GENERAL SUM FORMULA FOR BI-PERIODIC FIBONACCI AND LUCAS NUMBERS

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Abstract

In this paper, we give a general sum formula for the bi-periodic Fibonacci and Lucas numbers. Also, we express the general sum formula of the bi-periodic Lucas numbers in terms of the bi-periodic Fibonacci numbers.

1. Introduction

The bi-periodic Fibonacci sequence $\{q_n\}$ is defined by Edson and Yayenie [2] as:

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \ n \geq 2 \tag{1}$$

with initial values $q_0 = 0$, $q_1 = 1$ and a, b are nonzero numbers. Its associative sequence, the bi-periodic Lucas sequence $\{p_n\}$, is defined by Bilgici [1] as:

$$p_n = \begin{cases} bp_{n-1} + p_{n-2}, & \text{if } n \text{ is even} \\ ap_{n-1} + p_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \ n \geq 2 \tag{2}$$

with the initial conditions $p_0 = 2$ and $p_1 = a$. These sequences can be seen as a generalization of the Fibonacci and Lucas sequences. If we take $a = b = 1$ in $\{q_n\}$, we get the classical Fibonacci sequence, and if we take $a = b = 1$ in $\{p_n\}$, we get the classical Lucas sequence.

Also, $\{q_n\}$ and $\{p_n\}$ both satisfy the following recurrence relation:

$$f_n = (ab + 2)\, f_{n-2} - f_{n-4}, \ n \geq 4. \tag{3}$$

The Binet formulas of the sequences $\{q_n\}$ and $\{p_n\}$ are given by

$$q_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \tag{4}$$

and

$$p_n = \frac{a^{\xi(n)}}{(ab)^{\left\lfloor \frac{n+1}{2} \right\rfloor}} \left(\alpha^n + \beta^n\right) \tag{5}$$

respectively, where $\alpha = \frac{ab+\sqrt{a^2b^2+4ab}}{2}$ and $\beta = \frac{ab-\sqrt{a^2b^2+4ab}}{2}$ that is, α and β are the roots of the polynomial $x^2 - abx - ab$ and $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function, i.e., $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd. Let $a^2b^2 + 4ab \neq 0$. Note that $\alpha + \beta = ab$, $\alpha - \beta = \sqrt{a^2b^2 + 4ab}$ and $\alpha\beta = -ab$. For some properties of these sequences, we refer to [2, 7, 1, 6], and for more general cases of these sequences see [3] and [4].

In this paper, we consider sums of certain products of bi-periodic Fibonacci and Lucas numbers. In particular, we give a general sum formula for the bi-periodic Fibonacci numbers that generalize the following results:

$$\sum_{k=1}^{2n} q_k q_{k+1} = \frac{1}{b}\left[q_{2n+1}^2 - 1\right], \tag{6}$$

$$\sum_{k=1}^{2n} \left(\frac{a}{b}\right)^{\xi(k)} q_k q_{k+2} = \frac{1}{b}\left[q_{2n+1} q_{2n+2} - a\right], \tag{7}$$

$$\sum_{k=1}^{2n} q_k q_{k+3} = \frac{1}{b}\left[q_{2n+1} q_{2n+3} - (ab+1)\right]. \tag{8}$$

Yayenie [7, Theorem 6] gave these results as a generalization of Rao's results in [5], and also he noted that result (8) provides a new result for the classical Fibonacci numbers by the reason of lack of reference. Motivated by results (6)-(8), here we obtain a more general sum formula for the bi-periodic Fibonacci numbers. Also, we state an analogous result for the bi-periodic Lucas numbers, and we express the sum of the products of bi-periodic Lucas numbers in terms of the bi-periodic Fibonacci numbers. Moreover, we give additional identities for the sums of the form $\sum_{k=1}^{2n} q_{k+r}$ and $\sum_{k=1}^{2n} p_{k+r}$, where r is a nonnegative integer.

2. Main Results

First, we start with giving a general sum formula for the bi-periodic Fibonacci numbers. Assume that r is a nonnegative integer.

