CHARACTERIZATION OF REPRESENTATION FUNCTIONS

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Abstract

Given a set A of natural numbers, i.e., nonnegative integers, the representation function of A is the function which to every natural number n associates the number $r_A(n)$ of ordered pairs (a,b) of elements a,b of A such that $a+b=n$. We characterize intrinsically the representation function, answering an open problem. .

1. Introduction

Let A denote a subset of the set $\mathbb{N}=\{0,1,2,\ldots\}$ of natural numbers. The *representation function* of A is defined by $r_A(n)=|\{(a,b)\in A\times A : a+b=n\}|$ for any $n\in\mathbb{N}$. It is an important function which completely determines the set A, and several authors have written about this topic. In particular, the consequences of the equality, or of the partial equality from some point on, of the representation functions $r_A(n)$ and $r_B(n)$ of two sets, A and B, of natural numbers have been studied rather extensively [9, 8, 12, 2, 1]. Other research has focused on studying the properties of representation functions, trying to characterize the class of representation functions and to determine which functions belong to this class. Also, many outstanding open problems and conjectures have been made in this respect [4, 10, 11, 3, 5, 6]. In particular, M. B. Nathanson highlights in one of his papers [10] the following problem: *"What functions are representation functions?"*

In the present paper, we answer this question by giving an intrinsic characterization of representation functions. We thus establish that a function $f:\mathbb{N}\longrightarrow\mathbb{N}$ is the representation function of a subset A of $\mathbb{N}$ if and only if it satisfies

$$f(n)=\frac{1}{2}\left(n+1-\sum_{k=0}^{n}(-1)^{f(2k)f(2(n-k))}\right),$$

for all $n\in\mathbb{N}$. We also describe the unique set A corresponding to such a function f, by showing that

$$A=\{n\in\mathbb{N} : f(2n)\equiv 1 \pmod 2\}.$$

While this chracterization might turn out practically unusable, it has the merit of providing the first answer to a long standing problem. In its defense, we quote G. H. Hardy in one of his papers [7]: "I need hardly say that this result does not pretend to be more than a curiosity, and that its value for any kind of application is *nil.* At the same time it seems worth while to point out that it is possible to write down a closed analytical expression which properly represents a function of x in Euler's sense, and which does possess the at first sight astonishing property stated above; and I am not aware that a similar formula has been given before." One has to only make a couple of obvious modifications in the last sentence to make it apply to our result.

2. Main Results

Definition 1. The *characteristic function* of a subset A of $\mathbb{N}$ is defined, for $n \in \mathbb{N}$, by

$$\chi_A(n) = \begin{cases} 1, & \text{if } n \in A; \\ 0, & \text{if } n \notin A. \end{cases} \tag{1}$$

The *representation function* of A is defined by

$$r_A(n) = |\{(a, b) \in A \times A : a + b = n\}|. \tag{2}$$

Lemma 1. *For any $n \in \mathbb{N}$, we have*

$$r_A(n) = \sum_{k=0}^{n} \chi_A(k)\chi_A(n-k). \tag{3}$$

Proof. It follows from the definitions that

$$\begin{aligned} r_A(n) &= |\{0 \le k \le n : k \in A \text{ and } n-k \in A\}| \\ &= |\{0 \le k \le n : \chi_A(k) = \chi_A(n-k) = 1\}| \\ &= \sum_{k=0}^{n} \chi_A(k)\chi_A(n-k). \end{aligned}$$

□

Lemma 2. *For any $n \in \mathbb{N}$, we have*

$$r_A(2n) \equiv \chi_A(n) \pmod 2. \tag{4}$$

Proof. In view of Lemma 1,

$$r_A(2n) = \sum_{k=0}^{2n} \chi_A(k)\chi_A(2n-k) = 2 \sum_{\substack{0 \le j < k \le 2n: \\ j+k=2n}} \chi_A(j)\chi_A(k) + \chi_A(n)^2 \equiv \chi_A(n) \pmod 2,$$

since $\chi_A(n)^2 = \chi_A(n)$. □

Corollary 1. *A natural number n lies in A if and only if $r_A(2n) \equiv 1 \pmod 2$.*

Definition 2. For an integer $a \in \mathbb{Z}$, let $res_2(a)$ denote the least non-negative residue of a modulo 2, i.e.,

$$res_2(a) = \begin{cases} 0, & \text{if } a \equiv 0 \pmod 2; \\ 1, & \text{if } a \equiv 1 \pmod 2. \end{cases} \tag{5}$$

