#A39 INTEGERS 17 (2017)

LOWER BOUNDS ON THE WEAK SCHUR NUMBERS UP TO 9 COLORS

Fanasina Rafilipojaona
L.M.P.A. Joseph Liouville, University of the Littoral Opal Coast, Calais, France
rafanasin@gmail.com

Received: 4/20/15, Revised: 8/11/16, Accepted: 8/28/17, Published: 9/14/17

Abstract

A set of integers is weakly sum-free if it does not contain a solution of $x+y=z$ with $x \neq y$. Given $n \geq 1$, the weak Schur number $WS(n)$ is the maximal integer N such that there exists an n-coloring of the set $\{1, 2, \ldots, N\}$ such that each monochromatic subset is weakly sum-free. We give new lower bounds on $WS(n)$ for $n = 7, 8$ and 9 by constructing highly structured n-colorings, with some computer help.

1. Introduction

The *Schur number* $S(n)$ is defined to be the largest N such that for all n-colorings of the integer interval $[1, N] = \{1, 2, \ldots, N\}$, there is no monochromatic solution of the equation $(E_1) : a + b = c$. Introduced by Schur [10], these numbers are exactly known up to $n = 4$:

$$S(1) = 1,\ S(2) = 4,\ S(3) = 13,\ S(4) = 44.$$

For $n = 5, 6$ and 7, it is known only that

$$160 \leq S(5) \leq 305, \quad S(6) \geq 536 \text{ and } S(7) \geq 1680.$$

And for general n, Schur gave the following two bounds on $S(n)$ for all n:

$$\frac{3^n - 1}{2} \leq S(n) < n! \cdot e.$$

In this paper, we shall be mostly concerned with a *weak* version of these numbers. The weak Schur number $WS(n)$ is defined to be the largest N such that for all n-colorings of $[1, N]$, there is no monochromatic solution of the equation $(E_2) : a+b = c$ with $a \neq b$.

Here again, these numbers are known only for $n \leq 4$:

$$WS(1) = 2,\ WS(2) = 8,\ WS(3) = 23,\ WS(4) = 66.$$

Similarly, for $n = 5$ and 6, it is known only that

$$WS(5) \geq 196 \text{ and } WS(6) \geq 582.$$

Note that for general n we have

$$WS(n) \geq S(n),$$

since all solutions of (E_2) are solutions of (E_1). No other lower bounds on $WS(n)$ are currently known besides the ones given above.

Our main contribution in this paper is to provide lower bounds on $WS(n)$ for $7 \leq n \leq 9$. Explicitly, we shall obtain the following bounds:

$$WS(7) \geq 1740, \quad WS(8) \geq 5201, \quad WS(9) \geq 15596.$$

This is achieved by focusing on very special n-partitions of integer intervals which are then implemented in a specialized search algorithm.

Here are the contents of this paper. In Section 2, we provide some general background on sum-free and weakly sum-free sets, and we recall a lower bound on $S(n+m)$ due to Abbott and Hanson. In Section 3, we establish a lower bound on $WS(n+1)$ in terms of $WS(n)$. In Section 4, we describe the very special structure of the parts to be used in our partitions. In Section 5, we give some formulas for the restricted sums of these special parts. In Section 6, we give global restrictions on the shape of the looked-for partitions. These restrictions are then assembled in Section 7 in search algorithms. Finally, in Section 8, we provide the partitions obtained by these algorithms and which establish the above-mentioned lower bounds on $WS(n)$ for $n = 7, 8$ and 9.

2. Background

Recall that for two subsets A, B of $\mathbb{N}$, their *sum* $A + B$ is defined as

$$A + B = \{a + b \mid a \in A, b \in B\},$$

and their *restricted sum* $A \dot{+} B$ as

$$A \dot{+} B = \{a + b \mid a \in A, b \in B, a \neq b\}.$$

Note that $A \dot{+} B \subseteq A + B$.

Let $S \subseteq \mathbb{N}$. We say that S is *sum-free* if $(S + S) \cap S$ is empty, and is *weakly sum-free* if $(S \dot{+} S) \cap S$ is empty.

Estimating $S(n)$ from below with an integer N is achieved by providing a partition of $[1, N]$ into n sum-free parts. Similarly, obtaining an inequality of the form

$M \leq WS(n)$ may be achieved by constructing a partition of $[1, M]$ into n weakly sum-free parts.

For all $n \geq 1$, we have $S(n) \leq WS(n)$ because all sum-free sets are also weakly sum-free. Thus any lower bound on $S(n)$, and in particular $S(n)$ itself, is automatically a lower bound for $WS(n)$.

Given $x, y \in \mathbb{N}$, we shall denote by $[x, y]$ the integer interval defined as

$$[x, y] = \{z \in \mathbb{N} \mid x \leq z \leq y\}.$$

For a finite set C, we denote as usual $Card(C)$ or $|C|$ its cardinality, i.e., the number of elements in C. Note that $[x, y] = \emptyset$ whenever $x > y$.

Schur [10] established the following bounds on $S(n)$:

$$\frac{3^n - 1}{2} \leq S(n) < n! \cdot e.$$

The lower bound is due to the following Schur [10] inequality for $n \geq 1$:

$$S(n+1) \geq 3 \cdot S(n) + 1. \tag{2.1}$$

Abbott and Hanson [1] generalized this inequality for all integers $n, m \geq 1$:

$$S(n+m) \geq 2 \cdot S(n) \cdot S(m) + S(n) + S(m). \tag{2.2}$$

It follows from the value $S(4) = 44$ and inequality (2.2) that for all integers $n \geq 1$, we have

$$S(n+4) \geq 89 \cdot S(n) + 44.$$

This yields the following lower bounds on the Schur numbers $S(n)$ for $n \geq 4$:

$$S(n) \geq c_4 \cdot 89^{\frac{n}{4}}, \text{ with } c_4 = \frac{44}{89}.$$

In this paper, we shall mainly seek good lower bounds on the weak Schur numbers $WS(n)$.

It is known that

$$WS(1) = 2, \; WS(2) = 8, \; WS(3) = 23, \; WS(4) = 66.$$

P. F. Blanchard, F. Harary, and R. Reis [2] proved the value of $WS(4)$ by computer. See also [6] and [8].

Example 1. Here is a partition $[1, 66] = A_1 \sqcup A_2 \sqcup A_3 \sqcup A_4$ of $[1, 66]$ into 4 weakly sum-free parts A_i, thereby yielding the lower bound $WS(4) \geq 66$:

$$\begin{aligned}
A_1 &= \{1, 2, 4, 8, 11, 22, 25, 50, 63\} \\
A_2 &= \{3\} \cup [5, 7] \cup \{19, 21, 23\} \cup [51, 53] \cup [64, 66] \\
A_3 &= \{9, 10\} \cup [12, 18] \cup \{20\} \cup [54, 62] \\
A_4 &= \{24\} \cup [26, 49].
\end{aligned}$$

For $n = 5$, Walker [6] stated in 1952 that $WS(5) = 196$ without proof, not even with a partition showing $WS(5) \geq 196$. S. Eliahou, J. M. Marín, M. P. Revuelta, and M. I. Sanz [3] established much later that lower bound, by providing an explicit partition of $[1, 196]$ into 5 weakly sum-free parts. It remains an open problem to show that 196 is the exact value of $WS(5)$.

For $n = 6$ and 7, H. Fredricksen and M. Sweet [5] established in 2000 the lower bounds $S(6) \geq 536$ and $S(7) \geq 1680$, thereby automatically yielding the same lower bounds on $WS(6)$ and $WS(7)$.

In 2012, the authors of [3] obtained $WS(6) \geq 572$, and one year later, Eliahou et al. [4] improved by the current record $WS(6) \geq 582$.

For $n = 7$, no better lower bound than $WS(7) \geq S(7) \geq 1680$ has been available so far. In this paper, we improve that bound by showing that $WS(7) \geq 1740$.

Finally, we also provide good lower bounds on $WS(8)$ and $WS(9)$, cases for which the only available lower bounds so far were the ones given by Schur's inequality (2.1), namely

$$WS(8) \geq S(8) \geq 5041, \quad WS(9) \geq S(9) \geq 15124.$$

In this paper, we obtain

$$WS(8) \geq 5201, \quad WS(9) \geq 15596.$$

3. Comparing $WS(n+1)$ and $WS(n)$

In this short section, we establish a lower bound on $WS(n+1)$ in terms of $WS(n)$. While this bound is not particularly strong, its main interest lies in the simplicity of the construction below.

Proposition 2. *For all integers $n \geq 1$, we have*

$$WS(n+1) \geq \frac{5}{2} WS(n) + 2. \tag{3.1}$$

Proof. Let $N = WS(n)$ and let $A_1, \dots, A_n$ be n weakly sum-free sets such that

$$[1, N] = A_1 \sqcup \dots \sqcup A_n.$$

Set $l = 3N + 3$ and $K = \frac{5}{2}N + 2$. We shall build $n + 1$ weakly sum-free subsets of $[1, K]$ which will cover it.

Let $C_i = A_i \cup (l - A_i)$ for $1 \leq i \leq n$, where

$$l - A_i = \{l\} - A_i = \{l - x \mid x \in A_i\}.$$

Suppose that C_i is not weakly sum-free and let $a, b, c \in C_i$ such that $a + b = c$ and $a \neq b$. First, since $A_i \subseteq [1, N]$ is weakly sum-free and $l - A_i \subseteq [2N + 3, 3N + 2]$, the only possibilities are that $a, c \in l - A_i$ and $b \in A_i$ or $b, c \in l - A_i$ and $a \in A_i$. Suppose the first one. Then there exist $a', c' \in A_i$ such that $a = l - a', c = l - c'$ and the equality $a + b = c$ is equivalent to $b' + c' = a'$. Since A_i is weakly sum-free, it follows that $c' = b$ and $a' = 2b$. This is the reason why, if we remove from C_i all elements of the form $l - x$ such that $x, 2x \in A_i$, then

$$C_i \setminus \{l - x \mid x, 2x \in A_i\}$$

is weakly sum-free, for all $1 \leq i \leq n$. Set $d_i = \max\{x \in [1, N] \mid x, 2x \in A_i\}$. Then $C_i \cap [1, l - d_i - 1]$ is weakly sum-free because

$$(C_i \cap [1, l - d_i - 1]) \subseteq C_i \setminus \{l - x \mid x, 2x \in A_i\}.$$

For all i, we have $d_i, 2d_i \in A_i \subseteq [1, N]$, and therefore $2d_i \leq N$, i.e., $d_i \leq \frac{1}{2}N$. Thus we have

$$l - d_i - 1 \geq l - \frac{1}{2}N - 1 = 3N + 3 - \frac{1}{2}N - 1 = \frac{5}{2}N + 2 = K.$$

We set $B_i = C_i \cap [1, K]$. Since $(C_i \cap [1, K]) \subseteq (C_i \cap [1, l - d_i - 1])$ for $1 \leq i \leq n$, it follows that B_i is weakly sum-free for $i = 1, \ldots, n$. Now,

$$\begin{aligned}
\bigcup_{i=1}^{n} C_i &= \bigcup_{i=1}^{n} (A_i \cup (l - A_i)) = \left(\bigcup_{i=1}^{n} A_i\right) \cup \left(\bigcup_{i=1}^{n} (l - A_i)\right) = [1, N] \cup (l - [1, N]) \\
&= [1, N] \cup [2N + 3, 3N + 2].
\end{aligned}$$

Thus we have

$$\begin{aligned}
\bigcup_{i=1}^{n} B_i &= \bigcup_{i=1}^{n} (C_i \cap [1, K]) = \left(\bigcup_{i=1}^{n} C_i\right) \cap [1, K] \\
&= ([1, N] \cup [2N + 3, 3N + 2]) \cap [1, K] \\
&= [1, N] \cup [2N + 3, K] \text{ because } 2N + 3 < K = 2N + 2 + \frac{1}{2}N < 3N + 2.
\end{aligned}$$

The set $B_{n+1} = [N + 1, 2N + 2]$ is also weakly sum-free. Since

$$B_1 \cup \cdots \cup B_n \cup B_{n+1} = ([1, N] \cup [2N + 3, K]) \cup [N + 1, 2N + 2] = [1, K],$$

we get $n + 1$ weakly sum-free parts covering $[1, K]$. □

Example 3.

- Combining $WS(4) = 66$ with inequality (3.1), we get $WS(5) \geq (5/2)66 + 2 = 167$.
- Using $WS(5) \geq 196$, Proposition 2 yields $WS(6) \geq \frac{5}{2}WS(5) + 2 \geq \frac{5}{2}196 + 2 = 492$.

- Similarly, the inequality $WS(6) \geq 582$ yields $WS(7) \geq \frac{5}{2}582 + 2 = 1457$.

Our aim here is to give much better lower bounds on these numbers, and mainly on $WS(n)$ for $n \leq 9$. While Blanchard et al. [2] achieved an exhaustive search for $WS(4)$, doing the same for $n \geq 5$ is currently unfeasible because of the size of the search space.

In light of that obstacle, one obvious approach consists of restricting the search to only a tiny proportion of all possible weakly sum-free partitions of a given interval $[1, N]$. The difficulty, of course, occurs in guessing what partitions have the potential to yield good lower bounds on $WS(n)$.

This is precisely our approach in this paper. We shall propose partitions made with very special parts only, which are highly structured globally and which allow us to attain the currently best known lower bounds on $WS(n)$ for all $1 \leq n \leq 9$.

4. Structure of the Parts

Here we begin the description of the ingredients construction of the very specific parts which will make up our partitions.

4.1. Special Punctured Intervals

Definition 4. A *punctured interval* is a set of integers of the form

$$I = [x, y] \setminus G,$$

where $x < y$ are positive integers and G is a subset of $[x+1, y-1]$ with $|G| \leq 2$. The elements of G will be referred to as the *gaps* of the punctured interval I. They are uniquely determined by I since $x = \min I$, $y = \max I$ and $G = [x, y] \setminus I$.

The motivation for *punctured intervals* will become clear after several developments, specifically in Remark 21 of Section 5.

In some occasions below, it will be practical to consider subsets of $\mathbb{N}$ of the form $[x, y] \setminus G$ without imposing any restrictions on G. Of course, any finite subset of $\mathbb{N}$ may be represented in the same way.