Theorem 1. *For $n > 0$, we have*

$$\sum_{k=1}^{2n} \left(\frac{a}{b}\right)^{\xi(k)\xi(r+1)} q_k q_{k+r} = \frac{1}{b}\left[q_{2n+1} q_{2n+r} - q_r\right]. \tag{9}$$

Proof. Assume that r is even. By using the Binet formula (4), we get

$$\begin{aligned} q_k q_{k+r} &= \frac{a^{\xi(k+1)+\xi(k+r+1)}}{(ab)^{\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k+r}{2} \rfloor}} \left(\frac{\alpha^{2k+r} + \beta^{2k+r} - (\alpha\beta)^k (\alpha^r + \beta^r)}{(\alpha - \beta)^2} \right) \\ &= \frac{a^{1+\xi(k+1)} b^{1-\xi(k+1)}}{(ab)^{\frac{r}{2}} (\alpha - \beta)^2} \left(\alpha^r \left(\frac{\alpha^2}{ab} \right)^k + \beta^r \left(\frac{\beta^2}{ab} \right)^k - (-1)^k (\alpha^r + \beta^r) \right). \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{k=1}^{2n} \left(\frac{a}{b} \right)^{\xi(k)\xi(r+1)} q_k q_{k+r} \\ &= \frac{a^2}{(ab)^{\frac{r}{2}} (\alpha - \beta)^2} \sum_{k=1}^{2n} \left(\alpha^r \left(\frac{\alpha^2}{ab} \right)^k + \beta^r \left(\frac{\beta^2}{ab} \right)^k - (-1)^k (\alpha^r + \beta^r) \right) \\ &= \frac{a^2}{(ab)^{\frac{r}{2}} (\alpha - \beta)^2} \left(\alpha^r \sum_{k=1}^{2n} \left(\frac{\alpha^2}{ab} \right)^k + \beta^r \sum_{k=1}^{2n} \left(\frac{\beta^2}{ab} \right)^k - (\alpha^r + \beta^r) \sum_{k=1}^{2n} (-1)^k \right) \\ &= \frac{a^2}{(ab)^{\frac{r}{2}} (\alpha - \beta)^2} \left(\frac{\alpha^{r+1}}{ab} \left[\left(\frac{\alpha^2}{ab} \right)^{2n} - 1 \right] + \frac{\beta^{r+1}}{ab} \left[\left(\frac{\beta^2}{ab} \right)^{2n} - 1 \right] \right) \\ &= \frac{a}{b\,(ab)^{\frac{r}{2}} (\alpha - \beta)^2} \left(\frac{\alpha^{4n+r+1} + \beta^{4n+r+1}}{(ab)^{2n}} - \left(\alpha^{r+1} + \beta^{r+1} \right) \right). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &q_{2n+1} q_{2n+r} - q_r \\ &= \frac{a}{(ab)^{\frac{r}{2}} (\alpha - \beta)^2} \left(\frac{\alpha^{4n+r+1} + \beta^{4n+r+1} - (\alpha\beta)^{2n} (\alpha\beta^r + \beta\alpha^r)}{(ab)^{2n}} - (\alpha - \beta)(\alpha^r - \beta^r) \right) \\ &= \frac{a}{(ab)^{\frac{r}{2}} (\alpha - \beta)^2} \left(\frac{\alpha^{4n+r+1} + \beta^{4n+r+1}}{(ab)^{2n}} - \left(\alpha^{r+1} + \beta^{r+1} \right) \right), \end{aligned}$$

which completes the proof for even r. Similarly, it can be proven for odd r. □

Note that for $r = 1, 2$ and 3 we obtain results (6), (7), and (8) respectively.

By using Theorem 1 and the d'Ocagne's identity

$$q_{2n+1} q_{2n+r} - q_r = \left(\frac{b}{a} \right)^{\xi(r)} q_{2n+r+1} q_{2n}$$

in [2, Theorem 5], we can easily obtain the following result.

Corollary 1. *For $n>0$, we have*

$$\sum_{k=1}^{2n}\left(\frac{a}{b}\right)^{\xi(k)\xi(r+1)} q_k q_{k+r} = \frac{1}{a^{\xi(r)}b^{\xi(r+1)}}\left[q_{2n+r+1}q_{2n}\right]. \tag{10}$$

Now, we give an analogous result for the bi-periodic Lucas numbers.

