PROOF OF THE PRIME LADDER CONJECTURE

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Abstract

A *prime labeling* of a graph $G = (V, E)$ is a labeling of the vertices with the numbers $1, 2, \ldots, |V|$ where each vertex gets a unique label and, for each edge uv, the labels for u an v are relatively prime. A graph is called *prime* if it has a prime labeling. The Ladder Conjecture states that every ladder $P_n \times P_2$ is prime. We prove this conjecture. The following result which is essential to the proof of this conjecture may be of independent interest as it relates to the celebrated Goldbach Conjecture. Define a *canonical partition* of an integer n as a representation of n as the sum of odd primes $p_1, p_2, \ldots, p_m$ where $p_j \geq 2\sum_{i=1}^{j-1} p_i + 3$ for all $j \in \{2, 3, \ldots, m\}$. Every integer $n \geq 50$ has a canonical partition.

1. Introduction

A *prime labeling* of a graph $G = (V, E)$ is a labeling of the vertices with the numbers $1, 2, \ldots, |V|$ where each vertex gets a unique label and, for each edge uv, the labels for u an v are relatively prime. A graph is called *prime* if it has a prime labeling. For example, every path is prime as demonstrated by using consecutive labels along the path. However, complete graphs of order at least 3 are not prime. There are numerous results on the existence of prime labellings and extensions of this idea (for examples, see [1], [9], [2], [4], [6], [8]). One of several outstanding problems in this area is the conjecture by Entringer made around 1980 [4] which asserts that *all trees are prime*. This paper focusses on a different problem.

Let ladder L_n be the graph $P_n \times P_2$ with vertex set $\{u_i, v_i\colon\ 1 \leq i \leq n\}$ and edge set $\{u_i v_i\colon\ 1 \leq i \leq n\} \cup \{u_i u_{i+1}, v_i v_{i+1}\colon\ 1 \leq i \leq n-1\}$. In a 2002 paper by Vilfered, Somasundaram and Nichola [9] it was shown that *n is prime implies L_n is prime*, and they conjectured that all ladders are prime. Sundaram, Ponraj and Somasundaram [8] proved that if n is the sum of two primes then L_n is prime.

We mention these results because the proof presented here is an extension of their proofs. We use a special partition of integers into primes to construct a suitable labeling. Our proof uses the following result which is related to the celebrated Goldbach Conjecture. Define a *canonical partition* of an integer n as a representation of n as the sum of odd primes $p_1, p_2, \ldots, p_m$ where $p_j \geq 2\sum_{i=1}^{j-1} p_i + 3$ for all

$j \in \{2, 3, \ldots, m\}$. We prove that every integer $n \geq 50$ has a canonical partition. This ultimately leads to a proposed strengthening of the Goldbach Conjecture.

2. Proof of the Conjecture

We will use the following theorem by Loo [7].

Theorem A. *For every positive integer n, there exists a prime p such that $n < p < 4(n+2)/3$.*

Now notice that 49 does not have a canonical partition. As Erdős would say, "there's just not enough small numbers." However, we still have the following lemma.

Lemma 1. *Every integer $n \geq 50$ has a canonical partition.*

Proof. The proof is by induction on n. The list of partitions displayed in Figures 1, 2 and 3 verifies the cases $50 \leq n \leq 470$.

Let $n > 470$ and assume the statement holds for all integers $< n$. By Theorem A there is a prime p such that

$$\left\lceil \frac{2n}{3} \right\rceil + 1 \leq p \leq \frac{4}{3}\left(\left\lceil \frac{2n}{3} \right\rceil + 2\right) - \frac{1}{3} < n.$$

So

$$n - p \geq n - \frac{4}{3}\left(\left\lceil \frac{2n}{3} \right\rceil + 2\right) + \frac{1}{3} \geq \frac{n-29}{9} \geq 49\frac{1}{3}.$$

Hence, $n - p$ has a canonical partition. Since $p \geq 2n/3 + 1$, $p \geq 2(n-p) + 3$. It follows that n has a canonical partition. $\square$

It is convenient to use the following observation.