As usual, the *Kronecker delta* symbol is defined as

$$\delta_{x,y} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

We shall use it when x, y are either both integers or both subsets of $\mathbb{N}$.

Definition 5. A *special punctured interval* $I_m(a,h)$, where $m \le a$ are positive integers and $h \in \{\delta_{m,a}, 2\}$, is a punctured interval of the form

$$I_m(a,h) = \begin{cases} I_m(m,1) = [m, 2m+1] \setminus \{m+1\} & \text{if } a = m \\ I_m(m,2) = [m, 2m+2] \setminus \{m+2, 2m+1\} & \text{if } a = m \\ I_m(a,0) = [a, a+m-1] & \text{if } a > m \\ I_m(a,2) = [a, a+m+1] \setminus \{a+1, a+m\} & \text{if } a > m. \end{cases}$$

We shall often write *sp-interval* instead of "special punctured interval."

Note that $a = \min I_m(a,h)$. Note also that, for all $m \ge 2$, the set $I_m(a,h)$ has h gaps and that

$$|I_m(a,h)| = m + \delta_{m,a} = \begin{cases} m+1 & \text{if } a = m, \\ m & \text{if } a > m. \end{cases}$$

For $m = 1 < a$, since

$$I_1(1,1) = \{1,3\},\ I_1(1,2) = \{1,2,4\},\ I_1(a,2) = \{a, a+2\},\ I_1(a,0) = \{a\},$$

we have $|I_1(1,1)| = 2$, $|I_1(1,2)| = 3$, $|I_1(a,2)| = 2$, $|I_1(a,0)| = 1$.

4.1.1. Properties of sp-intervals

We shall show here that the sets $I_m(a,h)$ are weakly sum-free and maximal in an appropriate sense.

Definition 6. Let $m \le a < b$ be three positive integers. The set $\texttt{Weak}_m[a,b]$ is defined to be the set of all subsets $F \subseteq [a,b]$ such that

1. $\{a,b\} \subseteq F \subseteq [a,b]$,
2. $F \cup \{m\}$ is weakly sum-free,
3. F is maximal with respect to the above conditions.

A set F satisfying the conditions 1,2 and 3 will be called *weakly sum-free m-maximal*. This notion is understood to be relative to the given triple (m,a,b), $m \le a < b$. In formula:

$$\texttt{Weak}_m[a,b] = \{F \subseteq \mathbb{N} \mid \min F = a,\ \max F = b,\ F \textit{ weakly sum-free } m\textit{-maximal.}\}$$

Let $F \in \texttt{Weak}_m[\min F, \max F]$, then if I is a set such that $\min I = \min F$ and $\max I = \max F$, with $I \cup \{m\}$ weakly sum-free and $F \subseteq I$, then $I = F$.

Proposition 7. *Let m and a be two positive integers such that $m \leq a$. We have*

$$\begin{aligned} I_m(a,0) &\in \mathtt{Weak}_m[a, a+m-1], & (4.1)\\ I_m(a,2) &\in \mathtt{Weak}_m[a, a+m+1], & (4.2)\\ I_m(m,1) &\in \mathtt{Weak}_m[m, 2m+1], & (4.3)\\ I_m(m,2) &\in \mathtt{Weak}_m[m, 2m+2]. & (4.4) \end{aligned}$$

Proof. Here, we only prove (4.2). The proof for the other affirmations is similar and left to the reader.

- Let $x, y \in I_m(a,2) = [a, a+m+1] \setminus \{a+1, a+m\}$, with $m < a \leq x < y$. Then $x + y \geq 2a + 2 > a + m + 1$ therefore $I_m(a,2)$ is weakly sum-free. We also have $m + x, m + y \in [a+m, a+2m+1] \setminus \{a+m+1, a+2m\}$, so $m + x, m + y \notin I_m(a,2) \cup \{m\}$. Thus $I_m(a,2) \cup \{m\}$ is weakly sum-free.

- Let $I \subseteq [a, a+m+1]$ be a set of integers such that $a, a+m+1 \in I$ and $I \cup \{m\}$ is weakly sum-free. So $a + m \notin I$ and $(a+m+1) - m = a+1 \notin I$, i.e., $I \subseteq [a, a+m+1] \setminus \{a+1, a+m\} = I_m(a,2)$.

Thus $I_m(a,2)$ is weakly sum-free m-maximal, such that

$$\{a, a+m+1\} \subseteq I_m(a,2) \subseteq [a, a+m+1]$$

and therefore $I_m(a,2) \in \mathtt{Weak}_m[a, a+m+1]$. □

In other words, this proposition gives properties of $I_m(a,h)$ which are:

1. $I_m(a,h) \subseteq [a, a+m+h-(1-\delta_{m,a})] = \begin{cases} [m, 2m+1] & \text{if } m = a \text{ and } h = 1, \\ [m, 2m+2] & \text{if } m = a \text{ and } h = 2, \\ [a, a+m-1] & \text{if } m < a \text{ and } h = 2, \\ [a, a+m+1] & \text{if } m < a \text{ and } h = 2. \end{cases}$
2. $a, a+m+h-(1-\delta_{m,a}) \in I_m(a,h)$.
3. $I_m(a,h) \cup \{m\}$ is weakly sum-free.
4. And $I_m(a,h)$ is maximal relative to the above properties.

Note also that we can write $I_m(a,h)$ on one line as

$$I_m(a,h) = [a, a+m+h-(1-\delta_{m,a})] \setminus \{a+h-(1-\delta_{m,a}), a+m+\delta_{m,a}(3-h)\}.$$

4.2. Semi-special Sets

We shall need unions of *sp-intervals*.

Definition 8. A *semi-special set* is a finite union of sp-intervals. In other words, it is a set A of the form

$$A = I_{m_1}(a_1, h_1) \cup I_{m_2}(a_2, h_2) \cup \ldots \cup I_{m_n}(a_n, h_n),$$

where the a_i's are pairwise distinct positive integers.

Example 9. The four parts A_1, A_2, A_3, A_4 of Example 1 are semi-special sets. Indeed, a straightforward comparison shows that

$$\begin{aligned}
A_1 &= I_1(1,2) \cup I_1(8,0) \cup I_1(11,0) \cup I_1(22,0) \cup I_1(25,0) \cup I_1(50,0) \cup I_1(63,0)\\
A_2 &= I_3(3,1) \cup I_3(19,2) \cup I_3(51,0) \cup I_3(64,0)\\
A_3 &= I_9(9,2) \cup I_9(54,0)\\
A_4 &= I_{24}(24,1).
\end{aligned}$$

Given an integer n, as in Example 9, we propose to use semi-special sets to build n weakly sum-free parts $A_1, \ldots, A_n$ of $[1, N]$, i.e., for $1 \le i \le n$, A_i is a semi-special set such that

$$A_i = I_{m_1^i}(a_1^i, h_1^i) \cup \cdots \cup I_{m_{n_i}^i}(a_{n_i}^i, h_{n_i}^i)$$

and $a_1^i < a_2^i < \cdots < a_{n_i}^i$. We have seen in Proposition 7 that $I_{m_j^i}(a_j^i, h_j^i)$ is weakly sum-free for each i, j, but that is not necessarily the case for their union. We shall need conditions on a union of special intervals ensuring that it is weakly sum-free.

4.2.1. Criteria for Weak Sum-freeness

The following assertions give some necessary conditions on a semi-special set to be weakly sum-free.

Assertion 10. *Let $A = I_{m_1}(a_1, h_1) \cup I_{m_2}(a_2, h_2) \cup \cdots \cup I_{m_n}(a_n, h_n)$ be a semi-special set with $m_1 = a_1 < a_2 < \cdots < a_n$.*

If A is weakly sum-free, then $m_j < a_j$ for $2 \le j \le n$.

Proof. Recall that $m_j \le a_j$ for all j by definition of sp-intervals. Suppose that there exists j with $2 \le j \le n$ such that $m_j = a_j$. We shall show that A is not weakly sum-free.

By hypothesis, $m_j = a_j > a_1 = m_1$, therefore we have $m_j - 1 \ge m_1$ because $j \ge 2$ and $m_1 \in A$.

- If $m_1 > 2$, we have $2 < m_1 \le m_j - 1$, which implies $m_j + 2 < m_j + m_1 \le 2m_j - 1$. Then $m_j, m_j + m_1 \in I_{m_j}(m_j, h_j) \subseteq A$ and $m_1 \in I_{m_1}(m_1, h_1) \subseteq A$, so A is not weakly sum-free.

- If $m_1 = 2$, we have:
  - If $h_j = 1$, then $m_j + 2 \in I_{m_j}(m_j, 1) \subseteq A$ and therefore A is not weakly sum-free because $m_1 = 2, m_j, m_j + 2 \in A$.
  - If $h_j = 2$, then $m_j + 1, m_j + 3 \in I_{m_j}(m_j, 2) \subseteq A$, so A is not weakly sum-free because $2, m_j + 1, m_j + 3 \in A$.
- If $m_1 = 1$, we have:
  - If $h_j = 1$, then $2m_j, 2m_j + 1 \in I_{m_j}(m_j, 1) \subseteq A$, so $1, 2m_j, 2m_j + 1 \in A$, i.e., A is not weakly sum-free.
  - If $h_j = 2$ therefore $m_j + 1 \in I_{m_j}(m_j, 2) \subseteq A$ and then A is not weakly sum-free because $1, m_j, m_j + 1 \in A$.

We conclude that if A is weakly sum-free, then for $2 \leq j \leq n$, we have $m_j < a_j$, i.e., $h_j \in \{0, 2\}$. □

Assertion 11. *Let* $A = I_{m_1}(a_1, h_1) \cup I_{m_2}(a_2, h_2) \cup \cdots \cup I_{m_n}(a_n, h_n)$ *be a semi-special set such that* $2 \leq m_1 = a_1 < a_2 < \cdots < a_n$.

If A is weakly sum-free, then $m_j \leq m_1$ for $2 \leq j \leq n$.

Proof. Suppose that there exists $j \in [2, n]$ such that $m_j > m_1$, i.e., $m_j - 1 \geq m_1$. According to Assertion 10, we have $m_j < a_j$, then $h_j \in \{0, 2\}$, i.e., $I_{m_j}(a_j, 0) = [a_j, a_j + m_j - 1]$ or $I_{m_j}(a_j, 2) = (\{a_j\} \cup [a_j + 2, a_j + m_j - 1] \cup \{a_j + m_j + 1\})$ is included in A.

By hypothesis $m_1 \geq 2$ so we have $2 \leq m_1 \leq m_j - 1$, which implies $a_j + 2 \leq a_j + m_1 \leq a_j + m_j - 1$. So $a_j + m_1 \in I_{m_j}(a_j, h_j) \subseteq A$ and therefore A is not weakly sum-free because $m_1, a_j, a_j + m_1 \in \left(I_{m_1}(m_1, h_1) \cup I_{m_j}(a_j, h_j)\right) \subseteq A$.

Thus if A is weakly sum-free we have $m_j \leq m_1$ for all $1 \leq j \leq n$. □

In summary, if a semi-special set $A = I_{m_1}(a_1, h_1) \cup I_{m_2}(a_2, h_2) \cup \cdots \cup I_{m_n}(a_n, h_n)$, with $2 \leq m_1 = a_1 < a_2 < \cdots < a_n$, is weakly sum-free, then for $i \geq 2$, we have $m_i < a_i$ and $m_i \leq m_1$.

4.3. Special Sets

We now focus on a particular semi-special set which verifies the necessary conditions to be weakly sum-free.

Definition 12. A *special set* A is a semi-special set such that all its sp-intervals $I_{m_i}(a_i, h_i)$ have the same index $m_i = \min A$.

In other words, a semi-special set $A = I_{m_1}(a_1, h_1) \cup I_{m_2}(a_2, h_2) \cup \cdots \cup I_{m_n}(a_n, h_n)$ is a *special set* if $m_1 = m_2 = \cdots = m_n = m = a_1 < a_2 < \cdots < a_n$. Therefore, A is of the form:

$$A = I_m(m, h_1) \cup I_m(a_2, h_2) \cup \cdots \cup I_m(a_n, h_n).$$

We shall use a specific notation for a special set.

Notation 13. Let $A = I_m(m, h_1) \cup I_m(a_2, h_2) \cup \cdots \cup I_m(a_n, h_n)$ be a special set, with $m = a_1 < a_2 < \cdots < a_n$. Then we shall denote A as follows:

$$A = \langle m, h_1\rangle\langle a_2, h_2\rangle \cdots \langle a_n, h_n\rangle.$$

Therefore $I_m(a_i, h_i) = \langle a_i, h_i\rangle$ for $1 \le i \le n$ and the index of $\langle a_i, h_i\rangle$ is $m = \min A$.

Example 14. Let us rewrite the parts A_1, A_2, A_3, A_4 of $[1, 66]$ in Example 9 with the new notation:

$$\begin{aligned}
A_1 &= \langle 1, 2\rangle\langle 8, 0\rangle\langle 11, 0\rangle\langle 22, 0\rangle\langle 25, 0\rangle\langle 50, 0\rangle\langle 63, 0\rangle\\
A_2 &= \langle 3, 1\rangle\langle 19, 2\rangle\langle 51, 0\rangle\langle 64, 0\rangle\\
A_3 &= \langle 9, 2\rangle\langle 54, 0\rangle\\
A_4 &= \langle 24, 1\rangle.
\end{aligned}$$

Here, we have $\langle 19, 2\rangle = I_3(19, 2) = [19, 23] \setminus \{20, 22\} = \{19, 21, 23\}$. If $a > 1$, we have $I_1(a, 0) = \langle a, 0\rangle = \{a\}$. By convenience for $m = 1$, instead of using the new notation, we simply enumerate $A_1 = \{1, 2, 4, 8, 11, 22, 25, 50, 63\}$.

Recall that a special set $A = I_{m_1}(m_1, h_1) \cup I_{m_2}(a_2, h_2) \cup \cdots \cup I_{m_n}(a_n, h_n)$, with $m_1 = a_1 < a_2 < \cdots < a_n$, fulfills the conditions $m_j = m \le m_1 = m$ for all j and $m_j < a_j$, if $j \ge 2$ which are necessary conditions on a semi-special set to be weakly sum-free. See Assertions 11 and 10.