Lemma 3. *For any $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$, we have*

$$res_2(a) = \frac{1 - (-1)^a}{2}, \tag{6}$$

$$res_2(-a) = res_2(a), \tag{7}$$

$$res_2(ab) = res_2(a) res_2(b), \tag{8}$$

$$res_2(a^n) = res_2(a)^n = res_2(a), \ \textit{for } n \geq 1, \tag{9}$$

$$res_2(a+b) = res_2(a) + (-1)^a res_2(b) = res_2(b) + (-1)^b res_2(a), \tag{10}$$

$$res_2(a-b) = res_2(a+b). \tag{11}$$

Proof. These properties are easily verified by considering all possible cases. □

Corollary 2. *For any $n \in \mathbb{N}$, we have*

$$\chi_A(n) = res_2(r_A(2n)) = \frac{1 - (-1)^{r_A(2n)}}{2}. \tag{12}$$

Proof. This follows directly from (4) and (6). □

Proposition 1. *For any $n \in \mathbb{N}$, we have*

$$r_A(n) = \frac{1}{2}\left(n + 1 - \sum_{k=0}^{n} (-1)^{r_A(2k) r_A(2n-2k)}\right). \tag{13}$$

Proof. Using (3), (12), (8), and (6), respectively, we get

$$\begin{aligned} r_A(n) &= \sum_{k=0}^{n} \chi_A(k)\chi_A(n-k) = \sum_{k=0}^{n} res_2(r_A(2k)) res_2(r_A(2(n-k))) \\ &= \sum_{k=0}^{n} res_2\left(r_A(2k) r_A(2(n-k))\right) = \sum_{k=0}^{n} \frac{1 - (-1)^{r_A(2k) r_A(2n-2k)}}{2} \\ &= \frac{1}{2}\left(n + 1 - \sum_{k=0}^{n} (-1)^{r_A(2k) r_A(2n-2k)}\right). \end{aligned}$$

□

Remark 3. It follows from (12) and (13) that the values of $r_A(2n) \pmod 2$ completely determine A and completely determine all values of $r_A(n)$. In other words, the representation function r_A of A is completely determined by the parity of its values at the even natural numbers. Moreover, the relation (13) characterizes the representation function, as will be seen below.

Theorem 4. *Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be a function satisying, for all $n \in \mathbb{N}$, the relation*

$$f(n) = \frac{1}{2}\left(n + 1 - \sum_{k=0}^{n} (-1)^{f(2k)f(2(n-k))}\right). \tag{14}$$

Then $f = r_A$ is the representation function of the subset A of $\mathbb{N}$ defined by

$$A = \{n \in \mathbb{N} : f(2n) \equiv 1 \pmod 2\}. \tag{15}$$

Proof. For any $n \in \mathbb{N}$, we have

$$\sum_{k=0}^{n} (-1)^{f(2k)f(2(n-k))} = \sum_{k\in I} 1 - \sum_{k\in J} 1 = |I| - |J|, \tag{16}$$

where

$$I = \{0 \le k \le n : f(2k) \equiv 0 \pmod 2 \ \text{ or } \ f(2(n-k)) \equiv 0 \pmod 2\}$$

and

$$J = \{0 \le k \le n : f(2k) \equiv f(2(n-k)) \equiv 1 \pmod 2\}.$$

By definition of A, we also have

$$I = \{0 \le k \le n : k \notin A \text{ or } (n-k) \notin A\}$$

and

$$J = \{0 \le k \le n : k \in A \text{ and } (n-k) \in A\}.$$

Clearly,

$$I \cup J = \{k \in \mathbb{N} : 0 \le k \le n\} \quad \text{and} \quad I \cap J = \emptyset,$$

so that

$$|I| + |J| = |I \cup J| = n + 1. \tag{17}$$

Combining (16) and (17), we get

$$\sum_{k=0}^{n} (-1)^{f(2k)f(2(n-k))} = |I| - |J| = n + 1 - 2|J|. \tag{18}$$

It follows from the definition of f, (18) and (3), that

$$\begin{aligned} f(n) &= \frac{1}{2}\left(n+1-\sum_{k=0}^{n}(-1)^{f(2k)f(2(n-k))}\right) \\ &= |J| \\ &= |\{0 \le k \le n : k \in A \text{ and } (n-k) \in A\}| \\ &= \sum_{k\in A,(n-k)\in A} 1 = \sum_{k=0}^{n} \chi_A(k)\chi_A(n-k) = r_A(n). \end{aligned}$$ □

Corollary 3. *A function $f : \mathbb{N} \longrightarrow \mathbb{N}$ is the representation function of a subset A of $\mathbb{N}$ if and only if it satisfies the relation*

$$f(n) = \frac{1}{2}\left(n+1-\sum_{k=0}^{n}(-1)^{f(2k)f(2(n-k))}\right), \quad \textit{for all } n \in \mathbb{N}. \tag{19}$$

Proof. This follows directly from Proposition 1 and Theorem 4. □

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