Lemma 2. *Suppose σ and j are positive integers and p is a prime satisfying $p \geq 2\sigma$. If $j < p - \sigma$ or $p - \sigma < j < p + \sigma$, then $(\sigma + j, p + \sigma + j) = 1$.*

Proof. Let $d = (\sigma + j, p + \sigma + j)$. Then $d | p$. Since p is prime, $d = 1$ or $d = p$. Assume $d = p$. If $j < p - \sigma$, then $p > \sigma + j$ and so $p \nmid \sigma + j$. If $p - \sigma < j < p + \sigma$, then $p < \sigma + j < p + 2\sigma \leq 2p$. Since this interval contains no multiple of p, $p \nmid \sigma + j$. $\square$

The examples given in Figures 5 and 6 should help to clarify the following lemma.

Theorem 1. *Every ladder L_n is prime.*

$50 = 37 + 13$	$51 = 37 + 11 + 3$	$52 = 41 + 11$	$53 = 37 + 13 + 3$
$54 = 41 + 13$	$55 = 41 + 11 + 3$	$56 = 53 + 3$	$57 = 41 + 13 + 3$
$58 = 47 + 11$	$59 = 41 + 13 + 5$	$60 = 47 + 13$	$61 = 47 + 11 + 3$
$62 = 59 + 3$	$63 = 47 + 13 + 3$	$64 = 47 + 17$	$65 = 47 + 13 + 5$
$66 = 53 + 13$	$67 = 53 + 11 + 3$	$68 = 61 + 7$	$69 = 53 + 13 + 3$
$70 = 59 + 11$	$71 = 53 + 13 + 5$	$72 = 59 + 13$	$73 = 59 + 11 + 3$
$74 = 61 + 13$	$75 = 59 + 13 + 3$	$76 = 59 + 17$	$77 = 59 + 13 + 5$
$78 = 59 + 19$	$79 = 61 + 13 + 5$	$80 = 73 + 7$	$81 = 59 + 19 + 3$
$82 = 59 + 23$	$83 = 67 + 13 + 3$	$84 = 67 + 17$	$85 = 71 + 11 + 3$
$86 = 83 + 3$	$87 = 67 + 17 + 3$	$88 = 83 + 5$	$89 = 73 + 13 + 3$
$90 = 83 + 7$	$91 = 73 + 13 + 5$	$92 = 89 + 3$	$93 = 73 + 17 + 3$
$94 = 89 + 5$	$95 = 79 + 11 + 5$	$96 = 83 + 13$	$97 = 73 + 19 + 5$
$98 = 91 + 7$	$99 = 83 + 13 + 3$	$100 = 97 + 3$	$101 = 73 + 23 + 5$
$102 = 97 + 5$	$103 = 73 + 23 + 7$	$104 = 97 + 7$	$105 = 91 + 11 + 3$
$106 = 103 + 3$	$107 = 91 + 13 + 3$	$108 = 103 + 5$	$109 = 91 + 13 + 5$