4.3.1. Criteria for Weak Sum-freeness

Here, we give necessary conditions on a special set to be weakly sum-free.

Assertion 15. *Let $A = I_m(m, h_1) \cup I_m(a_2, h_2) \cup \cdots \cup I_m(a_n, h_n)$ be a special set such that $3 \le m = a_1 < a_2 < \cdots < a_n$.*

If A is weakly sum-free, then for $1 \le i < n$ we have

$$\begin{cases} a_{i+1} > \max I_m(a_i, h_i) \\ \quad\textit{or} \\ I_m(a_i, h_i) \supseteq I_m(a_{i+1}, h_{i+1}). \end{cases}$$

Note that: $(a_{i+1} > \max I_m(a_i, h_i))$ implies $(I_m(a_i, h_i) \cap I_m(a_{i+1}, h_{i+1}) = \emptyset)$.

Proof. Suppose that there exists $i \in [1, n-1]$ such that $a_{i+1} \leq \max I_m(a_i, h_i)$. We shall show that $I_m(a_i, h_i) \supseteq I_m(a_{i+1}, h_{i+1})$ or A is not weakly sum-free.

1. If $I_m(a_i, h_i) \cap I_m(a_{i+1}, h_{i+1}) = \emptyset$, then $a_{i+1} \notin I_m(a_i, h_i)$. According to Proposition 7, $I_m(a_i, h_i) \cup \{a_{i+1}\}$ is not weakly sum-free because $a_i < a_{i+1} \leq \max I_m(a_i, h_i)$. This is due to the properties of Section 4.1.1 about $I_m(a_i, h_i)$ which is a weakly sum-free m-maximal subset of $[a_i, \max I_m(a_i, h_i)]$. Therefore, A is not weakly sum-free.

2. If $I_m(a_i, h_i) \cap I_m(a_{i+1}, h_{i+1}) \neq \emptyset$: since $a_i < a_{i+1}$, we have $a_{i+1} - 1 \in I_m(a_i, h_i)$ or $a_{i+1} - 2 \in I_m(a_i, h_i)$. By hypothesis $m \geq 3$ so we have $a_{i+1} + 1 \leq a_{i+1} + m - 2 \leq a_{i+1} + m - 1$ and $a_{i+1} + m - 1 \in A$.

   - If $a_{i+1} - 1 \in I_m(a_i, h_i)$, we have:
     - ∘ If $a_{i+1} - 1 \neq m$, then A is not weakly sum-free because $m, a_{i+1} - 1, a_{i+1} + m - 1 \in A$.
     - ∘ If $a_{i+1} - 1 = m$, then $i = 1$ and $a_2 = m + 1$, we have:
       - ∗ If $a_2 + m = 2m + 1 \in A$, then A is not weakly sum-free because $m, m+1, 2m+1 \in A$.
       - ∗ If $2m + 1 \notin A$, then $2m + 1 \notin I_m(m, h_1)$. Therefore $h_1 = 2$ and $a_2 + m + 1 = 2m + 2 \in A$.
         - ⋆ If $m + 2 = a_2 + 1 \in A$, then $m, m+2, 2m+2 \in A$, so A is not weakly sum-free.
         - ⋆ If $m + 2 = a_2 + 1 \notin A$, then $m + 2 \notin I_m(m, h_1)$; therefore $h_1 = 2$, and $a_2 + 1 \notin I_m(a_2, h_2)$, so $h_2 = 2$ and we get the case $I_m(m, 2) \supseteq I_m(m+1, 2)$.
   - If $a_{i+1} - 1 \notin I_m(a_i, h_i)$, then $a_{i+1} - 2 \in I_m(a_i, h_i)$:
     - ∘ If $a_{i+1} - 2 \neq m$, we have
       - ∗ If $a_{i+1} + m - 2 \in A$, then A is not weakly sum-free because $m, a_{i+1} - 2, a_{i+1} + m - 2 \in A$.
       - ∗ If $a_{i+1} + m - 2 \notin A$, then $a_{i+1} + m - 2 = a_{i+1} + 1$; therefore $m = 3$ and $a_{i+1} + m + 1 = a_{i+1} + 4 \in I_3(a_{i+1}, 2) = \{a_{i+1}, a_{i+1} + 2, a_{i+1} + 4\} \subseteq A$. For $h_1 = 1, 2$, we have $2m = 6 \in I_3(3, h_1) \subseteq A$.
         - ⋆ If $a_{i+1} - 2 \neq 6$, then A is not weakly sum-free because $6, a_{i+1} - 2, a_{i+1} + 4 \in A$.
         - ⋆ If $a_{i+1} - 2 = 6$, i.e., $a_{i+1} = 8$, then $\max I_3(3, h_1) \geq 8$. This implies that $h_1 = 2$ and $I_3(3, 2) = \{3, 4, 6, 8\} \subseteq A$. So A is not weakly sum-free because $4, a_{i+1}, a_{i+1} + 4 \in A$.
     - ∘ If $a_{i+1} - 2 = m$, then $i = 1$ and $a_i = a_1 = m$. Since $a_2 - 1 = m + 1 \notin I_m(m, h_1)$, we have $h_1 = 1$ and $I_m(m, 1) \subseteq A$.

$*$ If $2m+3 = a_2+m+1 \in A$, then A not weakly sum-free because $a_1 = m, m+3, 2m+3 \in A$.

$*$ If $a_2+m+1 \notin A$, then $h_2 = 0$, so we have $I_m(m,1) \supseteq I_m(m+2,0)$.

□

Note that the hypothesis $m \geq 3$ is essential in Assertion 15. Here are counter-examples for $m \leq 2$.

For $m = 2$, for all integers $a_i > 7$, the special set $A = I_2(2,1) \cup I_2(a_i,2) \cup I_2(a_i+3,2) = \{2,4,5,a_i,a_i+3,a_i+6\}$ is weakly sum-free, with $a_i+3 \in I_2(a_i,2) \cap I_2(a_i+3,2)$ and $a_i+6 \in I_2(a_i+3,2) \setminus I_2(a_i,2)$.

Similarly, for $m = 1$, for all integers $a_i > 4$, the special set $A = I_1(1,1) \cup I_1(a_i,2) \cup I_1(a_i+2,2) = \{1,3,a_i,a_i+2,a_i+4\}$ is weakly sum-free, with $a_i+2 \in I_1(a_i,2) \cap I_1(a_i+2,2)$ and $a_i+4 \in I_1(a_i+2,2) \setminus I_1(a_i,2)$.

4.4. Special Partitions

We introduce some definitions related to the partitions which exclusively use special sets.

Definition 16. Let n, M be two positive integers. A *special n-partition* $\mathcal{P}$ of $[1,M]$ is a partition of $[1,M]$ into n weakly sum-free special sets parts.

Definition 17. Let n be a positive integer. We define the number $N(n)$ as the largest integer M such that there exists a special n-partition of $[1,M]$.

We have

$$N(n) = \max\{M \in \mathbb{N} \mid \text{ there exists } \mathcal{P} \text{ special } n\text{-partition of } [1,M]\}.$$

Of course, for all $n \in \mathbb{N}$, we have

$$N(n) \leq WS(n).$$

A *maximum special n-partition* is a special n-partition of $[1,N(n)]$. An integer M is a lower bound of $N(n)$ if there exists a special n-partition of $[1,M]$.

- We have $I_1(1,2) = \{1,2,4\} \supseteq [1,2]$. Thus $N(1) = 2$ because $WS(1) = 2$.
- Let $A_1 = I_1(1,2) \cup I_1(8,0)$ and $A_2 = I_3(3,1)$, we have $[1,8] = A_1 \sqcup A_2$ whence $N(2) \geq 8$. Since $WS(2) = 8$, it follows that $N(2) = 8$.
- We also have $N(3) = 23$ because $WS(3) = 23$ and we have a special 3-partition of $[1,23]$ into 3 parts A_1, A_2, A_3:

$$\begin{aligned} A_1 &= I_1(1,2) \cup I_1(8,0) \cup I_3(11,0) \cup I_1(22,0) \\ A_2 &= I_3(3,1) \cup I_3(19,2) \\ A_3 &= I_9(9,2). \end{aligned}$$

We remark that for $1 \leq n \leq 3$ there is only one maximum special n-partition of $[1, N(n)]$.

- We know that $WS(4) = 66$. In Example 9 we have $66 \leq N(4)$, whence $N(4) = 66$. We note that there are four maximum special 4-partitions of $[1, 66]$.

In summary, for $n \leq 4$, we have $WS(n) = N(n)$. For $n = 5$, we will see that $N(5) = 196$ (by computer).

In this paper, we shall give lower bounds on $N(n)$, and hence also on $WS(n)$, for $6 \leq n \leq 9$.

5. Formulas for Some Restricted Sums

In this section, we study the restricted sum of two finite sets, which are written in the form $[x, y] \setminus G$, and in particular the restricted sum of two specials sets. The resulting formulas were implemented in our search algorithm in order to greatly accelerate it.

5.1. Restricted Sum of Two Finite Sets

Let $I = [x, y] \setminus G$ and $J = [r, s] \setminus H$ be two finite sets with $G \subseteq [x + 1, y - 1]$ and $H \subseteq [r+1, s-1]$ such that $|G|$ and $|H|$ are not necessarily less than two. Note that if $\max(|G|, |H|) \leq 2$, then I and J are punctured intervals.

By definition, $I \cup J$ is weakly sum-free if

$$\big((I \cup J) \dot{+} (I \cup J)\big) \cap (I \cup J) = \emptyset.$$

We suppose that $I \cap J = \emptyset$ because if $L = I \cap J \neq \emptyset$, we may consider the finite set $I' = I \setminus L = [x', y'] \setminus G'$ with $G' \subseteq [x' + 1, y' - 1]$ and

$$I \cup J = (I \setminus L) \cup J = I' \cup J \text{ with } I' \cap J = \emptyset.$$

So here we suppose $I \cap J = \emptyset$ and we have

$$(I \cup J) \dot{+} (I \cup J) = (I \dot{+} I) \cup (I \dot{+} J) \cup (J \dot{+} J).$$

Then $I \cup J$ is weakly sum-free if for the restricted sums $S = I \dot{+} I, J \dot{+} J, I \dot{+} J$ so S and $I \cup J$ are disjoint.

We give the form of $I \dot{+} J$ when $I = J$ or $I \cap J = \emptyset$.

Proposition 18. *Let $I = [x, y] \setminus G$ and $J = [r, s] \setminus H$ be two finite sets of integers such that $I = J$ or $I \cap J = \emptyset$ with $G \subseteq [x + 1, y - 1]$ and $H \subseteq [r + 1, s - 1]$.*

Then there exists $K \subseteq ((x + H) \cup (G + s))$ such that

$$I \dot{+} J = [x + r + \delta_{I,J}, y + s - \delta_{I,J}] \setminus K.$$

Proof. As $x \in I \subseteq [x, y]$ and $s \in J \subseteq [r, s]$, we have

$$\left((x \dot{+} J) \cup (I \dot{+} s)\right) \subseteq I \dot{+} J \subseteq \left([x, y] \dot{+} [r, s]\right),$$

with

$$x \dot{+} J = \begin{cases} [x+r, x+s] \setminus (x+H) & \text{if } I \cap J = \emptyset \\ [x+r+1, x+s] \setminus (x+H) & \text{if } I = J, \end{cases}$$

and

$$I \dot{+} s = \begin{cases} [x+s, y+s] \setminus (G+s) & \text{if } I \cap J = \emptyset \\ [x+s, y+s-1] \setminus (G+s) & \text{if } I = J. \end{cases}$$

Then

$$(x \dot{+} J) \cup (I \dot{+} s) = \begin{cases} [x+r, y+s] \setminus \left((x+H) \cup (G+s)\right) & \text{if } I \cap J = \emptyset \\ [x+r+1, y+s-1] \setminus \left((x+H) \cup (G+s)\right) & \text{if } I = J. \end{cases}$$

We also have:

$$[x, y] \dot{+} [r, s] = \begin{cases} [x+r, y+s] & \text{if } I \cap J = \emptyset \\ [x+r+1, y+s-1] & \text{if } I = J. \end{cases}$$

We get:

$$\begin{cases} \Big([x+r, y+s] \setminus \left((x+H) \cup (G+s)\right)\Big) \subseteq I \dot{+} J \subseteq [x+r, y+s], & \text{if } I \cap J = \emptyset \\ \Big([x+r+1, y+s-1] \setminus \left((x+H) \cup (G+s)\right)\Big) \subseteq I \dot{+} J \subseteq [x+r+1, y+s-1], & \text{if } I = J. \end{cases}$$

Since by hypothesis $I = J$ or $I \cap J = \emptyset$, it follows that $\delta_{I,J} = 0$ whenever $I \cap J = \emptyset$. So we have

$$\Big([x+r+\delta_{I,J}, y+s-\delta_{I,J}] \setminus \left((x+H) \cup (G+s)\right)\Big) \subseteq I \dot{+} J \subseteq [x+r+\delta_{I,J}, y+s-\delta_{I,J}].$$

Therefore, there exists $K \subseteq (x+H) \cup (G+s)$ such that

$$I \dot{+} J = [x+r+\delta_{I,J}, y+s-\delta_{I,J}] \setminus K. \qquad \square$$

Let $I, J \subseteq \mathbb{N}$ be two finite subsets of the form $I = [x, y] \setminus G$ and $J = [r, s] \setminus H$ such that $G \subseteq [x+1, y-1]$ and $H \subseteq [r+1, s-1]$ with $I = J$ or $I \cap J = \emptyset$. Then there exists $K \subseteq ((x+H) \cup (G+s))$ such that $I \dot{+} J = [x+r+\delta_{I,J}, y+s-\delta_{I,J}] \setminus K$. Let $t \in [x+r+\delta_{I,J}, y+s-\delta_{I,J}]$, then

$$t \in K \Leftrightarrow t \notin I \dot{+} J.$$

As $K \subseteq ((x+H) \cup (G+s))$, then to determine $I \dot{+} J$, we have to know when an element of $((x+H) \cup (G+s))$ belongs to K. This is the purpose of the next proposition.

Note that if $(z, v) \in ((\{x\} \times H) \cup (G \times \{s\}))$ then $z + v \in ((x+H) \cup (G+s))$.

