of β and a rooted tree associated with f. We order those elements by their distance from the root. (Those elements having the same distance from the root can be put in any order with respect to each other.) We do this for every rooted tree associated with f to impose an order on all of β. Doing the same to each part will result in a singleton partition A above B satisfying the rules of f. □

Corollary 2. *Let f be an acyclic function and $\{B_1, B_2, \ldots, B_k\}$ the set of singleton partitions satisfying the rules R_f. Then the collection of all ordered partitions of $[n]$ which satisfy R_f forms a subposet of $\mathcal{P}_n$ corresponding to a (non-disjoint) union $\bigcup_{i=1}^{k} \mathfrak{B}_i$ where $\mathfrak{B}_i$ is the $(n-1)$-cube associated with B_i.*

Proof. By the above lemma, we can account for all the ordered partitions by only considering the singleton partitions satisfying the acyclic function, and all the ordered partitions above them. The result follows since the ordered partitions above a singleton partition form an $(n-1)$-cube. □

Lemma 4 and Corollary 2 tell us that once we know which singleton partitions appear in S_f, then we know that S_f is precisely those singletons and all the ordered partitions above them in $\mathcal{P}_n$. We will start to label the singleton partitions without slashes, e.g. $1/2/3/4 \to 1234$, unless we need to distinguish them from the ordered partition with one part. In the next section, we turn our attention to a geometric object isomorphic to $\mathcal{P}_n$: the permutahedron.

4. The Permutahedron

In this section we show how the alternating sums giving c_f when f is acyclic are related to the Euler characteristic of the permutahedron and use this correspondence to show that $c_f = 0$. It is well-known that the poset of the ordered partitions is isomorphic to the face lattice of the permutahedron [8, Fact 4.1]. Specifically, the rank i elements of $\mathcal{P}_n$ correspond to the faces of dimension i. Note that two vertices in the permutahedron are adjacent if one can be obtained from the other by a single swap of consecutive elements. For example, 315624 is adjacent to 351624. Figure 4 shows the transformation from the poset on 3 elements to the permutahedron on 3 elements, which in this case is a hexagon. Figure 5 shows the poset on 4 elements as the ordinary permutahedron (truncated octahedron).

In Figure 5, we only label the singleton partitions at the vertices, but the labeling of the other ordered partitions would be similar to Figure 4.

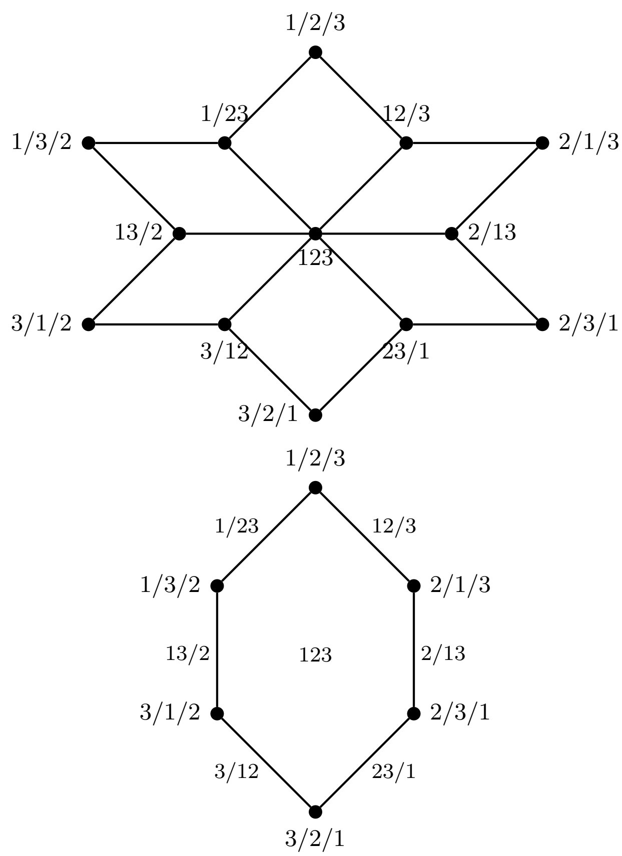

Figure 4: Top view of the poset of ordered partitions on 3 elements (top) and as a permutahedron (bottom)

