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ARITHMETIC PROGRESSIONS OF POLYGONAL NUMBERS WITH COMMON DIFFERENCE A POLYGONAL NUMBER

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Abstract

For integers $z \geq 1$ and $s \geq 3$, the zth s–gonal number is the positive integer $\left(z^2(s-2) - z(s-4)\right)/2$. Let $\mathcal{A} = (a, b, c)$ be a 3-term arithmetic progression (AP for short) of distinct positive integers. We say $\mathcal{A}$ is an (s,t)*–AP* if a, b, c are s-gonal numbers, and the common difference is a t-gonal number. Fermat proved that no $(4,4)$–AP's exist, and much later, Sierpiński showed that there exist infinitely many $(3,3)$–AP's. More recently, Ide and Jones have shown that no $(3,4)$–AP's exist, but that there do exist infinitely many $(4,3)$–AP's. In this article, we extend these results by showing that no $(s,4)$–AP's exist for any s, and that there do exist infinitely many (s,t)–AP's, for various other values of s and t.

1. Introduction

Throughout this article, all arithmetic progressions are assumed to have a positive common difference. The fact that no 4–term arithmetic progression of squares exists was first published posthumously by Euler in 1780. More recently, Brown, Dunn and Harrington [1] have extended Euler's result by showing that no 4–term arithmetic progression of s-gonal numbers exists for any s. Consequently, any mention of an arithmetic progression of polygonal numbers refers to a 3–term progression. This article is concerned with an investigation into such arithmetic progressions whose common difference is also a polygonal number. Historically, Fermat provided a proof that no arithmetic progression of squares exists whose common difference is also a square. However, in contrast, Sierpiński [5] constructed an infinite family of arithmetic progressions of triangular numbers whose common difference is a triangular number. More recently, Ide and Jones [3] have shown that no arithmetic progression of triangular numbers exists whose common difference is a square, but that there do exist infinitely many arithmetic progressions of squares whose common difference is a triangular number. Before stating the specific extensions of these results established in this article, we present some basic nomenclature.

For integers $z \geq 1$ and $s \geq 3$, the zth s–gonal number is the positive integer $\left(z^2(s-2) - z(s-4)\right)/2$. Observe that the zth s–gonal number can be written as

$$\frac{\left(2z(s-2)-(s-4)\right)^2-(s-4)^2}{8(s-2)}. \tag{1}$$

Let $\mathcal{A} = (a, b, c)$ be a 3-term arithmetic progression (AP for short) of distinct positive integers. We say $\mathcal{A}$ *is an* (s,t)*–AP* if a, b, c are s-gonal numbers, and the common difference is a t–gonal number. Using this notation, we provide in Table 1 a summary of the previously-known results concerning the existence of (s,t)–AP's.

Result	Proven by
No $(4,4)$–AP's	Fermat
Infinitely many $(3,3)$–AP's	Sierpiński
No $(3,4)$–AP's	Ide and Jones
Infinitely many $(4,3)$–AP's	Ide and Jones

Table 1: Previously-known results on the existence of (s,t)–AP's

In this article, we establish the following extensions of the information given in Table 1.

Theorem 1. *For any $s \geq 3$, no $(s,4)$–AP's exist.*

Theorem 2. *For any s and $t \neq 4$ such that $3 \leq s,t \leq 6$, there exist infinitely many (s,t)–AP's.*

2. Preliminaries

Proposition 1. *[6] If (x,y) is a rational point on a nonsingular cubic curve*

$$y^2 = x^3 + ax^2 + bx + c, \quad \textit{where} \quad a, b, c \in \mathbb{Z},$$

then

$$x = \frac{m}{e^2} \quad \textit{and} \quad y = \frac{n}{e^3},$$

for some $m, n, e \in \mathbb{Z}$ with $\gcd(m,e) = \gcd(n,e) = 1$.

The proof of Theorem 2 relies on the following standard facts from the theory of Pell equations.