Proposition 19. *Let $I = [x, y] \setminus G$, $J = [r, s] \setminus H$ be two non-empty finite sets such that $I \cap J = \emptyset$ or $I = J$ with $G \subseteq [x+1, y-1]$ and $H \subseteq [r+1, s-1]$.*

Let K be the subset of $((x + H) \cup (G + s))$ such that $I \dot{+} J = [x + r + \delta_{I,J}, y + s - \delta_{I,J}] \setminus K$. Let $(z, v) \in (\{x\} \times H) \cup (G \times \{s\})$, then we have

$$z + v \in K \Leftrightarrow [x - z, y - z] \cap [v - s, v - r] \setminus \left((G - z) \cup (v - H) \cup \{\frac{1}{2}(v - z)\}\right) = \emptyset.$$

Proof. Let $(z, v) \in (\{x\} \times H) \cup (G \times \{s\}) \subseteq [x, y - 1] \times [r + 1, s]$.

We know that $z + v \in I \dot{+} J$ if and only if $z + v \notin K$. In other words, there exists an integer i such that $(z+i, v-i) \in I \times J$ because $z+v = (z+i)+(v-i)$. If $I = J$ and $z + i = v - i$, then $(z + i) + (v - i) \notin I \dot{+} J$. So we have to add the condition $z + i \neq v - i$. Note that we do not need this condition when $I \cap J = \emptyset$. This means that $i \in \big((I - z) \cap (v - J)\big) \setminus \{\frac{1}{2}(v - z)\}$. This condition is also equal to:

$$(I-z)\cap(v-J)\setminus\{(v-z)/2\} = [x-z, y-z]\cap[v-s, v-r]\setminus\big((G - z) \cup (v - H) \cup \{(v - z)/2\}\big) \neq \emptyset.$$

Then we get:

$$z + v \notin K \Leftrightarrow [x - z, y - z] \cap [v - s, v - r] \setminus \left((G - z) \cup (v - H) \cup \{\frac{1}{2}(v - z)\}\right) \neq \emptyset.$$

□

Remark 20. Note that if $I \cap J = \emptyset$, the condition to have $z + v \in K$ is

$$[x - z, y - z] \cap [v - s, v - r] \setminus ((G - z) \cup (v - H)\}) = \emptyset.$$

Remark 21. Given an integer $M_0 \geq 2$, let $I = [x, y] \setminus G$, $J = [r, s] \setminus H$ be two finite sets, such that $|G|, |H| \leq M_0$ with $G \subseteq [x + 1, y - 1]$ and $H \subseteq [r + 1, s - 1]$. According to Proposition 18, there exists $K \subseteq ((x + H) \cup (G + s))$ such that $I \dot{+} J = [x + r + \delta_{I,J}, y + s - \delta_{I,J}] \setminus K$. Then $|K| \leq |G| + |H| \leq 2M_0$. Therefore to determine $I \dot{+} J$, we apply at most $2M_0$ times Proposition 19 on the elements of $((x + H) \cup (G + s))$. Thus the cost of the implementation of the restricted sum of $I \dot{+} J$ depends on M_0, but does not depend on the cardinal of I and J. That is why we chose $M_0 = 2$, giving rise to the sp-intervals.

5.2. Restricted Sum of Two Special Intervals in a Special Set

Here, we consider a special set

$$A = I_m(m, h_1) \cup I_m(a_2, h_2) \cup \cdots \cup I_m(a_n, h_n) = \bigcup_{i=1}^{n} I_m(a_i, h_i)$$

such that $m = a_1 < a_2 < \cdots < a_n$ and $h_i \in \{\delta_{i,1}, 2\}$ for $1 \leq i \leq n$ with $I_m(a_i, h_i) \cap I_m(a_j, h_j) = \emptyset$ if $i \neq j$.

The restricted sum of A with itself is:

$$\begin{aligned} A\dot{+}A &= \left(\bigcup_{i=1}^{n} I_m(a_i,h_i)\right) \dot{+} \left(\bigcup_{j=1}^{n} I_m(a_j,h_j)\right) \\ &= \bigcup_{i=1}^{n}\bigcup_{j=1}^{n} \left(I_m(a_i,h_i)\dot{+}I_m(a_j,h_j)\right). \end{aligned}$$

Then

$$\begin{aligned} (A\dot{+}A)\cap A &= \left(\bigcup_{i=1}^{n}\bigcup_{j=1}^{n} \left(I_m(a_i,h_i)\dot{+}I_m(a_j,h_j)\right)\right) \bigcap \left(\bigcup_{k=1}^{n} I_m(a_k,h_k)\right) \\ &= \bigcup_{i=1}^{n}\bigcup_{j=1}^{n}\bigcup_{k=1}^{n} \left(\left(I_m(a_i,h_i)\dot{+}I_m(a_j,h_j)\right)\cap I_m(a_k,h_k)\right). \end{aligned} \tag{5.1}$$

Assertion 22. *The special set $A=\bigcup_{i=1}^{n} I_m(a_i,h_i)$ is weakly sum-free if and only if for all integers $1\le i\le j\le k\le n$, we have*

$$(I_m(a_i,h_i)\dot{+}I_m(a_j,h_j))\cap I_m(a_k,h_k)=\emptyset. \tag{5.2}$$

Proof. Suppose that $A=\bigcup_{i=1}^{n} I_m(a_i,h_i)$ is weakly sum-free set, so using (5.1), we get

$$\bigcup_{i=1}^{n}\bigcup_{j=1}^{n}\bigcup_{k=1}^{n}(I_m(a_i,h_i)\dot{+}I_m(a_j,h_j))\cap I_m(a_k,h_k)=\emptyset. \tag{5.3}$$

Then we have $(I_m(a_i,h_i)\dot{+}I_m(a_j,h_j))\cap I_m(a_k,h_k)=\emptyset$, for all $1\le i,j,k\le n$.

Reciprocally, suppose that for $1\le i\le j\le k\le n$, we have

$$(I_m(a_i,h_i)\dot{+}I_m(a_j,h_j))\cap I_m(a_k,h_k)=\emptyset.$$

However, by construction $\min I_m(a_j,h_j) > \max I_m(a_k,h_k)$ if $j>k$. So for all $k<\max\{i,j\}$, we always have $(I_m(a_i,h_i)\dot{+}I_m(a_j,h_j))\cap I_m(a_k,h_k)=\emptyset$ because $\min(I_m(a_i,h_i)\dot{+}I_m(a_j,h_j)) > \max I_m(a_k,h_k)$.

Therefore, for all $1\le i,j,k\le n$, we have

$$(I_m(a_i,h_i)\dot{+}I_m(a_j,h_j))\cap I_m(a_k,h_k)=\emptyset.$$

□

This Assertion 22 shows that $A=\bigcup_{i=1}^{n} I_m(a_i,h_i)$ is not weakly sum-free if and only if there exist $1\le i\le j\le k\le n$ such that

$$(I_m(a_i,h_i)\dot{+}I_m(a_j,h_j))\cap I_m(a_k,h_k)\neq\emptyset. \tag{5.4}$$

Then we need to know $I_m(a_i,h_i)\dot{+}I_m(a_j,h_j)$ for all $1 \le i \le j \le n$. This restricted sum is of the form:

$$\begin{aligned}&I_m(a_i,h_i)\dot{+}I_m(a_i,h_i) \text{ for } 1 \le i \le n, \text{ or}\\ &I_m(a_i,h_i)\dot{+}I_m(a_j,h_j) \text{ for } i \ne j, \text{ i.e., if } I_m(a_i,h_i)\cap I_m(a_j,h_j)=\emptyset.\end{aligned}$$

Using Propositions 18 and 19, we give formulas on the restricted sum of two special intervals. Firstly we will give the restricted sum of $I_m(a,h)$ with itself. And after that, we will determine the restricted sum of two disjoint special intervals. Recall that the special interval $I_m(a,h)=\langle a,h\rangle$ is defined by

$$\begin{cases} I_m(m,1) &= [m,2m+1]\setminus\{m+1\} & \text{if } a=m,\\ I_m(m,2) &= [m,2m+2]\setminus\{m+2,2m+1\} & \text{if } a=m,\\ I_m(a,0) &= [a,a+m-1] & \text{if } a>m,\\ I_m(a,2) &= [a,a+m+1]\setminus\{a+1,a+m\} & \text{if } a>m.\end{cases}$$

Proposition 23. *Let m and a be two positive integers with $m<a$. Then*

$$I_m(a,0)\dot{+}I_m(a,0)=[2a+1,2a+2m-3].$$

$$I_m(a,2)\dot{+}I_m(a,2)=\begin{cases}[2a+2,2a+2m]\setminus\{2a+1,2a+m+2\} & \textit{if } m\le 4,\\ [2a+2,2a+2m] & \textit{if } m>4.\end{cases} \tag{5.5}$$

$$I_m(m,1)\dot{+}I_m(m,1)=\begin{cases}\{2m+2,4m+1\}\setminus\{3m+2\} & \textit{if } m\le 2,\\ [2m+2,4m+1] & \textit{if } m>2.\end{cases}$$

$$I_m(m,2)\dot{+}I_m(m,2)=\begin{cases}[2m+1,4m+2]\setminus\{2m+2,3m+1,3m+4\} & \textit{if } m\in\{1,2\},\\ [2m+1,4m+2]\setminus\{2m+2,3m+1\} & \textit{if } m\in\{3,4\},\\ [2m+1,4m+2]\setminus\{2m+2\} & \textit{if } m>4.\end{cases}$$

Proof. We choose to show (5.5) only. The proofs are similar for the other cases.

According to Proposition 18, there exists a set $K\subseteq\{2a+1,2a+m,2a+m+2,2a+2m+1\}$, such that

$$I_m(a,2)\dot{+}I_m(a,2)=[2a+1,2a+2m+1]\setminus K.$$

Let $(z,c)\in\big(\{a\}\times\{a+1,a+m\}\big)\cup\big(\{a+1,a+m\}\times\{a+m+1\}\big)$. According to Proposition 19, we have $z+c\in K$ if and only if

$$[a-z,a+m+1-z]\cap[c-(a+m+1),c-a]\setminus\{a+1-z,a+m-z,c-(a+1),c-(a+m),\tfrac{1}{2}(c-z)\}=\emptyset.$$

Here, $z+c\in\{2a+1,2a+m,2a+m+2,2a+2m+1\}=\big(a+\{a+1,a+m\}\big)\cup\big(\{a+1,a+m\}+(a+m+1)\big)$.

- For $z+c=a+(a+1)=2a+1$, applying Proposition 19 we have:
  $[a-a,(a+m+1)-a]\cap[a+1-(a+m+1),a+1-a]\setminus\{a+1-a,a+m-a,a+1-(a+1),a+1-(a+m),\frac{1}{2}(a+1-a)\}$
  $=[0,m+1]\cap[-m,1]\setminus\{1,m,0,1-m,\frac{1}{2}\}=\emptyset$ for all integers m.
  Then $a+a+1=2a+1\in K$ for all integers m.

- For $z+c = a+(a+m) = 2a+m$, we have:
  $[a-a,(a+m+1)-a] \cap [a+m-(a+m+1), a+m-a] \setminus \{a+1-a, a+m-a, a+m-(a+1), a+m-(a+m), \frac{1}{2}(a+m-a)\}$
  $= [0, m+1] \cap [-1, m] \setminus \{1, m, m-1, 0, \frac{m}{2}\} = \emptyset$ if $m \leq 4$.
  Then according to Proposition 19, $a+(a+m) = 2a+m \in K$ if $m \leq 4$.

- For $z+c = (a+1)+(a+m+1) = 2a+m+2$, we have:
  $[a-(a+1), (a+m+1)-(a+1)] \cap [a+m+1-(a+m+1), a+m+1-a] \setminus \{a+1-(a+1), a+m-(a+1), a+m+1-(a+1), a+m+1-(a+m), \frac{1}{2}(a+m+1-(a+1))\}$
  $= [-1, m] \cap [0, m+1] \setminus \{0, m-1, m, 1, \frac{m}{2}\} = \emptyset$ if $m \leq 4$.
  Then $(a+1)+(a+m+1) = 2a+m+2 \in K$ if $m \leq 4$.

- For $z+c = (a+m)+(a+m+1) = 2a+m+1$, we have:
  $[a-(a+m), (a+m+1)-(a+m)] \cap [a+m+1-(a+m+1), a+m+1-a] \setminus \{a+1-(a+m), a+m-(a+m), a+m+1-(a+1), a+m+1-(a+m), \frac{1}{2}(a+m+1-(a+m))\}$
  $= [-m, 1] \cap [0, m+1] \setminus \{1-m, 0, m, 1\} = \emptyset$ for all integers m.
  Then for all $m \in \mathbb{N}$ we have $(a+m)+(a+m+1) = 2a+2m+1 \in K$.

In summary, $K = \begin{cases} \{2a+1, 2a+m, 2a+m+2, 2a+2m+1\} & \text{if } m \leq 4 \\ \{2a+1, 2a+2m+1\} & \text{if } m > 4, \end{cases}$

and therefore

$$I_m(a,2) \dot{+} I_m(a,2) =$$
$$= \begin{cases} [2a+1, 2a+2m+1] \setminus \{2a+1, 2a+m, 2a+m+2, 2a+2m+1\} & \text{if } m \leq 4, \\ [2a+1, 2a+2m+1] \setminus \{2a+1, 2a+2m+1\} & \text{if } m > 4. \end{cases}$$

□

We will now see the restricted sum $I_m(a,h) \dot{+} I_m(a',h')$ where $m \leq a, a'$ and $I_m(a,h) \neq I_m(a',h')$, i.e., according to Assertion 15, $I_m(a,h) \cap I_m(a',h') = \emptyset$. As $a \in I_m(a, \delta_{m,a}) \cap I_m(a,2)$ for $m \leq a$, then we have seven possible types of restricted sums of two disjoint sp-intervals in a weakly sum-free special set.