$110 = 103 + 7$	$111 = 91 + 17 + 3$	$112 = 109 + 3$	$113 = 91 + 17 + 5$
$114 = 109 + 5$	$115 = 91 + 19 + 5$	$116 = 109 + 7$	$117 = 91 + 19 + 7$
$118 = 113 + 5$	$119 = 91 + 23 + 5$	$120 = 113 + 7$	$121 = 97 + 19 + 5$
$122 = 109 + 13$	$123 = 91 + 29 + 3$	$124 = 113 + 11$	$125 = 91 + 29 + 5$
$126 = 113 + 13$	$127 = 91 + 29 + 7$	$128 = 109 + 19$	$129 = 91 + 31 + 7$
$130 = 113 + 17$	$131 = 97 + 31 + 3$	$132 = 113 + 19$	$133 = 91 + 31 + 11$
$134 = 103 + 31$	$135 = 97 + 31 + 7$	$136 = 113 + 23$	$137 = 97 + 29 + 11$
$138 = 109 + 29$	$139 = 97 + 31 + 11$	$140 = 109 + 31$	$141 = 97 + 31 + 13$
$142 = 113 + 29$	$143 = 127 + 13 + 3$	$144 = 113 + 31$	$145 = 127 + 13 + 5$
$146 = 109 + 37$	$147 = 127 + 17 + 3$	$148 = 101 + 47$	$149 = 127 + 19 + 3$
$150 = 113 + 37$	$151 = 137 + 11 + 3$	$152 = 109 + 43$	$153 = 137 + 13 + 3$
$154 = 113 + 41$	$155 = 139 + 13 + 3$	$156 = 113 + 43$	$157 = 139 + 13 + 5$
$158 = 127 + 31$	$159 = 139 + 17 + 3$	$160 = 113 + 47$	$161 = 139 + 19 + 3$
$162 = 109 + 53$	$163 = 139 + 17 + 7$	$164 = 127 + 37$	$165 = 139 + 23 + 3$
$166 = 113 + 53$	$167 = 139 + 23 + 5$	$168 = 131 + 37$	$169 = 139 + 23 + 7$
$170 = 127 + 43$	$171 = 139 + 29 + 3$	$172 = 131 + 41$	$173 = 139 + 29 + 5$
$174 = 131 + 43$	$175 = 139 + 29 + 7$	$176 = 173 + 3$	$177 = 139 + 31 + 7$
$178 = 137 + 41$	$179 = 139 + 29 + 11$	$180 = 137 + 43$	$181 = 139 + 31 + 11$
$182 = 139 + 43$	$183 = 139 + 31 + 13$	$184 = 131 + 53$	$185 = 139 + 41 + 5$
$186 = 127 + 59$	$187 = 149 + 31 + 7$	$188 = 181 + 7$	$189 = 139 + 37 + 13$
$190 = 179 + 11$	$191 = 173 + 13 + 5$	$192 = 181 + 11$	$193 = 179 + 11 + 3$
$194 = 191 + 3$	$195 = 181 + 11 + 3$	$196 = 193 + 3$	$197 = 181 + 13 + 3$
$198 = 193 + 5$	$199 = 181 + 13 + 5$	$200 = 197 + 3$	$201 = 181 + 17 + 3$