Theorem 2. *For $n>0$, we have*

$$\sum_{k=1}^{2n}\left(\frac{b}{a}\right)^{\xi(k)\xi(r+1)} p_k p_{k+r} = \frac{1}{a^{\xi(r+1)}b^{\xi(r)}}\left[p_{2n}p_{2n+r+1}-2p_{r+1}\right]. \tag{11}$$

Proof. Here, we prove the theorem for the case of odd r. It can be proven similarly for the case of even r. By using the Binet formula (5), we get

$$\begin{aligned} p_k p_{k+r} &= \frac{a}{(ab)^{k+\frac{r+1}{2}}}\left(\alpha^{2k+r}+\beta^{2k+r}+(-1)^k\left(\alpha^r+\beta^r\right)\right) \\ &= \frac{a}{(ab)^{\frac{r+1}{2}}}\left(\alpha^r\left(\frac{\alpha^2}{ab}\right)^k+\beta^r\left(\frac{\beta^2}{ab}\right)^k+(-1)^k\left(\alpha^r+\beta^r\right)\right). \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{k=1}^{2n} p_k p_{k+r} &= \frac{a}{(ab)^{\frac{r+1}{2}}}\sum_{k=1}^{2n}\left(\alpha^r\left(\frac{\alpha^2}{ab}\right)^k+\beta^r\left(\frac{\beta^2}{ab}\right)^k+(-1)^k\left(\alpha^r+\beta^r\right)\right) \\ &= \frac{a}{(ab)^{\frac{r+1}{2}}}\left(\alpha^r\sum_{k=1}^{2n}\left(\frac{\alpha^2}{ab}\right)^k+\beta^r\sum_{k=1}^{2n}\left(\frac{\beta^2}{ab}\right)^k+\left(\alpha^r+\beta^r\right)\sum_{k=1}^{2n}(-1)^k\right) \\ &= \frac{a}{(ab)^{\frac{r+1}{2}}}\left(\frac{\alpha^{r+1}}{ab}\left[\left(\frac{\alpha^2}{ab}\right)^{2n}-1\right]+\frac{\beta^{r+1}}{ab}\left[\left(\frac{\beta^2}{ab}\right)^{2n}-1\right]\right) \\ &= \frac{1}{b\,(ab)^{\frac{r+1}{2}}}\left(\frac{\alpha^{4n+r+1}+\beta^{4n+r+1}}{(ab)^{2n}}-\left(\alpha^{r+1}+\beta^{r+1}\right)\right). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &p_{2n}p_{2n+r+1}-2p_{r+1} \\ &= \frac{1}{(ab)^{2n+\frac{r+1}{2}}}\left(\alpha^{4n+r+1}+\beta^{4n+r+1}+(ab)^{2n}\left(\alpha^{r+1}+\beta^{r+1}\right)-2\,(ab)^{2n}\left(\alpha^{r+1}+\beta^{r+1}\right)\right) \\ &= \frac{1}{(ab)^{\frac{r+1}{2}}}\left(\frac{\alpha^{4n+r+1}+\beta^{4n+r+1}}{(ab)^{2n}}-\left(\alpha^{r+1}+\beta^{r+1}\right)\right). \end{aligned}$$

Then

$$\sum_{k=1}^{2n} p_k p_{k+r} = \frac{1}{b}\left(p_{2n}p_{2n+r+1}-2p_{r+1}\right)$$

which proves the desired result. ∎

In the following corollary, we can easily express the general sum formula for the bi-periodic Lucas numbers in terms of the products of bi-periodic Fibonacci numbers by using Theorem 2 and the identity

$$p_{2n}p_{2n+r+1} - 2p_{r+1} = \left(\frac{b}{a}\right)^{\xi(r)} (ab+4)\, q_{2n+r+1}q_{2n}$$

in [1, Corollary 3].