Proposition 24. *Let m, a, a' be three positive integers such that $m < a, a'$ and $a \neq a'$. We have*

$$I_m(m,1) \dot{+} I_m(a,0) = \begin{cases} [a+1, a+3] \setminus \{a+2\} & \textit{if } m = 1, \\ [a+m, a+3m] & \textit{if } m > 1. \end{cases}$$

$$I_m(m,1) \dot{+} I_m(a,2) = \begin{cases} [a+m, a+3m+2] \setminus \{a+m+1, a+2m+2\} & \textit{if } m \leq 2, \\ [a+m, a+3m+2] \setminus \{a+m+1\} & \textit{if } m > 2. \end{cases} \tag{5.6}$$

$$I_m(m,2)\dot{+}I_m(a,0)=\begin{cases}[a+m,a+3m+1]\setminus\{a+2m+1\} & \text{if } m\le 2,\\ [a+m,a+3m+1] & \text{if } m>2.\end{cases}$$

$$I_m(m,2)\dot{+}I_m(a,2)=\begin{cases}[a+m,a+3m+3]\setminus\{a+2m+3,a+3m+2\} & \text{if } m=1,3,\\ [a+2,a+9]\setminus\{a+4,a+7,a+8\} & \text{if } m=2,\\ [a+m,a+3m+3]\setminus\{a+3m+2\} & \text{if } m>3.\end{cases}$$

$$I_m(a,0)\dot{+}I_m(a',0)=[a+a',a+a'+2m-2].$$

$$I_m(a,0)\dot{+}I_m(a',2)=\begin{cases}[a+a',a+a'+2m]\setminus\{a+a'+m\} & \text{if } m\le 2,\\ [a+a',a+a'+2m] & \text{if } m>2.\end{cases}$$

And finally, the value of $I_m(a,2)\dot{+}I_m(a',2)$ *is*

$$\begin{cases}[a+a',a+a'+2m+2]\setminus\{a+a'+1,a+a'+m,a+a'+m+2,a+a'+2m+1\} & \\ & \text{if } m\le 3,\\ [a+a',a+a'+2m+2]\setminus\{a+a'+1,a+a'+2m+1\} & \\ & \text{if } m>3.\end{cases}$$

Proof. Here, we show equality (5.6) only. The proofs of the other cases are similar. According to Proposition 18, there exists a set $K\subseteq\{a+m+1,a+2m,a+2m+2\}$ such that $I_m(m,1)\dot{+}I_m(a,2)=[a+m,a+3m+2]\setminus K$ because $I_m(m,1)\cap I_m(a,2)=\emptyset$.

Let $(z,c)\in(\{m\}\times\{a+1,a+m\})\cup(\{m+1\}\times\{a+m+1\})$. Using Proposition 19, we have $z+c\in K$ if and only if

$$[m-z,(2m+1)-z]\cap[c-(a+m+1),c-a]\setminus\{m+1-z,c-(a+1),c-(a+m)\}=\emptyset.$$

Here, $z+c\in\{a+m+1,a+2m,a+2m+2\}$.

- For $z+c=m+(a+1)=a+m+1$, we have:
  $[m-m,(2m+1)-m]\cap[a+1-(a+m+1),a+1-a]\setminus\{m+1-m,a+1-(a+1),a+1-(a+m)\}$
  $=[0,m+1]\cap[-m,1]\setminus\{1,0,1-m\}=\emptyset$ for all integers m.
  Then $m+a+1\in K$ for all integers m.

- For $z+c=m+(a+m)=a+2m$, we have:
  $[m-m,(2m+1)-m]\cap[a+m-(a+m+1),a+m-a]\setminus\{m+1-m,a+m-(a+1),a+m-(a+m)\}$
  $=[0,m+1]\cap[-1,m]\setminus\{1,m-1,0\}=\emptyset$ if $m\le 1$.
  Then $m+(a+m)=a+2m\in K$ if $m=1$. (same as the previous case when $m=1$).

- For $z+c=(m+1)+(a+m+1)=a+2m+2$, we have:
  $[m-(m+1),(2m+1)-(m+1)]\cap[a+m+1-(a+m+1),a+m+1-a]\setminus\{m+1-(m+1),a+m+1-(a+1),a+m+1-(a+m)\}$

$= [-1, m] \cap [0, m+1] \setminus \{0, 1, m\} = \emptyset$ if $m \leq 2$.
Then $(m+1) + (a+m+1) = a + 2m + 2 \in K$ if $m \leq 2$.

$$\text{In summary, } K = \begin{cases} \{a+m+1, a+2m+2\} & \text{if } m \leq 2, \\ \{a+m+1\} & \text{if } m > 2. \end{cases}$$

So $I_m(m, 1) \dot{+} I_m(a, 2) = [a+m, 2a+3m+2] \setminus K$

$$= \begin{cases} [a+m, 2a+3m+2] \setminus \{a+m+1, a+2m+2\} & \text{if } m \leq 2, \\ [a+m, 2a+3m+2] \setminus \{a+m+1\} & \text{if } m > 2. \end{cases}$$

□

6. Global Structure

In this section, we shall construct a word corresponding to a partition which parts are weakly sum-free special sets. We shall use this correspondence to restrict the field of partitions search.

6.1. Combinatorics on Words

Let us recall some background on words. See, e.g., book [7] for more information. Let L be a finite set. We view L as an *alphabet* and refer to its elements as *letters*. A finite sequence of elements of L is called a *word* on L (finite word). We denote by juxtaposition

$$a_1 a_2 \ldots a_n$$

the sequence $(a_1, a_2, \ldots, a_n)$ with $a_i \in L$ for $1 \leq i \leq n$. Set ϵ the empty word (the empty sequence). The length of $w = a_1 a_2 \ldots a_n$ is $|w| = n$ and $|\epsilon| = 0$. We denote L^* the set of all words on L:

$$L^* = \{ \text{ words on } L\} = \{w = a_1 a_2 \ldots a_m \mid m \in \mathbb{N}, a_i \in L, 1 \leq i \leq m\}.$$

There is a natural product on L^* defined by concatenation. Given two words $w = a_1 a_2 \ldots a_m$ and $v = b_1 b_2 \ldots b_n$, their product is the word

$$wv = a_1 a_2 \ldots a_m b_1 b_2 \ldots b_n.$$

This operation is associative and ϵ is the neutral element.
For $w, v \in L^*$, we say that w is a *left factor* of v if there exists a word $u \in L^*$ such that

$$v = wu.$$

The relation "*to be a left factor of*" is a partial order on words called the *prefix order* and denoted by $\leq$. Thus for instance, we have

$$\begin{array}{ll} \epsilon \leq w \leq wu \leq wuu \leq wuuw & (\text{ with } w, u \in L^*) \\ \epsilon \leq a \leq aab \leq aabbab & (\text{ with } a, b \in L). \end{array}$$

6.2. A Contraction Map on Words

Let $R \subseteq L$. We define a map

$$\begin{aligned} \lambda_R : & L \longrightarrow L^* \\ & x \longmapsto \lambda_R(x) = \begin{cases} \epsilon & \text{if } x \in R, \\ x & \text{if } x \notin R. \end{cases} \end{aligned}$$

The image $Im(\lambda_R) = (L \setminus R) \cup \{\epsilon\}$ is the set of words with at most one letter in $(L \setminus R)$. We define the extension

$$\begin{aligned} \rho_R : & L^* \longrightarrow (L \setminus R)^* \\ & w \longmapsto \rho_R(w) = \lambda_R(a_1)\lambda_R(a_2)\ldots\lambda_R(a_n). \end{aligned}$$

In other words, $\rho_R(w)$ is the word that we obtain when we delete all letters from R in w. For example, if $w = 124367162523 \in [1,7]^*$, then $\rho_{[1,3]}(w) = 46765$.

6.3. The Palindrome $s(a, b)$

A word $w \in L^*$ is a *palindrome* if $w = w^R$ where w^R denotes the reversal (or mirror image) of w. That is, if $w = a_1a_2\ldots a_n \in L^*$, then $w^R = a_n\ldots a_2a_1$. In other words, a word $w = a_1a_2\ldots a_n$ is a palindrome if $a_i = a_{n+1-i}$ for all $1 \le i \le n$.

We now introduce a particular palindrome which will play a key role in our constructions. Given an integer n, we consider the alphabet $L = [1, n]$. For $a, b \in [1, n]$ such that $a \le b$, we define the *palindrome* $s(a, b)$ recursively by:

$$\begin{aligned} s(a, a) &= a & \text{if } b = a, \\ s(a, b) &= s(a, b-1)\, b\, s(a, b-1) & \text{if } b > a. \end{aligned}$$

Note that $|s(a, b)| = 2^{b-a} - 1$.

For example, $s(2, 5) = s(2, 4)\, 5\, s(2, 4) = 232423252324232$ and $s(4, 6) = 4546454$.

6.4. Color Sequence of a Partition

Let $n \in \mathbb{N}$. Here, we shall introduce the word corresponding to a special n-partition.

Let $\mathcal{P}$ be a partition of $[1, N]$ into n parts $A_1, A_2, \ldots, A_n$ which are all weakly sum-free *special sets*. For $1 \le i \le n$, we have

$$\begin{aligned} A_i &= \langle m_i, h_1^i\rangle\langle a_2^i, h_2^i\rangle \cdots \langle a_{n_i}^i, h_{n_i}^i\rangle \\ &= I_{m_i}(m_i, h_1^i) \cup I_{m_i}(a_2^i, h_2^i) \cup \cdots \cup I_{m_i}(a_{n_i}^i, h_{n_i}^i) \end{aligned}$$

with $m_i = a_1^i < a_2^i < \cdots < a_{n_i}^i$ and $h_j^i \in \{\delta_{j,1}, 2\}$ for $1 \le j \le n_i$. We suppose also that

$$\min A_1 = m_1 < \min A_2 = m_2 < \cdots < \min A_n = m_n.$$

We identify each sp-interval $I_{m_i}(a_j^i, h_j^i)$ by its minimum a_j^i. Then we consider the set $\{a_j^i \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n_i\}$ of the minima of all sp-intervals in $\mathcal{P}$. Now, we define the increasing sequence $(b_k)_{1 \leq k \leq l}$ of the elements of this set, where $l = \sum_{i=1}^{n} n_i$, i.e., we have:

$$\begin{cases} [1, N] = \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} I_{m_i}(a_j^i, h_j^i) \\ \{b_1, \ldots, b_l\} = \{a_1^1, \ldots, a_{n_1}^1, a_1^2, \ldots, a_{n_2}^2, \ldots, a_1^n, \ldots, a_{n_n}^n\} \\ \qquad = \{a_j^i \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n_i\} \\ b_1 < b_2 < \cdots < b_l. \end{cases}$$

By assigning color i to all elements in A_i, we define an n-coloring

$$\begin{aligned} \chi : [1,N] &\rightarrow [1, n] \\ x &\longmapsto \chi(x) = i \text{ if } x \in A_i. \end{aligned}$$

We consider the color sequence $(\chi(b_k))_{1 \leq k \leq m}$ with $\chi(b_k) \in [1, n]$ for $1 \leq k \leq m$. We are now ready to introduce the word associated to the partition $\mathcal{P}$:

Definition 25. The word $w(\mathcal{P})$ associated to the special n-partition $\mathcal{P}$ is defined by

$$w(\mathcal{P}) = \chi(b_1)\chi(b_2) \cdots \chi(b_m).$$

Therefore $w(\mathcal{P}) \in [1, n]^*$ and $|w(\mathcal{P})| = m = \sum_{i=1}^{n} n_i$.

In Example 9, we have a special 4-partition $\mathcal{P}'$ of $[1, 66]$, then the word $w(\mathcal{P}') \in [1, 4]^*$ associated to $\mathcal{P}'$ is

$$w(\mathcal{P}') = 12131214121312.$$

By definition, the contraction of $w(\mathcal{P}')$ on $[2, n]$ is $\rho_{\{1\}}(\mathcal{P}') = 2324232$. On the other hand, we have the palindrome $s(2, 4) = 2324232$, then

$$\rho_{\{1\}}(\mathcal{P}') = s(2, 4).$$

6.5. Constrained Partition

Now, we propose to do the inverse process. We introduce a new definition on partitions related to a given palindrome $s(a, n)$.

Definition 26. Let $n, a \in \mathbb{N}$ with $a \in [2, n]$. A special n-partition $\mathcal{P}$ is said to be *constrained by* the palindrome $s(a, n)$ if

$$\rho_{[1,a-1]}(w(\mathcal{P})) \leq s(a, n).$$

We use a specific algorithm to search special n-partitions constrained by $s(a, n)$.

7. The Partition Search Algorithm

In this section we give two search methods of partitions. The first consists in finding special n-partitions of $[1,N]$. The second searches only special n-partitions of $[1,N]$ constrained by $s(a,n)$, for a given $a \in [1,n-1]$.

7.1. Coverings and Partitions

Fix an integer $n \geq 1$. Let N be a positive integer. Recall that $A_1,\dots,A_n \subseteq \mathbb{N}$ is a *covering* $\mathcal{C}$ of $[1,N]$ if

$$[1,N] \subseteq A_1 \cup \cdots \cup A_n.$$

Note that for any covering $\mathcal{C}$, we may extract a partition $\mathcal{P}$ by removing duplicate elements, for instance, as follows:

$$\begin{cases} B_1 = A_1 \cap [1,N], \\ B_i = \left(A_i \setminus \bigcup_{j<i} A_j \right) \cap [1,N] \quad \text{for } 2 \leq i \leq n. \end{cases}$$

In this section, we search coverings and then we get corresponding partitions by the above formulas.

Let $\mathcal{P}_1, \mathcal{P}_2$ be two partitions (resp., coverings). We denote by

$$\mathcal{P}_1 \preceq \mathcal{P}_2 \ \text{ if for all } A \in \mathcal{P}_1, \text{ there exists } B \in \mathcal{P}_2 \text{ such that } A \subseteq B.$$

Let $n \in \mathbb{N}$. An *extremum special n-partition* $\mathcal{P}_1$ is a special n-partition such that if there exists a special n-partition $\mathcal{P}_2$ with $\mathcal{P}_1 \preceq \mathcal{P}_2$, then we have $\mathcal{P}_2 = \mathcal{P}_1$.