Figure 1: Cases $50 \leq n \leq 201$ for Lemma 1

$202 = 199 + 3$	$203 = 181 + 19 + 3$	$204 = 199 + 5$	$205 = 191 + 11 + 3$
$206 = 199 + 7$	$207 = 193 + 11 + 3$	$208 = 197 + 11$	$209 = 193 + 13 + 3$
$210 = 199 + 11$	$211 = 197 + 11 + 3$	$212 = 199 + 13$	$213 = 199 + 11 + 3$
$214 = 211 + 3$	$215 = 199 + 13 + 3$	$216 = 211 + 5$	$217 = 199 + 13 + 5$
$218 = 211 + 7$	$219 = 199 + 17 + 3$	$220 = 197 + 23$	$221 = 199 + 19 + 3$
$222 = 211 + 11$	$223 = 199 + 19 + 5$	$224 = 211 + 13$	$225 = 211 + 11 + 3$
$226 = 223 + 3$	$227 = 211 + 13 + 3$	$228 = 223 + 5$	$229 = 211 + 13 + 5$
$230 = 227 + 3$	$231 = 211 + 17 + 3$	$232 = 229 + 3$	$233 = 211 + 19 + 3$
$234 = 229 + 5$	$235 = 211 + 19 + 5$	$236 = 233 + 3$	$237 = 223 + 11 + 3$
$238 = 233 + 5$	$239 = 223 + 13 + 3$	$240 = 233 + 7$	$241 = 227 + 11 + 3$
$242 = 239 + 3$	$243 = 229 + 11 + 3$	$244 = 241 + 3$	$245 = 229 + 13 + 3$
$246 = 241 + 5$	$247 = 233 + 11 + 3$	$248 = 241 + 7$	$249 = 233 + 13 + 3$
$250 = 239 + 11$	$251 = 233 + 13 + 5$	$252 = 241 + 11$	$253 = 239 + 11 + 3$
$254 = 251 + 3$	$255 = 241 + 11 + 3$	$256 = 251 + 5$	$257 = 241 + 13 + 3$
$258 = 251 + 7$	$259 = 241 + 13 + 5$	$260 = 257 + 3$	$261 = 241 + 17 + 3$
$262 = 257 + 5$	$263 = 241 + 19 + 3$	$264 = 257 + 7$	$265 = 251 + 11 + 3$
$266 = 263 + 3$	$267 = 251 + 13 + 3$	$268 = 263 + 5$	$269 = 251 + 13 + 5$
$270 = 263 + 7$	$271 = 257 + 11 + 3$	$272 = 269 + 3$	$273 = 257 + 13 + 3$
$274 = 271 + 3$	$275 = 257 + 13 + 5$	$276 = 271 + 5$	$277 = 263 + 11 + 3$
$278 = 271 + 7$	$279 = 263 + 13 + 3$	$280 = 277 + 3$	$281 = 263 + 13 + 5$
$282 = 277 + 5$	$283 = 269 + 11 + 3$	$284 = 281 + 3$	$285 = 271 + 11 + 3$
$286 = 283 + 3$	$287 = 271 + 13 + 3$	$288 = 283 + 5$	$289 = 271 + 13 + 5$
$290 = 283 + 7$	$291 = 277 + 11 + 3$	$292 = 281 + 11$	$293 = 277 + 13 + 3$
$294 = 283 + 11$	$295 = 281 + 11 + 3$	$296 = 293 + 3$	$297 = 283 + 11 + 3$
$298 = 293 + 5$	$299 = 283 + 13 + 3$	$300 = 293 + 7$	$301 = 283 + 13 + 5$
$302 = 283 + 19$	$303 = 283 + 17 + 3$	$304 = 293 + 11$	$305 = 283 + 19 + 3$
$306 = 293 + 13$	$307 = 293 + 11 + 3$	$308 = 277 + 31$	$309 = 293 + 13 + 3$
$310 = 307 + 3$	$311 = 293 + 13 + 5$	$312 = 307 + 5$	$313 = 293 + 17 + 3$
$314 = 311 + 3$	$315 = 293 + 19 + 3$	$316 = 313 + 3$	$317 = 293 + 19 + 5$
$318 = 313 + 5$	$319 = 293 + 23 + 3$	$320 = 317 + 3$	$321 = 307 + 11 + 3$
$322 = 317 + 5$	$323 = 307 + 13 + 3$	$324 = 317 + 7$	$325 = 311 + 11 + 3$
$326 = 313 + 13$	$327 = 313 + 11 + 3$	$328 = 317 + 11$	$329 = 313 + 13 + 3$
$330 = 317 + 13$	$331 = 317 + 11 + 3$	$332 = 313 + 19$	$333 = 317 + 13 + 3$
$334 = 331 + 3$	$335 = 317 + 13 + 5$	$336 = 331 + 5$	$337 = 317 + 17 + 3$
$338 = 331 + 7$	$339 = 317 + 19 + 3$	$340 = 337 + 3$	$341 = 317 + 19 + 5$
$342 = 337 + 5$	$343 = 317 + 23 + 3$	$344 = 337 + 7$	$345 = 331 + 11 + 3$
$346 = 317 + 29$	$347 = 331 + 13 + 3$	$348 = 337 + 11$	$349 = 331 + 13 + 5$
$350 = 347 + 3$	$351 = 337 + 11 + 3$	$352 = 349 + 3$	$353 = 337 + 13 + 3$