Corollary 2. *For $n > 0$, we have*

$$\sum_{k=1}^{2n} \left(\frac{b}{a}\right)^{\xi(k)\xi(r+1)} p_k p_{k+r} = \frac{ab+4}{a} \left[q_{2n+r+1}q_{2n}\right]. \tag{12}$$

2.1. The Case of $r = 1$

We express the sum formula (11) in terms of the bi-periodic Fibonacci numbers in a simple way. First, consider

$$\begin{aligned} \sum_{k=1}^{2n} q_k q_{k+1} &= q_1q_2 + q_2q_3 + q_3q_4 + q_4q_5 + \cdots + q_{2n-1}q_{2n} + q_{2n}q_{2n+1} \\ &= q_2\left(q_1+q_3\right) + q_4\left(q_3+q_5\right) + \cdots + q_{2n}\left(q_{2n-1}+q_{2n+1}\right). \end{aligned}$$

By using the identity $q_{n-1} + q_{n+1} = p_n$ in [1, Theorem 3], we get

$$\sum_{k=1}^{2n} q_k q_{k+1} = p_2q_2 + p_4q_4 + \cdots + p_{2n}q_{2n} = \sum_{k=1}^{n} p_{2k}q_{2k}$$

and from (6), we obtain

$$\sum_{k=1}^{n} p_{2k}q_{2k} = \frac{1}{b}\left[q_{2n+1}^2 - 1\right].$$

Similarly, if we consider

$$\begin{aligned} \sum_{k=1}^{2n} p_k p_{k+1} &= p_1p_2 + p_2p_3 + p_3p_4 + p_4p_5 + \cdots + p_{2n-1}p_{2n} + p_{2n}p_{2n+1} \\ &= p_2\left(p_1+p_3\right) + p_4\left(p_3+p_5\right) + \cdots + p_{2n}\left(p_{2n-1}+p_{2n+1}\right) \end{aligned}$$

and use the identity $p_{n-1} + p_{n+1} = (ab+4)\, q_n$ in [1, Theorem 3], we get

$$\begin{aligned} \sum_{k=1}^{2n} p_k p_{k+1} &= (ab+4)\left[p_2q_2 + p_4q_4 + \cdots + p_{2n}q_{2n}\right] \\ &= (ab+4)\sum_{k=1}^{n} p_{2k}q_{2k}. \end{aligned}$$

As a result, we obtain

$$\sum_{k=1}^{2n} p_k p_{k+1} = \frac{ab+4}{b}\left[q_{2n+1}^2 - 1\right]. \tag{13}$$

Note that this result can also be obtained by using Corollary 2.

Also, we can express formulas (10) and (12) as a product of the bi-periodic Fibonacci and Lucas numbers. By using the identity $q_{2n} = q_n p_n$ in [1, Corollary 5], we can easily get the results

$$\sum_{k=1}^{2n} q_k q_{k+1} = \frac{1}{a}\left[p_n p_{n+1} q_n q_{n+1}\right] \tag{14}$$

$$\sum_{k=1}^{2n} p_k p_{k+1} = \frac{ab+4}{a}\left[p_n p_{n+1} q_n q_{n+1}\right]. \tag{15}$$

3. Another Sum Formula for Bi-periodic Fibonacci and Lucas Numbers

In this section, we present an additional theorem for the sums of the form $\sum_{k=1}^{2n} q_{k+r}$ and $\sum_{k=1}^{2n} p_{k+r}$, where r is a nonnegative integer.

Theorem 3. *For $n > 0$, we have*

$$\sum_{k=1}^{2n} q_{k+r} = \frac{1}{b}\left(q_{2\left(n+\left\lfloor \frac{r}{2}\right\rfloor\right)+1} - q_{2\left\lfloor \frac{r}{2}\right\rfloor+1}\right) + \frac{1}{a}\left(q_{2\left(n+\left\lfloor \frac{r+1}{2}\right\rfloor\right)} - q_{2\left\lfloor \frac{r+1}{2}\right\rfloor}\right), \tag{16}$$

$$\sum_{k=1}^{2n} p_{k+r} = \frac{1}{a}\left(p_{2\left(n+\left\lfloor \frac{r}{2}\right\rfloor\right)+1} - p_{2\left\lfloor \frac{r}{2}\right\rfloor+1}\right) + \frac{1}{b}\left(p_{2\left(n+\left\lfloor \frac{r+1}{2}\right\rfloor\right)} - p_{2\left\lfloor \frac{r+1}{2}\right\rfloor}\right). \tag{17}$$