Recall that a special n-partition (resp., covering) $\mathcal{P}$ of $[1,N]$ is of the form $\mathcal{P} = \{A_1,\dots,A_n\}$ such that for $1 \leq i \leq n$ we have

$$A_i = I_{m_i}(a_1^i, h_1^i) \cup I_{m_i}(a_2^i, h_2^i) \cup \cdots \cup I_{m_i}(a_{n_i}^i, h_{n_i}^i)$$

where $m_i = a_1^i < a_2^i < \cdots < a_{n_i}^i$ and $a_{j+1}^i > \max I_{m_i}(a_j^i, h_j^i)$ for $1 \leq j < n_i$ (according to Assertion 15). We suppose also that

$$\min A_1 = m_1 = 1 < \min A_2 = m_2 < \cdots < \min A_n = m_n.$$

As we have seen above we can write $I_m(a,h) = \langle a,h \rangle$ where the index m is the minimum of the part. So with a computer, we optimize the memory because the program code corresponding to our algorithm uses only two integers a,h to define the entry $\langle a,h \rangle$, even if $\mathrm{Card}(\langle a,h \rangle) = m + \delta_{m,a}$ for all $m \geq 2$.

7.2. Initialization

We introduce some definitions relative to partition into n weakly sum-free parts.

Definition 27. Let n be a positive integer. We define the number $K(n)$ to be the largest currently known integer such that there exists a partition $\mathcal{P}_n$ of $[1, K(n)]$ into n weakly sum-free parts.

Now, we turn to special partitions.

Definition 28. Let n be a positive integer. We define the number $K_{sp}(n)$ to be the largest currently known integer such that there exists a special n-partition (*or* special n-covering) $\mathcal{P}_n$ of $[1, K_{sp}(n)]$.

By these definitions, for all positive integers n, we have

$$K_{sp}(n) \leq K(n) \leq WS(n)$$
$$K_{sp}(n) \leq N(n) \leq WS(n).$$

Let $n \in \mathbb{N}$. Let $r(n) \geq 0$ be an integer. We build a special $(n+1)$-covering $\mathcal{P}_{n+1}$ of $[1, N]$, by completing a special n-covering $\mathcal{P}_n$ of $[1, N']$ such that

$$K_{sp}(n) - r(n) \leq N' \leq K_{sp}(n).$$

In other words, we initialize our search of special $(n+1)$-coverings of $[1, N]$ from all known special n-coverings $\mathcal{P}_n$ of $[1, N']$, with $N' \in [K_{sp}(n) - r(n), K_{sp}(n)]$.

The choice of $r(n)$ is important because a special $(n+1)$-covering of $[1, K_{sp}(n+1)]$ is not necessarily an extension of a special n-covering of $[1, K_{sp}(n)]$. For example, for $n = 5$, we have $K_{sp}(5) = 196$. Any special 6-covering of $[1, N]$, which extends a special 5-covering of $[1, 196]$, happens to satisfy $N \leq 575$. However, special 6-coverings of $[1, 582]$ may be obtained as extensions of special 5-coverings of $[1, 195]$. This shows that we need to take $r(5) \geq 1$ in our search algorithm.

If we choose $r(i) = K_{sp}(i)$, for $1 \leq i \leq n$, we will get all special $(n+1)$-coverings (resp., $(n+1)$-partitions). In this case, we say that we initialize the search with the empty covering (resp., partition). For $2 \leq n \leq 5$, to obtain all special n-coverings, we have chosen $r(n-1) = K_{sp}(n-1)$.

We remark that if $\mathcal{P}_{n+1} = \{A_1, \ldots, A_{n+1}\}$ with

$$\min A_1 < \min A_2 < \cdots < \min A_{n+1},$$

then for $1 \leq i \leq n$ we have

$$r(i) \geq K_{sp}(i) - \min A_{i+1} + 1. \tag{7.1}$$

A posteriori, for $6 \leq n \leq 9$, to have best special n-coverings, using inequality (7.1), it would have sufficed to take

$$r(5) = 1,\ r(6) = 2,\ r(7) = 5,\ r(8) = 2.$$

But we have explored up to

$$r(5) = 11,\ r(6) = 12,\ r(7) = 15,\ r(8) = 12,$$

without improving the results.

7.3. Search Algorithm of Special Partitions

In this algorithm, we search special n-coverings (without additional constraints).

`(Start)`: We suppose that we have a special n-covering $\mathcal{C}$ of $[1, N]$ into n parts $A_1, \ldots, A_n$. For $i = 1, \ldots, n$, we have

$$\begin{aligned} A_i &= I_{m_i}(a_1^i, h_1^i) \cup I_{m_i}(a_2^i, h_2^i) \cup \cdots \cup I_{m_i}(a_{n_i}^i, h_{n_i}^i) \\ &= \langle a_1^i, h_1^i \rangle \langle a_2^i, h_2^i \rangle \cdots \langle a_{n_i}^i, h_{n_i}^i \rangle \end{aligned}$$

with $a_1^i = m_i < a_2^i < \ldots < a_{n_i}^i$(because $a_{j+1}^i > \max I_{m_i}(a_j^i, h_j^i)$), and $h_j^i \in \{\delta_{1,j}, 2\}$ for all i, j. Suppose that N is the integer such that

$$[1, N] \subseteq \bigcup_{i=1}^{n} A_i,$$

and $N + 1 \notin \bigcup_{i=1}^{n} A_i$. In other words, N is the largest integer (depending on $\mathcal{C}$) such that $[1, N]$ is covered by $\mathcal{C}$.

For each $k \in [1, n]$ and $h(k) \in \{\delta_{A_k, \emptyset}, 2\}$, we consider the covering $\mathcal{C}^{(k,h(k))} = \{A_i^{(k,h(k))} \mid 1 \leq i \leq n\}$ such that, for $1 \leq i \leq n$,

$$A_i^{(k,h(k))} = \begin{cases} A_i & \text{if } i \neq k, \\ A_k \cup I_{m_k}(N+1, h(k)) & \text{if } i = k, \end{cases}$$

where $m_k = N + 1$ if $A_k = \emptyset$. We have

$$|\{\mathcal{C}^{(k,h(k))} \mid (k, h(k)) \in [1, n] \times \{\delta_{A_k, \emptyset}, 2\}\}| = 2n.$$

As $N + 1 \in A_k^{(k,h(k))}$, there exists $N^{(k,h(k))} \geq N + 1$, the largest integer (depending on $\mathcal{C}^{(k,h(k))}$) such that

$$[1, N^{(k,h(k))}] \subseteq \bigcup_{i=1}^{n} A_i^{(k,h(k))}.$$

By construction, $A_i^{(k,h(k))} = A_i$ is weakly sum-free for $i \neq k$. Therefore, the covering $\mathcal{C}^{(k,h(k))}$ is a special n-covering of $[1, N^{(k,h(k))}]$ if and only if $A_k^{(k,h(k))}$ is weakly sum-free. Using Assertion 22, we determine if $A_k^{(k,h(k))}$ is weakly sum-free. But we have supposed that A_k is already weakly sum-free (if it is not empty), i.e., for $1 \leq i \leq j \leq l \leq n_k$,

$$(I_{m_k}(a_i^k, h_i^k) \dot{+} I_{m_k}(a_j^k, h_j^k)) \cap I_{m_k}(a_l^k, h_l^k) = \emptyset.$$

So $A_k^{(k,h(k))} = A_k \cup I_{m_k}(N + 1, h(k))$ is weakly sum-free if and only if

$$(I_{m_k}(a_i^k, h_i^k) \dot{+} I_{m_k}(a_j^k, h_j^k)) \cap I_{m_k}(N + 1, h(k)) = \emptyset,$$

for $1 \leq i \leq j \leq n_k + 1$, where $a_{n_k+1}^k = N + 1$ and $h_{n_k+1}^k = h(k)$. Let

$$L_w = \{(k, h(k)) \in [1, n] \times \{\delta_{A_k, \emptyset}, 2\} \mid A_k^{(k,h_{(k)})} \text{ is weakly sum-free}\}.$$

CASE 1: $L_w \neq \emptyset$. We have $L_w = \{(k_1, h(k_1)), (k_2, h(k_2)), \ldots, (k_{|L_w|}, h(k_{|l_w|}))\}$. Then we obtain $|L_w|$ new special n-coverings

$$\mathcal{C}^{(k_1,h(k_1))}, \cdots, \mathcal{C}^{(k_{|L_w|},h(k_{|L_w|}))} \in \{\mathcal{C}^{(k,h(k))} \mid (k, h(k)) \in [1, n] \times \{\delta_{A_k,\emptyset}, 2\}\}$$

such that for $i = 1, \ldots, |L_w|$ we have:

(a) $\mathcal{C} \preceq \mathcal{C}^{(k_i,h(k_i)}$,

(b) $A_1^{(k_i,h(k_i))}, A_2^{(k_i,h(k_i))}, \ldots, A_n^{(k_i,h(k_i))}$ are weakly sum-free special sets.

Let us return to `(start)` with the special n-covering $\mathcal{C} = \mathcal{C}^{(k_i,h(k_i))}$ of $[1, N]$, where $N = N^{(k_i,h(k_i))}$, for $1 \leq i \leq |L_w|$.

CASE 2: $L_w = \emptyset$. Then the special n-covering $\mathcal{C}$ is extremum. And we `stop` the search over the special n-covering $\mathcal{C}$.

Remark 29. If we start this algorithm with the empty n-covering, we list all special n-coverings.

7.4. Search Algorithm of Special n-partitions Constrained by the Palindrome $s(a, n)$

We now present our method to build a special n-covering (partition) $\mathcal{P}$ which corresponding word $w(\mathcal{P})$ follows a given palindrome $s(a, n)$ with $a \in [1, n-1]$.

`(Start):` We suppose that we have a special n-covering $\mathcal{C}$ of $[1, N]$ into n parts $A_1, \ldots, A_n$ constrained by $s(a, n)$. In other words, we have

1. $\rho_{[1,a-1]}(w(\mathcal{C})) \leq s(a, n)$,
2. $A_i = I_{m_i}(a_1^i, h_1^i) \cup I_{m_i}(a_2^i, h_2^i) \cup \cdots \cup I_{m_i}(a_{n_i}^i, h_{n_i}^i)$ for $1 \leq i \leq n$, with $a_1^i = m_i < a_2^i < \ldots < a_{n_i}^i$, and $h_j^i \in \{\delta_{1,j}, 2\}$ for all $1 \leq j \leq n_i$.

Suppose that N is the integer such that

$$[1, N] \subseteq \bigcup_{i=1}^{n} A_i,$$

and $N + 1 \notin \bigcup_{i=1}^{n} A_i$. In other words, N is the largest integer such that $[1, N]$ is covered by $\mathcal{C}$.

Recall that the word $w(\mathcal{C})$ corresponding to $\mathcal{C}$ is defined by

$$w(\mathcal{C}) = \chi(b_1)\chi(b_2) \ldots \chi(b_m),$$

where $m = \sum_{i=1}^{n} n_i$ and $\{b_i \mid 1 \leq i \leq m\} = \{a_j^i \mid 1 \leq i \leq n, \ 1 \leq j \leq n_i\}$ such that

$$b_1 < b_2 < \cdots < b_m.$$

We have supposed that:

$$\rho_{[1,a-1]}(w(\mathcal{C})) \leq s(a,n) = c_{(1)}c_{(2)}\dots c_{(2^{n-a}-1)},$$

i.e., there exists $l \leq 2^{n-a}-1$ such that

$$\begin{aligned}\rho_{[1,a-1]}(w(\mathcal{C})) &= c_{(1)}c_{(2)}\dots c_{(l)},\\ l &= |\rho_{[1,a-1]}(w(\mathcal{C}))|.\end{aligned}$$

According to the length l of $\rho_{[1,a-1]}(w(\mathcal{C}))$, we define the integer

$$f(l) = \begin{cases} c_{(l+1)} & \text{if } l < 2^{n-a}-1,\\ a-1 & \text{if } l = 2^{n-a}-1.\end{cases}$$

For $k \in [1,a-1] \cup \{f(l)\}$ and $h(k) \in \{\delta_{A_k,\emptyset}, 2\}$, we consider the covering

$$\mathcal{C}^{(k,h(k))} = \{A_i^{(k,h(k))} \mid 1 \leq i \leq n\}$$

such that

$$A_i^{(k,h(k))} = \begin{cases} A_i & \text{if } i \neq k,\\ A_k \cup I_{m_k}(N+1, h(k)) & \text{if } i = k,\end{cases}$$

for $1 \leq i \leq n$, where $m_k = N+1$ if $A_k = \emptyset$. We have

$$\begin{aligned}|\{\mathcal{C}^{(k,h(k))} \mid (k,h(k)) \in ([1,a-1] \cup \{f(l)\}) \times \{\delta_{A_k,\emptyset},2\}\}| &= \begin{cases} 2a & \text{if } l < 2^{n-a}-1\\ 2(a-1) & \text{if } l = 2^{n-a}-1\end{cases}\\ &= 2(a-\delta_{l,2^{n-a}-1}).\end{aligned}$$

As $N+1 \in A_k^{(k,h(k))}$, there exists $N^{(k,h(k))} \geq N+1$, the largest integer (depending on $\mathcal{C}^{(k,h(k))}$) such that

$$[1, N^{(k,h(k))}] \subseteq \bigcup_{i=1}^{n} A_i^{(k,h(k))}.$$

By construction, $k \in [1,a-1] \cup \{f(l)\}$ with $f(l) = c_{(l+1)} \geq a$ or $f(l) = a-1$, i.e., $k = c_{(l+1)}$ or $k < a$. We have

$$\chi(N+1) = \chi(a^k_{n_k+1}) = k$$

and

$$w(\mathcal{C}^{(k,h(k))}) = w(\mathcal{C})\chi(N+1) = w(\mathcal{C})k.$$

As $\rho_{[1,a-1]}(w(\mathcal{C})) = c_{(1)}c_{(2)}\dots c_{(l)} \leq s(a,n)$, then

$$\begin{aligned}\rho_{[1,a-1]}(w(\mathcal{C}^{(k,h(k))})) &= \rho_{[1,a-1]}(w(\mathcal{C}))\lambda_{[1,a-1]}(k)\\ &= \begin{cases} c_{(1)}c_{(2)}\dots c_{(l)} & \text{if } k < a,\\ c_{(1)}c_{(2)}\dots c_{(l)}c_{(l+1)} & \text{if } k \geq a.\end{cases}\end{aligned}$$

Therefore,

$$\rho_{[1,a-1]}(w(\mathcal{C}^{(k,h(k))})) \le s(a,n).$$

By hypothesis, $A_i^{(k,h(k))} = A_i$ is weakly sum-free for $i \neq k$. Then the covering $\mathcal{C}^{(k,h(k))}$ is a special n-covering of $[1, N^{(k,h(k))}]$ if and only if $A_k^{(k,h(k))}$ is weakly sum-free. Like in the previous method, the set $A_k^{(k,h(k))} = A_k \cup I_{m_k}(N+1, h(k))$ is weakly sum-free if and only if

$$(I_{m_k}(a_i^k, h_i^k) \dot{+} I_{m_k}(a_j^k, h_j^k)) \cap I_{m_k}(N+1, h(k)) = \emptyset$$

for $1 \le i \le j \le n_k + 1$, where $a_{n_k+1}^k = N+1$ and $h_{n_k+1}^k = h(k)$. Let

$$L_w = \{(k, h(k)) \in ([1, a-1] \cup \{f(l)\}) \times \{\delta_{A_k,\emptyset}, 2\} \mid A_k^{(k,h_{(k)})} \textit{ weakly sum-free}\}.$$

Here, we reduce the field of partition search because $|L_w| \le 2(a - \delta_{l,2^{n-a}-1}) \le 2a$.