Definition 1. Let $D > 1$ be a square-free positive integer, and let $N \neq 0$ be an integer. We define a *(generalized) Pell equation* to be a Diophantine equation of the form

$$x^2 - Dy^2 = N. \tag{2}$$

For arbitrary N, the solutions of (2) are intimately related to the solutions of

$$x^2 - Dy^2 = 1, \tag{3}$$

which we indicate in the following theorem. For a proof and a more extensive treatment, see [4].

Proposition 2. *Let $D > 1$ be a square-free positive integer, and let $N \neq 0$ be an integer.*

1. *There are infinitely many solutions (x_n, y_n) to the equation (3), and they are all given by*
$$x_n + y_n\sqrt{D} = \left(x_1 + y_1\sqrt{D}\right)^n, \quad n \in \mathbb{Z},$$
where (x_1, y_1) is the solution to (3) such that y_1 is the smallest nonnegative value of y among all solutions (x, y) to (3). The solution (x_1, y_1) is known as the fundamental solution to (3).

2. *Let (x_1, y_1) be the fundamental solution to (3) and let (γ, δ) be a solution to (2). Then, for each $n \geq 1$, the ordered pair (g_n, h_n) is a solution to (2), where g_n and h_n are defined by*
$$g_n + h_n\sqrt{D} = \left(x_1 + y_1\sqrt{D}\right)^n \left(\gamma + \delta\sqrt{D}\right).$$

3. The Proof of Theorem 1

Proof. From (1) and the result of Brown, Dunn and Harrington [1], we see that an arithmetic progression $\mathcal{A}$ of s–gonal numbers can be written as

$$\mathcal{A} = \left(\frac{a^2 - (s-4)^2}{8(s-2)}, \frac{b^2 - (s-4)^2}{8(s-2)}, \frac{c^2 - (s-4)^2}{8(s-2)}\right), \tag{4}$$

where a, b, c are positive integers with $a < b < c$. We assume that $\mathcal{A}$ has common difference d^2 and proceed toward a contradiction. It is easy to see from (4) that

$$a^2 = b^2 - 8(s-2)d^2 \qquad \text{and} \qquad c^2 = b^2 + 8(s-2)d^2. \tag{5}$$

Without loss of generality, we can assume

$$\gcd(a,d)=\gcd(b,d)=\gcd(c,d)=1, \tag{6}$$

since, for example, if $\gcd(a,d)=k>1$, then $\gcd(b,d)=\gcd(c,d)=k$ and a smaller arithmetic progression can be produced with common difference $(d/k)^2$ satisfying the assumed gcd property in (6). Multiplying together the two equations in (5) gives

$$(ac)^2=b^4-64(s-2)^2d^4. \tag{7}$$

Dividing (7) by d^4 yields the quartic curve

$$v^2=u^4-64(s-2)^2, \quad \text{where} \quad u=\frac{b}{d} \quad \text{and} \quad v=\frac{ac}{d^2}. \tag{8}$$

Using standard techniques [2], we arrive at the birationally equivalent nonsingular elliptic curve

$$\mathcal{E}: y^2=x^3+256(s-2)^2x, \quad \text{where} \quad u=\frac{y}{4} \quad \text{and} \quad v=\frac{8x-y^2}{16}. \tag{9}$$

From Proposition 1, we let

$$(x,y)=\left(\frac{m}{e^2},\frac{n}{e^3}\right)=\left(\frac{2^kM}{e^2},\frac{n}{e^3}\right),$$

where $m=2^kM$ with $M\equiv 1 \pmod 2$, and

$$\gcd(m,e)=\gcd(n,e)=1. \tag{10}$$

Then $\mathcal{E}$ can be rewritten as

$$n^2=2^{3k}M^3+2^{k+8}(s-2)^2e^4M. \tag{11}$$

From (9), we have that

$$u=\frac{n}{4e^3} \quad \text{and} \quad v=\frac{2^{k+3}Me^4-n^2}{16e^6}. \tag{12}$$

We divide the proof into the three cases: $k=0$, $k=1$ and $k\geq 2$. Suppose first that $k=0$, so that $n\equiv 1 \pmod 2$. Then, we deduce from (6), (8), (10) and (12) that

$$b=n, \quad d=4e^3 \quad \text{and} \quad ac=2^3Me^4-n^2. \tag{13}$$

Substituting the quantities from (13) into (7) and rearranging gives

$$e^4\left(2^2M^2+2^{10}(s-2)^2e^4\right)=Mn^2, \tag{14}$$

which is impossible since the left-hand side of (14) is even, but the right-hand side is odd.