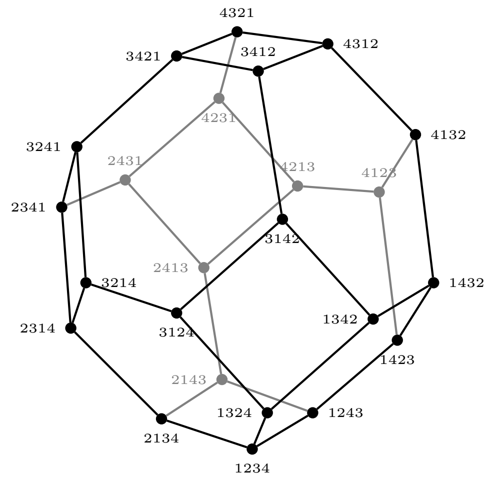

Figure 5: Poset of ordered partitions on 4 elements as a permutahedron

The permutahedron Π_n is often defined as the convex hull of the points

$$P_\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$$

for every $\sigma \in \mathfrak{S}_n$. It is a convex polytope, and in particular, it is contractible to a point. Thus, the permutahedron has Euler characteristic $\chi(\Pi_n) = 1$ [8]. Since the ordered partitions are in bijection with the faces of the permutahedron, the alternating sum of the ordered partitions by rank is precisely the Euler characteristic of the permutahedron, and this gives us a second proof of Lemma 3.

We adopt a slightly different convention, relabeling each vertex in P_σ of the permutahedron Π_n to $P_{\sigma^{-1}}$. We will denote this relabeled permutahedron by Π'_n. Figure 6 shows Π_3 in $\mathbb{R}^3$, the relabeled Π'_3, and the correspondence between the half-space $x_1 \leq x_2$ and the rule $1 \preceq 2$. In general we can see that Π_n, and thus Π'_n, is an $(n-1)$-dimensional object, since all the points lie in the hyperplane $x_1 + x_2 + \cdots + x_n = \binom{n+1}{2}$. In general $x_i \leq x_j$ in Π_n corresponds to $i \preceq j$ in Π'_n.

Lemma 5. *The singleton partitions in Π'_n satisfying the precedence rule $i \preceq j$ are contained in the corresponding half-space $x_i \leq x_j$ in Π_n.*

Proof. A permutation σ^{-1} satisfies $\sigma_i^{-1} \preceq \sigma_j^{-1}$ precisely when $\sigma(i) \leq \sigma(j)$. Hence we see that if the permutation (or singleton partition) σ^{-1} satisfies the precedence relation $i \preceq j$ then the vertex $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is in the half-space $x_i \leq x_j$ and vice versa. □

We are now ready to prove the main result of this section.

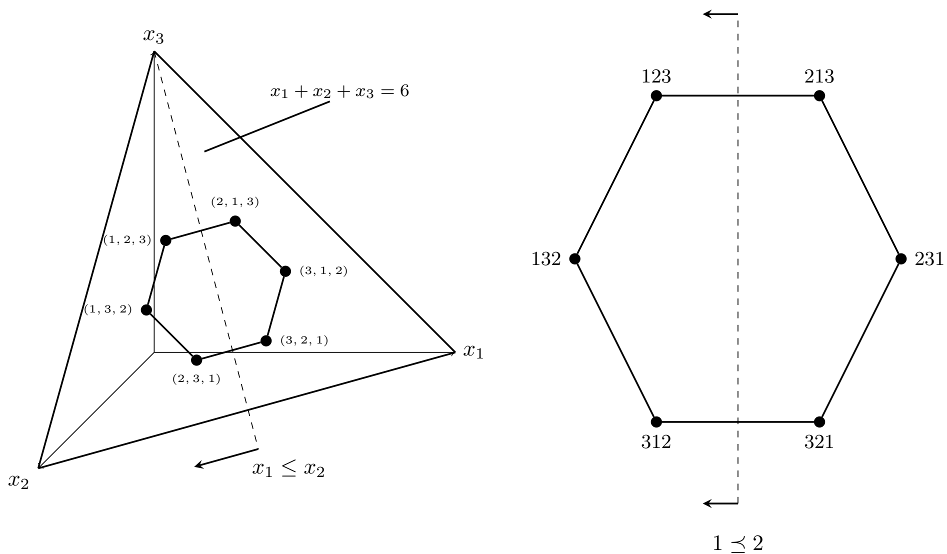

Figure 6: The permutahedra Π_3 (left) and Π'_3 (right)

Proposition 1. *If $f : [n] \to [n]$ is not bijective, then the coefficient $c_f = 0$ is zero.*

Proof. Let $f : [n] \to [n]$ be an acyclic function. Let $R_f = \{f(j) = i \preceq j\}$ denote the set of precedence rules determined by f. To each precedence rule $f(j) = i \preceq j$ in R_f we can assign a half-space $H_{ij} := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_{f(j)} = x_i \leq x_j\}$, and the intersection of these half-spaces with the permutahedron Π'_n defines a convex polytope, which we will denote by Π_f.