CASE 1: If $L_w \neq \emptyset$. We have $L_w = \{(k_1, h(k_1)), (k_2, h(k_2)), \ldots, (k_{|L_w|}, h(k_{|l_w|}))\}$. Then we obtain $|L_w|$ new special n-coverings

$$\mathcal{C}^{(k_1,h(k_1))}, \ldots, \mathcal{C}^{(k_{|L_w|}, h(k_{|L_w|}))} \in \Big\{\mathcal{C}^{(k,h(k))} \,|\, (k, h(k)) \in ([1, a-1] \cup \{f(l)\}) \times \{\delta_{A_k,\emptyset}, 2\}\Big\}$$

such that for $i = 1, \ldots, |L_w|$ we have:

(a) $\mathcal{C} \preceq \mathcal{C}^{(k_i,h(k_i))}$,

(b) $A_1^{(k_i,h(k_i))}, A_2^{(k_i,h(k_i))}, \ldots, A_n^{(k_i,h(k_i))}$ are weakly sum-free special sets,

(c) $\rho_{[1,a-1]}(w(\mathcal{C}^{(k_i,h(k_i))})) \le s(a,n)$.

Let's return to `(start)` with the special n-covering $\mathcal{C} = \mathcal{C}^{(k_i,h(k_i))}$ of $[1, N]$, where $N = N^{(k_i,h(k_i))}$, for $1 \le i \le |L_w|$.

CASE 2: If $L_w = \emptyset$. Then the covering $\mathcal{C}$ is an extremum special n-covering such that $\rho_{[1,a-1]}(w(\mathcal{C}^{(k,h(k))})) \le s(a,n)$. And we `stop` the search over the special n-partition $\mathcal{C}$ constrained by $s(a,n)$.

8. Best Known Partitions

In this section, we shall give the partitions which we have found by applying the above search methods. We remark that

$$K(n) = K_{sp}(n) \le N(n) \text{ for } 1 \le n \le 9.$$

We present the results with Notation 13. We used the first Search method 7.3 for $n = 5$ and 6. The second Search method 7.4 was applied for $n = 7, 8$ and 9 with the constraint $s(2, n)$.

8.1. 5- and 6-colorings

The bounds $WS(5) \geq 196$ and $WS(6) \geq 582$ were respectively obtained in [3] and [4]. While not improving these two bounds, we have found partitions of $[1, 196]$ and $[1, 582]$ respectively into 5 and 6 parts, which are weakly sum-free special sets, using the first method 7.3.

For $n = 5$, we initialized the search with the empty partition. We recorded 913337131 extremum special 5-partitions. An exhaustive search revealed that there exists no covering of $[1, 197]$ into 5 parts which are weakly sum-free special sets, and we have three special 5-coverings of $[1, 196]$ whence

$$N(5) = 196.$$

The C++ program corresponding to the algorithm has taken 26 hours to get all results, using one core of a laptop processor i7-2.6 Ghz. The size of the file containing all extremum special 5-coverings is more than 190 Go.

Proposition 30. *We have $WS(5) \geq N(5) = 196$.*

Proof. Here, we show that $WS(5) \geq 196$ by giving one special 5-partition of $[196]$.

$$\begin{aligned}
A_1 &= \{1, 2, 4, 8, 11, 22, 25, 53, 63, 69, 135, 140, 150, 155, 178, 183, 196\} \\
A_2 &= \langle 3, 1\rangle\langle 19, 2\rangle\langle 50, 0\rangle\langle 64, 0\rangle\langle 137, 0\rangle\langle 151, 0\rangle\langle 180, 0\rangle\langle 193, 0\rangle \\
A_3 &= \langle 9, 2\rangle\langle 54, 0\rangle\langle 141, 0\rangle\langle 184, 0\rangle \\
A_4 &= \langle 24, 1\rangle\langle 154, 2\rangle \\
A_5 &= \langle 67, 2\rangle.
\end{aligned}$$

□

As we have mentioned above, we have found three special 5-partitions of $[1, 196]$. One of these had already been given in [3]. It remains an open problem to show that $WS(5) = 196$. We now turn to 6-colorings.

Proposition 31. *We have $WS(6) \geq 582$.*

Proof. It suffices to give a special 6-partition of $[1, 582]$.

$$\begin{aligned}
A_1 &= \left[\begin{array}{l} \{1, 2, 4, 8, 11, 22, 25, 53, 63, 68, 136, 149, 154, 177, 182, 192, 198, 393, 407, 412, 435, 440, 450, \\ 455, 521, 526, 536, 541, 564, 569, 582\} \end{array}\right. \\
A_2 &= \left[\begin{array}{l} \langle 3, 1\rangle\langle 19, 2\rangle\langle 50, 0\rangle\langle 64, 0\rangle\langle 137, 0\rangle\langle 150, 0\rangle\langle 179, 0\rangle\langle 193, 0\rangle\langle 395, 0\rangle\langle 408, 0\rangle\langle 437, 0\rangle\langle 451, 0\rangle \\ \langle 523, 0\rangle\langle 537, 0\rangle\langle 566, 0\rangle\langle 579, 0\rangle \end{array}\right. \\
A_3 &= \langle 9, 2\rangle\langle 54, 0\rangle\langle 140, 0\rangle\langle 183, 0\rangle\langle 398, 0\rangle\langle 441, 0\rangle\langle 527, 0\rangle\langle 570, 0\rangle \\
A_4 &= \langle 24, 1\rangle\langle 153, 2\rangle\langle 411, 2\rangle\langle 540, 2\rangle \\
A_5 &= \langle 67, 1\rangle\langle 454, 2\rangle \\
A_6 &= \langle 196, 2\rangle.
\end{aligned}$$

□

We have found thirteen special 6-partitions of $[1, 582]$. Observe that the 6-partition establishing $WS(6) \geq 582$ given in [4] is not made up of special sets. For instance, its sixth part is $B_6 = [196, 392] \setminus \{197, 252, 292, 304, 342, 368, 370\}$ which is *not* a special set, whereas our sixth part is the special set $A_6 = \langle 196, 2\rangle = [196, 394] \setminus \{198, 393\}$.

Remark 32. For $n \leq 6$, the word corresponding to a special n-partition $\mathcal{P}_n$ of $[1, K(n)]$ verifies

$$\rho_{\{1\}}(w(\mathcal{P}_n)) \leq s(2, n).$$

In the following results, we have used the search algorithm 7.4 using as constraint the palindrome $s(2, n)$ to restrict the field of search.

8.2. 7-colorings

The best lower bound we currently have for $WS(7)$ is obtained from $WS(7) \geq S(7) \geq 1680$, that was established in [5]. Here, we improve this bound as follows.

Proposition 33. *We have $WS(7) \geq 1740$.*

Proof. It suffices to give a special 7-partition of $[1, 1740]$:

$$A_1 = \begin{cases} \{1, 2, 4, 8, 11, 22, 25, 50, 63, 68, 136, 149, 154, 177, 182, 192, 197, 397, 407, 412, 435, 440, 450, \\ 455, 521, 526, 536, 541, 564, 569, 582, 585, 1170, 1180, 1185, 1208, 1213, 1223, 1228, 1294, 1299, \\ 1309, 1314, 1337, 1351, 1356, 1551, 1565, 1570, 1593, 1598, 1608, 1613, 1679, 1684, 1694, 1699, \\ 1722, 1727, 1737\} \end{cases}$$

$$A_2 = \begin{cases} \langle 3,1\rangle\langle 19,2\rangle\langle 51,0\rangle\langle 64,0\rangle\langle 137,0\rangle\langle 150,0\rangle\langle 179,0\rangle\langle 193,0\rangle\langle 394,0\rangle\langle 408,0\rangle\langle 437,0\rangle\langle 451,0\rangle\langle 523,0\rangle \\ \langle 537,0\rangle\langle 566,0\rangle\langle 579,0\rangle\langle 1167,0\rangle\langle 1181,0\rangle\langle 1210,0\rangle\langle 1224,0\rangle\langle 1296,0\rangle\langle 1310,0\rangle\langle 1339,0\rangle\langle 1352,0\rangle \\ \langle 1553,0\rangle\langle 1566,0\rangle\langle 1595,0\rangle\langle 1609,0\rangle\langle 1681,0\rangle\langle 1695,0\rangle\langle 1724,0\rangle\langle 1738,0\rangle \end{cases}$$

$$A_3 = \begin{cases} \langle 9,2\rangle\langle 54,0\rangle\langle 140,0\rangle\langle 183,0\rangle\langle 398,0\rangle\langle 441,0\rangle\langle 527,0\rangle\langle 570,0\rangle\langle 1171,0\rangle\langle 1214,0\rangle\langle 1300,0\rangle\langle 1342,0\rangle \\ \langle 1556,0\rangle\langle 1599,0\rangle\langle 1685,0\rangle\langle 1728,0\rangle \end{cases}$$

$$A_4 = \langle 24,1\rangle\langle 153,2\rangle\langle 411,2\rangle\langle 540,2\rangle\langle 1184,2\rangle\langle 1313,2\rangle\langle 1569,2\rangle\langle 1698,2\rangle$$

$$A_5 = \langle 67,1\rangle\langle 454,2\rangle\langle 1227,2\rangle\langle 1612,2\rangle$$

$$A_6 = \langle 196,1\rangle\langle 1355,2\rangle$$

$$A_7 = \langle 583,2\rangle.$$

□

Note that we have found four different special 7-partitions of $[1, 1740]$. We also note that we get these special 7-partitions of $[1740]$ from special 6-partitions of $[1, 582]$, previously obtained by the same algorithm.

8.3. 8-colorings

Proposition 34. *We have $WS(8) \geq 5201$.*

Proof. Here is one special 8-partition of $[1, 5201]$.

$$A_1 = \left[\begin{array}{l}\{1, 2, 4, 8, 11, 22, 25, 50, 63, 68, 139, 149, 154, 177, 182, 192, 197, 397, 407, 412, 435, 440, \\ 453, 524, 534, 539, 562, 567, 577, 582, 1167, 1177, 1182, 1205, 1210, 1223, 1294, 1304, 1309, \\ 1332, 1337, 1347, 1352, 1547, 1552, 1562, 1567, 1590, 1595, 1605, 1679, 1689, 1694, 1717, 1722, \\ 1732, 1738, 3473, 3478, 3488, 3493, 3516, 3521, 3534, 3605, 3615, 3620, 3643, 3648, 3658, 3663, \\ 3858, 3863, 3873, 3878, 3901, 3906, 3916, 3990, 4000, 4005, 4028, 4033, 4043, 4048, 4628, 4633, \\ 4643, 4648, 4671, 4676, 4686, 4760, 4770, 4775, 4798, 4803, 4813, 4818, 5013, 5018, 5028, 5033, \\ 5056, 5061, 5071, 5145, 5155, 5160, 5183, 5188, 5198\}\end{array}\right.$$

$$A_2 = \left[\begin{array}{l}\langle 3, 1\rangle\langle 19, 2\rangle\langle 51, 0\rangle\langle 64, 0\rangle\langle 136, 0\rangle\langle 150, 0\rangle\langle 179, 0\rangle\langle 193, 0\rangle\langle 394, 0\rangle\langle 408, 0\rangle\langle 437, 0\rangle\langle 450, 0\rangle \\ \langle 521, 0\rangle\langle 535, 0\rangle\langle 564, 0\rangle\langle 578, 0\rangle\langle 1164, 0\rangle\langle 1178, 0\rangle\langle 1207, 0\rangle\langle 1220, 0\rangle\langle 1291, 0\rangle\langle 1305, 0\rangle\langle 1334, 0\rangle \\ \langle 1348, 0\rangle\langle 1549, 0\rangle\langle 1563, 0\rangle\langle 1592, 0\rangle\langle 1606, 0\rangle\langle 1676, 0\rangle\langle 1690, 0\rangle\langle 1719, 0\rangle\langle 1733, 0\rangle\langle 3475, 0\rangle \\ \langle 3489, 0\rangle\langle 3518, 0\rangle\langle 3531, 0\rangle\langle 3602, 0\rangle\langle 3616, 0\rangle\langle 3645, 0\rangle\langle 3659, 0\rangle\langle 3860, 0\rangle\langle 3874, 0\rangle\langle 3903, 0\rangle \\ \langle 3917, 0\rangle\langle 3987, 0\rangle\langle 4001, 0\rangle\langle 4030, 0\rangle\langle 4044, 0\rangle\langle 4630, 0\rangle\langle 4644, 0\rangle\langle 4673, 0\rangle\langle 4687, 0\rangle\langle 4757, 0\rangle \\ \langle 4771, 0\rangle\langle 4800, 0\rangle\langle 4814, 0\rangle\langle 5015, 0\rangle\langle 5029, 0\rangle\langle 5058, 0\rangle\langle 5072, 0\rangle\langle 5142, 0\rangle\langle 5156, 0\rangle\langle 5185, 0\rangle \\ \langle 5199, 0\rangle\end{array}\right.$$