Figure 2: Cases $202 \leq n \leq 353$ for Lemma 1

$354 = 349 + 5$	$355 = 337 + 13 + 5$	$356 = 353 + 3$	$357 = 337 + 17 + 3$
$358 = 353 + 5$	$359 = 337 + 19 + 3$	$360 = 353 + 7$	$361 = 347 + 11 + 3$
$362 = 359 + 3$	$363 = 349 + 11 + 3$	$364 = 359 + 5$	$365 = 349 + 13 + 3$
$366 = 359 + 7$	$367 = 353 + 11 + 3$	$368 = 349 + 19$	$369 = 353 + 13 + 3$
$370 = 367 + 3$	$371 = 353 + 13 + 5$	$372 = 367 + 5$	$373 = 359 + 11 + 3$
$374 = 367 + 7$	$375 = 359 + 13 + 3$	$376 = 373 + 3$	$377 = 359 + 13 + 5$
$378 = 373 + 5$	$379 = 359 + 17 + 3$	$380 = 373 + 7$	$381 = 367 + 11 + 3$
$382 = 379 + 3$	$383 = 367 + 13 + 3$	$384 = 379 + 5$	$385 = 367 + 13 + 5$
$386 = 383 + 3$	$387 = 373 + 11 + 3$	$388 = 383 + 5$	$389 = 373 + 13 + 3$
$390 = 383 + 7$	$391 = 373 + 13 + 5$	$392 = 389 + 3$	$393 = 379 + 11 + 3$
$394 = 389 + 5$	$395 = 379 + 13 + 3$	$396 = 389 + 7$	$397 = 383 + 11 + 3$
$398 = 379 + 19$	$399 = 383 + 13 + 3$	$400 = 397 + 3$	$401 = 383 + 13 + 5$
$402 = 397 + 5$	$403 = 389 + 11 + 3$	$404 = 401 + 3$	$405 = 389 + 13 + 3$
$406 = 401 + 5$	$407 = 389 + 13 + 5$	$408 = 401 + 7$	$409 = 389 + 17 + 3$
$410 = 397 + 13$	$411 = 397 + 11 + 3$	$412 = 409 + 3$	$413 = 397 + 13 + 3$
$414 = 409 + 5$	$415 = 401 + 11 + 3$	$416 = 409 + 7$	$417 = 401 + 13 + 3$
$418 = 401 + 17$	$419 = 401 + 13 + 5$	$420 = 409 + 11$	$421 = 401 + 17 + 3$
$422 = 419 + 3$	$423 = 409 + 11 + 3$	$424 = 421 + 3$	$425 = 409 + 13 + 3$
$426 = 421 + 5$	$427 = 409 + 13 + 5$	$428 = 421 + 7$	$429 = 409 + 17 + 3$
$430 = 419 + 11$	$431 = 409 + 19 + 3$	$432 = 421 + 11$	$433 = 419 + 11 + 3$
$434 = 431 + 3$	$435 = 421 + 11 + 3$	$436 = 433 + 3$	$437 = 421 + 13 + 3$
$438 = 433 + 5$	$439 = 421 + 13 + 5$	$440 = 433 + 7$	$441 = 421 + 17 + 3$
$442 = 439 + 3$	$443 = 421 + 19 + 3$	$444 = 439 + 5$	$445 = 431 + 11 + 3$
$446 = 443 + 3$	$447 = 433 + 11 + 3$	$448 = 443 + 5$	$449 = 433 + 13 + 3$
$450 = 443 + 7$	$451 = 433 + 13 + 5$	$452 = 449 + 3$	$453 = 439 + 11 + 3$
$454 = 449 + 5$	$455 = 439 + 13 + 3$	$456 = 449 + 7$	$457 = 443 + 11 + 3$
$458 = 439 + 19$	$459 = 443 + 13 + 3$	$460 = 457 + 3$	$461 = 443 + 13 + 5$
$462 = 457 + 5$	$463 = 449 + 11 + 3$	$464 = 461 + 3$	$465 = 449 + 13 + 3$
$466 = 463 + 3$	$467 = 449 + 13 + 5$	$468 = 463 + 5$	$469 = 449 + 17 + 3$
$470 = 467 + 3$			

Figure 3: Cases $354 \leq n \leq 470$ for Lemma 1

Proof. The statement has been verified in [8] for every integer $n \leq 116$. We only need to assume $n \geq 50$, and hence n has a canonical partition (Lemma 1). We use this partition in the labeling of L_n defined by Figure 4. There are m tables $T_1, T_2, \ldots, T_m$, one for each prime in the sum $n = p_1 + p_2 + \cdots + p_m$. The rungs of the ladder follow the ordering $T_1, T_2, \ldots, T_m$ of the tables. For example, the first rung is assigned the labels p_1 and $p_1 + 1$, and the last rung is assigned the labels $2n$ and $2n - p_m$. Associated with each cell in the tables is a number in brackets [] which gives the order in which the numbers $1, 2, \ldots, 2n$ appear. For example, 1 refers to 1, [2] refers to $2, 3, \ldots, p_1 - 1$, [3] refers to p_1, [4] refers to $p_1 + 1$, and so on until the integer $2n$ appears in cell [$6m$]. Hence, each integer from 1 to $2n$ is used exactly once.