The faces of Π_f fall into two disjoint subsets: those faces that correspond to the ordered partitions in S_f and those that do not. Let Γ_f denote the faces of Π_f which correspond to elements of S_f and let Δ_f denote the faces of Π_f that do not. The set of faces Δ_f are precisely the faces of Π_f which lie entirely on the boundary of at least one half-space $x_i = x_j$ because they resulted from intersecting Π'_n with one of the half-spaces H_{ij}. Thus each face of Δ_f is a convex polytope, and we see that Δ_f is a union of convex polytopes. Each of these convex polytopes contains the point $(x_1, x_2, \ldots, x_n)$ where $x_1 = x_2 = \cdots = x_n$, so Δ_f is a contractible space. Hence Δ_f has Euler characteristic $\chi(\Delta_f) = 1$.

Since Π_f is a convex polytope and $\Pi_f = \Gamma_f \sqcup \Delta_f$ we see that

$$1 = \chi(\Pi_f) = \chi(\Gamma_f) + \chi(\Delta_f) = 0 + 1.$$

It follows that $\chi(\Gamma_f) = 0$. Since the summands of the sum $\sum_{B \in S_f} (-1)^{n-|B|}$ correspond with the faces of Γ_f we see that $c_f = \sum_{B \in S_f} (-1)^{n-|B|} = \chi(\Gamma_f) = 0$ as desired. □

We end this section with an example demonstrating the proof of Proposition 1.

Example 3. Figure 7 shows Π_f with Γ_f bold and Δ_f dashed for the terms $a_f = a_{11}a_{12}a_{13}$, which has the rules $1 \preceq 2$ and $1 \preceq 3$ and $a_f = a_{11}a_{12}a_{33}a_{34}$, which has the rules $1 \preceq 2$ and $3 \preceq 4$.

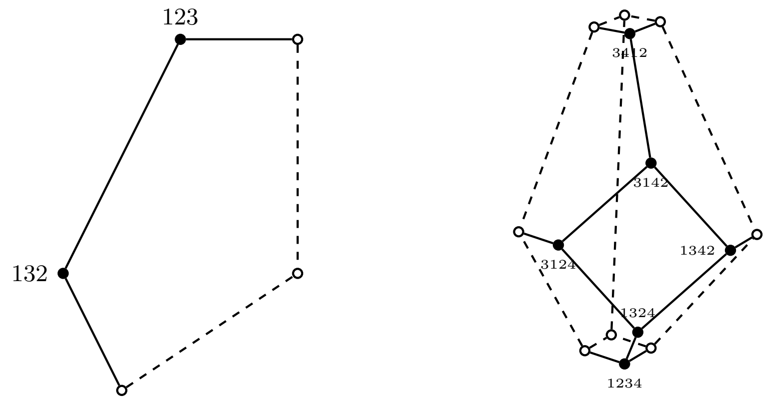

Figure 7: $a_f = a_{11}a_{12}a_{13}$ (left) and $a_f = a_{11}a_{12}a_{33}a_{34}$ (right)

5. Multivariate Finite Operator Calculus

This ordered partition expansion of the determinant came from a conjecture about a transfer formula in *multivariate finite operator calculus* (MFOC). In this section we give a very brief overview of the objects of study in MFOC and the conjecture that gives this expansion. The interested reader is encouraged to read [11] for a more comprehensive description of this subject matter.