If $k = 1$, then we see from (11) that

$$n^2 = 2^3\left(M^3 + 2^6(s-2)^2e^4M\right), \tag{15}$$

which is impossible since $M^3 + 2^6(s-2)^2e^4M \equiv 1 \pmod 2$ implies that the right-hand side of (15) is not a square.

Hence, we may assume that $k \geq 2$. Then, we have from (11) that $n \equiv 0 \pmod 4$. Proceeding as in the case of $k = 0$, we conclude from (6), (8), (10) and (12) that

$$b = \frac{n}{2^2}, \quad d = e^3 \quad \text{and} \quad ac = \frac{2^{k+3}Me^4 - n^2}{2^4}. \tag{16}$$

Substituting the quantities from (16) into (7) gives, after some algebra and rearranging,

$$e^4\left(2^{2k+6}M^2 + 2^{14}(s-2)^2e^4\right) = 2^{k+4}Mn^2, \tag{17}$$

which implies that $e^4 = 1$ by (10). Substituting the expression for n^2 from (11) (with $e^4 = 1$) into (17) and rearranging yields

$$2^{14}(s-2)^2 = 2^{2k+6}\left(2^{2k-2}M^4 + 2^6(s-2)^2M^2 - M^2\right). \tag{18}$$

Since $k \geq 2$ and $M \equiv 1 \pmod 2$, we have that

$$2^{2k-2}M^4 + 2^6(s-2)^2M^2 - M^2 \equiv 3 \pmod 4,$$

which implies that the right-hand side of (18) is not a square. This final contradiction completes the proof of the theorem. □

4. The Proof of Theorem 2

Theorem 2 is really just an application of the following theorem that gives sufficient conditions for the existence of infinitely many (s,t)–AP's. Using these conditions as an algorithm, the proof of Theorem 2 is then simply a computer search employing this algorithm. The strategy used in this algorithm is, in part, similar to techniques used for some of the results in [3], and also represents a slight modification of the method employed by Sierpiński in [5]. A partial summary of the details of our computer search, indicating the results in the statement of Theorem 2, is provided in Table 2. We have included $(s,t) \in \{(3,3),(4,3)\}$ in Table 2 since the infinite families found here differ from the infinite family for $(3,3)$ found by Sierpiński [5] and the infinite family for $(4,3)$ found by Ide and Jones [3].

Theorem 3. *Let $s, t \geq 3$ be integers with $t \neq 4$. Suppose that $(x, y, z, u) = (x_0, y_0, z_0, u_0)$ is a solution to the system of Diophantine equations*

$$x^2 + z^2 = 2y^2 \quad \text{and} \quad (t-2)\left(y^2 - x^2\right) = (s-2)\left(u^2 - (t-4)^2\right), \tag{19}$$

such that $0 < x_0 < y_0 < z_0$, $D := u_0^2 - (t-4)^2$ is not a square,

$$\begin{aligned} x_0 \equiv y_0 \equiv z_0 \equiv -(s-4) \pmod{2(s-2)} \\ \text{and} \quad u_0 \equiv -(t-4) \pmod{2(t-2)}. \end{aligned} \tag{20}$$

Let (α, β) be the fundamental solution to (3). If there exists a congruence class $\mathcal{C}$ for which

$$g_n + h_n\sqrt{D} = \left(\alpha + \beta\sqrt{D}\right)^n \left(u_0 + \sqrt{D}\right) \tag{21}$$

is such that

$$g_n \equiv -(t-4) \pmod{2(t-2)} \quad \text{and} \quad h_n \equiv 1 \pmod{2(s-2)}$$

for all $n \in \mathcal{C}$, with $n \geq 1$, then there exist infinitely many (s, t)–AP's.