$$A_3 = \left[\begin{array}{l}\langle 9, 2\rangle\langle 54, 0\rangle\langle 140, 0\rangle\langle 183, 0\rangle\langle 398, 0\rangle\langle 441, 0\rangle\langle 525, 0\rangle\langle 568, 0\rangle\langle 1168, 0\rangle\langle 1211, 0\rangle\langle 1295, 0\rangle \\ \langle 1338, 0\rangle\langle 1553, 0\rangle\langle 1596, 0\rangle\langle 1680, 0\rangle\langle 1723, 0\rangle\langle 3479, 0\rangle\langle 3522, 0\rangle\langle 3606, 0\rangle\langle 3649, 0\rangle\langle 3864, 0\rangle \\ \langle 3907, 0\rangle\langle 3991, 0\rangle\langle 4034, 0\rangle\langle 4634, 0\rangle\langle 4677, 0\rangle\langle 4761, 0\rangle\langle 4804, 0\rangle\langle 5019, 0\rangle\langle 5062, 0\rangle\langle 5146, 0\rangle \\ \langle 5189, 0\rangle\end{array}\right.$$

$$A_4 = \left[\begin{array}{l}\langle 24, 1\rangle\langle 153, 2\rangle\langle 411, 2\rangle\langle 538, 2\rangle\langle 1181, 2\rangle\langle 1308, 2\rangle\langle 1566, 2\rangle\langle 1693, 2\rangle\langle 3492, 2\rangle\langle 3619, 2\rangle\langle 3877, 2\rangle \\ \langle 4004, 2\rangle\langle 4647, 2\rangle\langle 4774, 2\rangle\langle 5032, 2\rangle\langle 5159, 2\rangle\end{array}\right.$$

$$A_5 = \langle 67, 1\rangle\langle 454, 0\rangle\langle 1224, 0\rangle\langle 1609, 0\rangle\langle 3535, 0\rangle\langle 3920, 0\rangle\langle 4690, 0\rangle\langle 5075, 0\rangle$$

$$A_6 = \langle 196, 1\rangle\langle 1351, 2\rangle\langle 3662, 2\rangle\langle 4817, 2\rangle$$

$$A_7 = \langle 581, 1\rangle\langle 4047, 2\rangle$$

$$A_8 = \langle 1736, 2\rangle.$$

□

Our C++ program has found six special 8-partitions of $[1, 5201]$. Each partition contains a special 7-partition of $[1, 1735]$ but not of $[1, 1740]$.

8.4. 9-colorings

Proposition 35. *We have $WS(9) \geq 15596$.*

Proof. We have found one special 9-partition of $[1, 15596]$:

$A_1 =$

$\{1, 2, 4, 8, 11, 22, 25, 50, 63, 68, 139, 149, 154, 177, 182, 192, 197, 397, 407, 412, 435, 440, 453,$
$524, 534, 539, 562, 567, 577, 582, 1167, 1177, 1182, 1205, 1210, 1223, 1294, 1304, 1309, 1332,$
$1337, 1347, 1352, 1547, 1552, 1562, 1567, 1590, 1595, 1605, 1679, 1689, 1694, 1717, 1722, 1732,$
$1737, 3477, 3487, 3492, 3515, 3520, 3533, 3604, 3614, 3619, 3642, 3647, 3657, 3662, 3857, 3862,$
$3872, 3877, 3900, 3905, 3915, 3989, 3999, 4004, 4027, 4032, 4042, 4047, 4627, 4632, 4642, 4647,$
$4670, 4675, 4685, 4759, 4769, 4774, 4797, 4802, 4812, 4817, 5012, 5017, 5027, 5032, 5055, 5060,$
$5070, 5144, 5154, 5159, 5182, 5187, 5197, 5203, 10403, 10408, 10418, 10423, 10446, 10451,$
$10464, 10535, 10545, 10550, 10573, 10578, 10588, 10593, 10788, 10793, 10803, 10808, 10831,$
$10836, 10846, 10920, 10930, 10935, 10958, 10963, 10973, 10978, 11558, 11563, 11573, 11578,$
$11601, 11606, 11616, 11690, 11700, 11705, 11728, 11733, 11743, 11748, 11943, 11948, 11958,$
$11963, 11986, 11991, 12001, 12075, 12085, 12090, 12113, 12118, 12128, 12133, 13868, 13873,$
$13883, 13888, 13911, 13916, 13926, 14000, 14010, 14015, 14038, 14043, 14053, 14058, 14253,$
$14258, 14268, 14273, 14296, 14301, 14311, 14385, 14395, 14400, 14423, 14428, 14438, 14443,$
$15023, 15028, 15038, 15043, 15066, 15071, 15081, 15155, 15165, 15170, 15193, 15198, 15208,$
$15213, 15408, 15413, 15423, 15428, 15451, 15456, 15466, 15537, 15550, 15555, 15578, 15583,$
$15593\}$

$A_2 =$

$\langle 3,1\rangle\langle 19,2\rangle\langle 51,0\rangle\langle 64,0\rangle\langle 136,0\rangle\langle 150,0\rangle\langle 179,0\rangle\langle 193,0\rangle\langle 394,0\rangle\langle 408,0\rangle\langle 437,0\rangle\langle 450,0\rangle$
$\langle 521,0\rangle\langle 535,0\rangle\langle 564,0\rangle\langle 578,0\rangle\langle 1164,0\rangle\langle 1178,0\rangle\langle 1207,0\rangle\langle 1220,0\rangle\langle 1291,0\rangle\langle 1305,0\rangle\langle 1334,0\rangle$
$\langle 1348,0\rangle\langle 1549,0\rangle\langle 1563,0\rangle\langle 1592,0\rangle\langle 1606,0\rangle\langle 1676,0\rangle\langle 1690,0\rangle\langle 1719,0\rangle\langle 1733,0\rangle\langle 3474,0\rangle$
$\langle 3488,0\rangle\langle 3517,0\rangle\langle 3530,0\rangle\langle 3601,0\rangle\langle 3615,0\rangle\langle 3644,0\rangle\langle 3658,0\rangle\langle 3859,0\rangle\langle 3873,0\rangle\langle 3902,0\rangle$
$\langle 3916,0\rangle\langle 3986,0\rangle\langle 4000,0\rangle\langle 4029,0\rangle\langle 4043,0\rangle\langle 4629,0\rangle\langle 4643,0\rangle\langle 4672,0\rangle\langle 4686,0\rangle\langle 4756,0\rangle$
$\langle 4770,0\rangle\langle 4799,0\rangle\langle 4813,0\rangle\langle 5014,0\rangle\langle 5028,0\rangle\langle 5057,0\rangle\langle 5071,0\rangle\langle 5141,0\rangle\langle 5155,0\rangle\langle 5184,0\rangle$
$\langle 5198,0\rangle\langle 10405,0\rangle\langle 10419,0\rangle\langle 10448,0\rangle\langle 10461,0\rangle\langle 10532,0\rangle\langle 10546,0\rangle\langle 10575,0\rangle\langle 10589,0\rangle$
$\langle 10790,0\rangle\langle 10804,0\rangle\langle 10833,0\rangle\langle 10847,0\rangle\langle 10917,0\rangle\langle 10931,0\rangle\langle 10960,0\rangle\langle 10974,0\rangle\langle 11560,0\rangle$
$\langle 11574,0\rangle\langle 11603,0\rangle\langle 11617,0\rangle\langle 11687,0\rangle\langle 11701,0\rangle\langle 11730,0\rangle\langle 11744,0\rangle\langle 11945,0\rangle\langle 11959,0\rangle$
$\langle 11988,0\rangle\langle 12002,0\rangle\langle 12072,0\rangle\langle 12086,0\rangle\langle 12115,0\rangle\langle 12129,0\rangle\langle 13870,0\rangle\langle 13884,0\rangle\langle 13913,0\rangle$
$\langle 13927,0\rangle\langle 13997,0\rangle\langle 14011,0\rangle\langle 14040,0\rangle\langle 14054,0\rangle\langle 14255,0\rangle\langle 14269,0\rangle\langle 14298,0\rangle\langle 14312,0\rangle$
$\langle 14382,0\rangle\langle 14396,0\rangle\langle 14425,0\rangle\langle 14439,0\rangle\langle 15025,0\rangle\langle 15039,0\rangle\langle 15068,0\rangle\langle 15082,0\rangle\langle 15152,0\rangle$
$\langle 15166,0\rangle\langle 15195,0\rangle\langle 15209,0\rangle\langle 15410,0\rangle\langle 15424,0\rangle\langle 15453,0\rangle\langle 15467,0\rangle\langle 15538,0\rangle\langle 15551,0\rangle$
$\langle 15580,0\rangle\langle 15594,0\rangle$

$A_3 =$

$\langle 9,2\rangle\langle 54,0\rangle\langle 140,0\rangle\langle 183,0\rangle\langle 398,0\rangle\langle 441,0\rangle\langle 525,0\rangle\langle 568,0\rangle\langle 1168,0\rangle\langle 1211,0\rangle\langle 1295,0\rangle\langle 1338,0\rangle$
$\langle 1553,0\rangle\langle 1596,0\rangle\langle 1680,0\rangle\langle 1723,0\rangle\langle 3478,0\rangle\langle 3521,0\rangle\langle 3605,0\rangle\langle 3648,0\rangle\langle 3863,0\rangle\langle 3906,0\rangle$
$\langle 3990,0\rangle\langle 4033,0\rangle\langle 4633,0\rangle\langle 4676,0\rangle\langle 4760,0\rangle\langle 4803,0\rangle\langle 5018,0\rangle\langle 5061,0\rangle\langle 5145,0\rangle\langle 5188,0\rangle$
$\langle 10409,0\rangle\langle 10452,0\rangle\langle 10536,0\rangle\langle 10579,0\rangle\langle 10794,0\rangle\langle 10837,0\rangle\langle 10921,0\rangle\langle 10964,0\rangle\langle 11564,0\rangle$
$\langle 11607,0\rangle\langle 11691,0\rangle\langle 11734,0\rangle\langle 11949,0\rangle\langle 11992,0\rangle\langle 12076,0\rangle\langle 12119,0\rangle\langle 13874,0\rangle\langle 13917,0\rangle$
$\langle 14001,0\rangle\langle 14044,0\rangle\langle 14259,0\rangle\langle 14302,0\rangle\langle 14386,0\rangle\langle 14429,0\rangle\langle 15029,0\rangle\langle 15072,0\rangle\langle 15156,0\rangle$
$\langle 15199,0\rangle\langle 15414,0\rangle\langle 15457,0\rangle\langle 15541,0\rangle\langle 15584,0\rangle$

$$A_4 = \left[\begin{array}{l} \langle 24,1\rangle\langle 153,2\rangle\langle 411,2\rangle\langle 538,2\rangle\langle 1181,2\rangle\langle 1308,2\rangle\langle 1566,2\rangle\langle 1693,2\rangle\langle 3491,2\rangle\langle 3618,2\rangle\langle 3876,2\rangle \\ \langle 4003,2\rangle\langle 4646,2\rangle\langle 4773,2\rangle\langle 5031,2\rangle\langle 5158,2\rangle\langle 10422,2\rangle\langle 10549,2\rangle\langle 10807,2\rangle\langle 10934,2\rangle \\ \langle 11577,2\rangle\langle 11704,2\rangle\langle 11962,2\rangle\langle 12089,2\rangle\langle 13887,2\rangle\langle 14014,2\rangle\langle 14272,2\rangle\langle 14399,2\rangle\langle 15042,2\rangle \\ \langle 15169,2\rangle\langle 15427,2\rangle \\ \langle 15554,2\rangle \end{array}\right.$$

$$A_5 = \left[\begin{array}{l} \langle 67,1\rangle\langle 454,0\rangle\langle 1224,0\rangle\langle 1609,0\rangle\langle 3534,0\rangle\langle 3919,0\rangle\langle 4689,0\rangle\langle 5074,0\rangle\langle 10465,0\rangle\langle 10850,0\rangle \\ \langle 11620,0\rangle\langle 12005,0\rangle\langle 13930,0\rangle\langle 14315,0\rangle\langle 15085,0\rangle\langle 15470,0\rangle \end{array}\right.$$

$$A_6 = \langle 196,1\rangle\langle 1351,2\rangle\langle 3661,2\rangle\langle 4816,2\rangle\langle 10592,2\rangle\langle 11747,2\rangle\langle 14057,2\rangle\langle 15212,2\rangle$$

$$A_7 = \langle 581,1\rangle\langle 4046,2\rangle\langle 10977,2\rangle\langle 14442,2\rangle$$

$$A_8 = \langle 1736,1\rangle\langle 12132,2\rangle$$

$$A_9 = \langle 5201,2\rangle.$$

□

We remark that this partition is an extension of a special 8-partition of $[1, 5200]$ (recall that we have an 8-partition of $[1, 5201]$).

8.5. The Case $n = 10$

For $n = 10$, inequality (2.2) of Abbott and Hanson

$$S(k+m) \geq 2 \cdot S(k) \cdot S(m) + S(k) + S(m),$$

with $k = m = 5$ and $S(5) \geq 160$ yields:

$$S(10) \geq 2 \cdot S(5) \cdot S(5) + S(5) + S(5) \geq 2 \times 160 \times 160 + 160 + 160 = 51520.$$

This bound $WS(10) \geq S(10) \geq 51520$ is currently the best one available for both $WS(10)$ and $S(10)$.

Acknowledgment. I thank my thesis advisor Shalom Eliahou, Jean Fromentin and Denis Robilliard for their help, and specifically Jean Fromentin for introducing me to C++ and to basic search algorithms. I also thank the anonymous referee for his very detailed comments and suggestions.

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[9] F. Rafilipojaona. Links to two folders $[F_1], [F_2]$ containing best special n-partitions for $n = 4, 5, 6, 7, 8$ and 9:
$[F_1]$ `https://drive.google.com/folderview?id=0B3t8vdlbd7b1fjBJWFUyUDI2NWVLX1dXbEpVTkJRZUJnNDJ1SnphT1FBY05QQUozZlI3RlU&usp=sharing`
$[F_2]$ `https://drive.google.com/folderview?id=0B3t8vdlbd7b1fmZBWVVIbkw0UjNfQWVRU1YxSUNCUjJSUmRtbWVfNDQ1cC0xaG1UWkhsWTg&usp=sharing`

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