Let k be a field. Let $\{\mathbf{e}_i\}_{1\leq i\leq \ell}$ denote the standard ℓ-dimensional basis of k^ℓ. The main objects of study in MFOC are polynomials $p \in k[x_1, \ldots, x_\ell]$ and operators $T \in k[[D_1, \ldots, D_\ell]]$, where D_i is the partial derivative with respect to x_i. A sequence of polynomials $b_{\mathbf{n}}(\mathbf{x}) = b_{n_1,\ldots,n_\ell}(x_1, ..., x_\ell)$ is called a *basic sequence* if there is a set of operators $\mathbf{B} = (B_1, \ldots, B_\ell)$ with $B_i = D_iP_i$ where each $B_i : b_{\mathbf{n}} \mapsto b_{\mathbf{n}-\mathbf{e}_i}$ and each P_i is an invertible operator. Such a set of operators is called a delta ℓ-tuple. The power series for an operator T is written as

$$T = \sum_{\mathbf{n}\geq 0} a_{\mathbf{n}}\mathbf{D}^{\mathbf{n}} = \sum_{n_1,\ldots,n_\ell\geq 0} a_{n_1,\ldots,n_\ell} D_1^{n_1}\cdots D_\ell^{n_\ell}. \tag{4}$$

These are all standard notations in any multivariate theory. However, the following notation is not completely standard in MFOC. Given a subset $A \subseteq [\ell]$ we define $X_A := \prod_{i\in A} X_i$.

The following is the Transfer Theorem from MFOC, and is essentially Theorem 1.3.6 in [11] with different notation.

Theorem 2 (Transfer Theorem)**.** *Let $\mathcal{J}$ denote the usual Jacobian matrix of a collection of polynomials. Suppose $\boldsymbol{B} = (B_1, B_2, \ldots, B_\ell)$ is a delta ℓ-tuple where $B_i = D_i P_i^{-1}$, then*

$$b_{\boldsymbol{n}}(\boldsymbol{x}) = \boldsymbol{P}^{\boldsymbol{n+1}} \mathcal{J}(B_1, B_2, \ldots, B_\ell) \frac{\boldsymbol{x}^{\boldsymbol{n}}}{\boldsymbol{n}!}$$

is the basic sequence for $\boldsymbol{B}$ written in terms of $\frac{\boldsymbol{x}^{\boldsymbol{n}}}{\boldsymbol{n}!}$.

Within the Jacobian, the partial derivatives $\frac{\partial B_i}{\partial D_j} = B_i\theta_j - \theta_j B_i$ are called *Pincherle derivatives*, where $\theta_j : p \to x_j p$ is the jth *umbral shift* operator that does not commute with the delta operators. Thus, there are many ways this transfer formula could be expanded. The following conjecture (based on the examples provided below) gives one such way.

Conjecture 1. *The basic sequence from the Transfer Theorem can also be calculated as*

$$b_{\boldsymbol{n}}(\boldsymbol{x}) = \sum_{B \vdash [\ell]} (-1)^{\ell - |B|} \left(\theta_\beta P_\beta\right)_B \frac{\boldsymbol{x}^{\boldsymbol{n-1}}}{\boldsymbol{n}!},$$

where B runs through all ordered partitions of $[\ell]$ and β runs through the partitions of B.

Because of our abuse of some notation, we give the following examples of Conjecture 1 when $\ell = 2$ and $\ell = 3$, along with an example explaining the meaning of $\left(\theta_\beta P_\beta\right)_B$.

$$b_{m,n}(u,v) = \left(uP_1^m vP_2^n + vP_2^n uP_1^m - uvP_1^m P_2^n\right) \frac{u^{m-1}v^{n-1}}{m!n!}$$

$$\begin{aligned} b_{m,n,p}(a,b,c) \quad = \quad & (aRbScT + aRcTbS + bSaRcT + bScTaR + cTaRbS + cTbSaR \\ & -aRbcST - bSacRT - cTabRS - abRScT - acRTbS - bcSTaR \\ & +abcRST) \frac{a^{m-1}b^{n-1}c^{p-1}}{m!n!p!} \end{aligned}$$

$$B = (\{2\}, \{1,3\}) \quad \Rightarrow \quad \left(\theta_\beta P_\beta\right)_B = bSacRT$$

The ordered partition expansion of the determinant, Theorem 1, follows by setting each $B_i = \mathbf{D}^{\mathbf{a}_i} = D_1^{a_{i1}} D_2^{a_{i2}} \cdots D_n^{a_{in}}$ in Conjecture 1.

6. Conclusions and Open Questions

We end with few open questions stemming from our work.

Q1. Now that Theorem 1 shows that Conjecture 1 is true for one term of an operator's power series, can Conjecture 1 be proven by a linearity argument?

Q2. Can the proof of Ryser's formula given by Horn and Johnson [5] be modified to give another proof of Theorem 1?

Q3. Can our proof of Theorem 1 be modified to prove Ryser's formula by using the topological/combinatorial properties of the cube instead of the permutahedron?

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