Proof. By using the Binet formula of the bi-periodic Fibonacci numbers, we get

$$\sum_{k=1}^{2n} q_{k+r} = \sum_{k=1}^{n} q_{2\left(k+\left\lfloor \frac{r}{2}\right\rfloor\right)} + \sum_{k=1}^{n} q_{2\left(k+\left\lfloor \frac{r-1}{2}\right\rfloor\right)+1}$$

$$= \sum_{k=1}^{n} \frac{a}{(ab)^{k+\left\lfloor \frac{r}{2}\right\rfloor}} \left(\frac{\alpha^{2\left(k+\left\lfloor \frac{r}{2}\right\rfloor\right)} - \beta^{2\left(k+\left\lfloor \frac{r}{2}\right\rfloor\right)}}{\alpha-\beta}\right)$$

$$+ \sum_{k=1}^{n} \frac{1}{(ab)^{k+\left\lfloor \frac{r-1}{2}\right\rfloor}} \left(\frac{\alpha^{2\left(k+\left\lfloor \frac{r-1}{2}\right\rfloor\right)+1} - \beta^{2\left(k+\left\lfloor \frac{r-1}{2}\right\rfloor\right)+1}}{\alpha-\beta}\right)$$

$$+ \frac{1}{\alpha-\beta}\left(\alpha \sum_{k=1}^{n} \left(\frac{\alpha^2}{ab}\right)^{k+\left\lfloor \frac{r-1}{2}\right\rfloor} - \beta \sum_{k=1}^{n} \left(\frac{\beta^2}{ab}\right)^{k+\left\lfloor \frac{r-1}{2}\right\rfloor}\right)$$

$$= \frac{a}{\alpha-\beta}\left(\frac{\alpha^{2\left\lfloor\frac{r}{2}\right\rfloor+1}}{(ab)^{\left\lfloor\frac{r}{2}\right\rfloor+1}}\left(\left(\frac{\alpha^2}{ab}\right)^n-1\right)-\frac{\beta^{2\left\lfloor\frac{r}{2}\right\rfloor+1}}{(ab)^{\left\lfloor\frac{r}{2}\right\rfloor+1}}\left(\left(\frac{\beta^2}{ab}\right)^n-1\right)\right)$$

$$+\frac{1}{\alpha-\beta}\left(\left(\frac{\alpha^2}{ab}\right)^{\left\lfloor\frac{r-1}{2}\right\rfloor+1}\left(\left(\frac{\alpha^2}{ab}\right)^n-1\right)-\left(\frac{\beta^2}{ab}\right)^{\left\lfloor\frac{r-1}{2}\right\rfloor+1}\left(\left(\frac{\beta^2}{ab}\right)^n-1\right)\right)$$

$$= a\frac{\alpha^{2\left(n+\left\lfloor\frac{r}{2}\right\rfloor\right)+1}-\beta^{2\left(n+\left\lfloor\frac{r}{2}\right\rfloor\right)+1}-(ab)^n\left(\alpha^{2\left\lfloor\frac{r}{2}\right\rfloor+1}-\beta^{2\left\lfloor\frac{r}{2}\right\rfloor+1}\right)}{(ab)^{n+\left\lfloor\frac{r}{2}\right\rfloor+1}(\alpha-\beta)}$$

$$+\frac{\alpha^{2\left(n+\left\lfloor\frac{r-1}{2}\right\rfloor+1\right)}-\beta^{2\left(n+\left\lfloor\frac{r-1}{2}\right\rfloor+1\right)}-(ab)^n\left(\alpha^{2\left(\left\lfloor\frac{r-1}{2}\right\rfloor+1\right)}-\beta^{2\left(\left\lfloor\frac{r-1}{2}\right\rfloor+1\right)}\right)}{(ab)^{n+\left\lfloor\frac{r-1}{2}\right\rfloor+1}(\alpha-\beta)}$$

$$=\frac{1}{b}\left(q_{2\left(n+\left\lfloor\frac{r}{2}\right\rfloor\right)+1}-q_{2\left\lfloor\frac{r}{2}\right\rfloor+1}\right)+\frac{1}{a}\left(q_{2\left(n+\left\lfloor\frac{r+1}{2}\right\rfloor\right)}-q_{2\left\lfloor\frac{r+1}{2}\right\rfloor}\right).$$

The result for the bi-periodic Lucas numbers can be proven similarly. □

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