To prove the relative primeness of each pair of numbers corresponding to the ends of a rung in the ladder, we apply Lemma 2 as follows. For column 2 of T_k with $3 \leq k \leq m$, let $\sigma = \sum_{i=1}^{k-2} p_i$ and $p = p_{k-1}$. For column 3 of T_k with $3 \leq k \leq m$, let $\sigma = \sum_{i=1}^{k-1} p_i$ and $p = p_k$. For column 5 of T_m, let $\sigma = \sum_{i=1}^{m-1} p_i$ and $p = p_m$.

It is trivial to verify the relative primeness of the labels on the ends of all other pairs of edges in L_n. $\square$

3. Remarks

Admittedly, the following conjecture is stronger than Goldbach's strong and weak conjectures. We have verified this conjecture by computer for $n \leq 5,000,000$.

Conjecture 1. *Every integer $n \geq 50$ has a canonical partition with at most three terms.*

The Goldbach Conjecture provides a partition of an odd integer n where the smallest prime has size at most $n/3$. However, for any canonical partition of n we have $n = p_1 + p_2 + x$, $p_2 \geq 2p_1 + 3$ and $x \geq 2(p_1 + p_2) + 3$ where x represents the sum of the remaining primes in the partition. So $n \geq 9p_1 + 12$. Similarly, for an even integer n, the Goldbach Conjecture implies $n \geq 2p_1$ while a canonical partition yields $n \geq 3p_1 + 3$. These observations motivate the following two refinements of the Goldbach Conjecture.

Conjecture 2. *Every even integer $n \geq 14$ is the sum of two odd primes, one of which is at most $n/3 - 1$.*

Conjecture 3. *Every odd integer $n \geq 51$ is the sum of three odd primes, one of which is at most $(n - 12)/9$.*

j	1	$1 < j < p_1$	p_1	$p_1 < j < p_2 - p_1$	$p_2 - p_1$
	T_1			T_2	
$f(u_j)$	[3] p_1	[2] j	[8] p_2	[10] $p_1 + p_2 + j$	[11] $2p_2$
$f(v_j)$	[4] $p_1 + 1$	[5] $p_1 + j$	[6] $2p_1$	[7] $p_1 + j$	[14] p_3

$T_{3,5,\ldots,2\lfloor m/2\rfloor-1}$ for $3 \le k$ odd $< m$

j	$p_{k-1} - \sum_{i=1}^{k-2} p_i < j \le \sum_{i=1}^{k-1} p_i$	$\sum_{i=1}^{k-1} p_i < j < p_k - \sum_{i=1}^{k-1} p_i$	$p_k - \sum_{i=1}^{k-1} p_i$
$f(u_j)$	$[12, 24, \ldots, 6k-6]$ $\sum_{i=1}^{k-1} p_i + j$	$[13, 25, \ldots, 6k-5]$ $\sum_{i=1}^{k-1} p_i + j$	$[20, 32, \ldots, 6k+2]$ p_{k+1}
$f(v_j)$	$[9, 21, \ldots, 6k-9]$ $\sum_{i=1}^{k-2} p_i + j$	$[16, 28, \ldots, 6k-2]$ $\sum_{i=1}^{k} p_i + j$	$[17, 29, \ldots, 6k-1]$ $2p_k$

$T_{4,6,\ldots,2\lfloor m/2\rfloor-2}$ for $4 \le k$ even $< m$

j	$p_{k-1} - \sum_{i=1}^{k-2} p_i < j \le \sum_{i=1}^{k-1} p_i$	$\sum_{i=1}^{k-1} p_i < j < p_k - \sum_{i=1}^{k-1} p_i$	$p_k - \sum_{i=1}^{k-1} p_i$
$f(u_j)$	$[15, 27, \ldots, 6k-9]$ $\sum_{i=1}^{k-2} p_i + j$	$[22, 34, \ldots, 6k-2]$ $\sum_{i=1}^{k} p_i + j$	$[23, 35, \ldots, 6k-1]$ $2p_k$
$f(v_j)$	$[18, 30, \ldots, 6k-6]$ $\sum_{i=1}^{k-1} p_i + j$	$19, 31, \ldots, 6k-5]$ $\sum_{i=1}^{k-1} p_i + j$	$[26, 38, \ldots, 6k+2]$ p_{k+1}