Proof. Let $n \in \mathcal{C}$, with $n \geq 1$, and note from part (2) of Proposition 2 that (g_n, h_n) is a solution to (2) with $N = (t-4)^2$. Let

$$(x, y, z, u) = (x_0 h_n, y_0 h_n, z_0 h_n, g_n). \tag{22}$$

Then

$$x^2 + z^2 = (x_0 h_n)^2 + (z_0 h_n)^2 = \left(x_0^2 + z_0^2\right) h_n = 2y_0^2 h_n^2 = 2y^2$$

and

$$\begin{aligned} (t-2)\left(y^2 - x^2\right) &= (t-2)\left((y_0 h_n)^2 - (x_0 h_n)^2\right) \\ &= h_n^2 (t-2)\left(y_0^2 - x_0^2\right) \\ &= h_n^2 (s-2)\left(u_0^2 - (t-4)^2\right) \\ &= (s-2)\left(g_n^2 - (t-4)^2\right) \\ &= (s-2)\left(u^2 - (t-4)^2\right), \end{aligned}$$

which proves that (22) is a solution to (19). Observe also that (22) satisfies the congruence conditions in (20). Thus, since $0 < x < y < z$, there exist positive integers $a < b < c$ and d such that

$$\begin{aligned} x &= 2a(s-2) - (s-4) \\ y &= 2b(s-2) - (s-4) \\ z &= 2c(s-2) - (s-4) \\ u &= 2d(t-2) - (t-4). \end{aligned}$$

Hence, by (1),

$$\mathcal{P}_x = \frac{x^2-(s-4)^2}{8(s-2)}, \quad \mathcal{P}_y = \frac{y^2-(s-4)^2}{8(s-2)} \quad \text{and} \quad \mathcal{P}_z = \frac{z^2-(s-4)^2}{8(s-2)}$$

are s–gonal numbers, and it follows that $\mathcal{A} = (\mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_z)$ is an arithmetic progression with common difference equal to the t–gonal number

$$\mathcal{P}_u = \frac{u^2-(t-4)^2}{8(t-2)}.$$ □

Table 2: Summary for some new infinite families of (s,t)–AP's

(s,t)	(x_0,y_0,z_0,u_0)	(α,β)	$\mathcal{C}$
(3,3)	(7,13,17,11)	(11,1)	$n \equiv 0 \pmod 2$
(3,5)	(5,145,205,251)	(251,1)	$n \equiv 0 \pmod 2$
(3,6)	(7,13,17,22)	(241,11)	$\mathbb{N}$
(4,3)	(28,52,68,31)	(31,1)	$n \equiv 0 \pmod 4$
(4,5)	(84,156,204,161)	(161,1)	$n \equiv 0 \pmod 4$
(4,6)	(28,52,68,62)	(1921,31)	$n \equiv 0 \pmod 2$
(5,3)	(89,149,191,69)	(69,1)	$n \equiv 0 \pmod 4$
(5,5)	(11471,13001,14369,6119)	(6119,1)	$n \equiv 0 \pmod 6$
(5,6)	(2093,6773,9347,7438)	(27661921,3719)	$n \equiv 0 \pmod 3$
(6,3)	(1302,3390,4614,1565)	(1565,1)	$n \equiv 0 \pmod 8$
(6,5)	(13838,111518,157102,95831)	(95831,1)	$n \equiv 0 \pmod 8$
(6,6)	(7230,31830,44430,30998)	(480438001,15499)	$n \equiv 0 \pmod 4$

5. Some Final Comments

There are three inherent practical weaknesses in the algorithm described in Theorem 3. For one, we require a "seed" AP. Such a seed was not found for $(s,t) = (6,8)$ in our computer search. Secondly, the fundamental solution (α,β) of (3) must be found. For example, this fundamental solution was not found for $(s,t) = (5,7)$ in our computer search. Thirdly, the congruence class $\mathcal{C}$ must exist. However, with these computational weaknesses aside, we see no apparent mathematical obstruction in extending Theorem 2 to other (s,t)–AP's. In fact, our computer search found many such pairs beyond the pairs listed in Theorem 2, and we conjecture that for all s and t, with $t \neq 4$, there exist infinitely many (s,t)–AP's.

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