T_m for $k = m$ odd

j	$p_{m-1} - \sum_{i=1}^{m-2} p_i < j \le \sum_{i=1}^{m-1} p_i$	$\sum_{i=1}^{m-1} p_i < j < p_m - \sum_{i=1}^{m-1} p_i$	$p_m - \sum_{i=1}^{m-1} p_i$	$p_m - \sum_{i=1}^{m-1} p_i < j \le \sum_{i=1}^{m} p_i$
$f(u_j)$	$[6m-6]$ $\sum_{i=1}^{m-1} p_i + j$	$[6m-5]$ $\sum_{i=1}^{m-1} p_i + j$	[1] 1	$[6m-3]$ $\sum_{i=1}^{m-1} p_i + j$
$f(v_j)$	$[6m-9]$ $\sum_{i=1}^{m-2} p_i + j$	$[6m-2]$ $\sum_{i=1}^{m} p_i + j$	$[6m-1]$ $2p_m$	$[6m]$ $\sum_{i=1}^{m} p_i + j$

T_m for $k = m$ even

j	$p_{m-1} - \sum_{i=1}^{m-2} p_i < j \le \sum_{i=1}^{m-1} p_i$	$\sum_{i=1}^{m-1} p_i < j < p_m - \sum_{i=1}^{m-1} p_i$	$p_m - \sum_{i=1}^{m-1} p_i$	$p_m - \sum_{i=1}^{m-1} p_i < j \le \sum_{i=1}^{m} p_i$
$f(u_j)$	$[6m-9]$ $\sum_{i=1}^{m-2} p_i + j$	$[6m-2]$ $\sum_{i=1}^{m} p_i + j$	$[6m-1]$ $2p_m$	$[6m]$ $\sum_{i=1}^{m} p_i + j$
$f(v_j)$	$[6m-6]$ $\sum_{i=1}^{m-1} p_i + j$	$[6m-5]$ $\sum_{i=1}^{m-1} p_i + j$	[1] 1	$[6m-3]$ $\sum_{i=1}^{m-1} p_i + j$

Figure 4: Labeling of L_n for Theorem 1

j	1	2	3	$3 < j < 8$	8
$f(u_j)$	3	2	11	$14 + j$	22
$f(v_j)$	4	5	6	$3 + j$	31

j	$8 < j \leq 14$	$14 < j < 17$	17	$17 < j \leq 45$
$f(u_j)$	$14 + j$	$14 + j$	1	$14 + j$
$f(v_j)$	$3 + j$	$45 + j$	62	$45 + j$

Figure 5: Labeling of $L_{3+11+31}$ from Theorem 1

j	1	2	3	$3 < j < 8$	8
$f(u_j)$	3	2	11	$14 + j$	22
$f(v_j)$	4	5	6	$3 + j$	31

j	$8 < j \leq 14$	$14 < j < 17$	17
$f(u_j)$	$14 + j$	$14 + j$	97
$f(v_j)$	$3 + j$	$45 + j$	62

j	$17 < j \leq 45$	$45 < j < 52$	52
$f(u_j)$	$14 + j$	$142 + j$	194
$f(v_j)$	$45 + j$	$45 + j$	293

j	$52 < j \leq 142$	$142 < j < 151$	151
$f(u_j)$	$142 + j$	$142 + j$	877
$f(v_j)$	$45 + j$	$435 + j$	586

j	$151 < j \leq 435$	$435 < j < 442$	442	$442 < j \leq 1312$
$f(u_j)$	$142 + j$	$1312 + j$	1754	$1312 + j$
$f(v_j)$	$435 + j$	$435 + j$	1	$435 + j$

Figure 6: Labeling of $L_{3+11+31+97+293+877}$ from Theorem 1

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