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NESTED INTERVAL SEQUENCES OF POSITIVE REAL NUMBERS

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Abstract

The nested interval sequence $NI(x)$ of a real number x in $(0,1]$ is defined and shown to yield an infinite sum $S(x)$ of reciprocals of positive integers, comparable to an Engels series. Conversely, every such sum represents such a number x, which is rational if and only if $NI(x)$ is eventually periodic. In selected cases, $S(x)$ is presented in terms of hypergeometric functions, Chebyshev polynomials, or Bessel functions. Also discussed are alternating nested interval sequences and associated sums.

1. Introduction

The purpose of this paper is to introduce nested interval sequences, which appear not to have been studied previously. We begin with an intuitive description. For any subinterval I of $(0,1]$, let S_I be the set of reciprocals $1/n$ of numbers in $N = \{1,2,3,\ldots\}$, scaled to I. Each pair $1/n_1, 1/(n_1+1)$ from $S_{(0,1]}$ determine the interval $I(n_1) = (1/(n_1+1), 1/n_1]$, and these intervals, for all n_1 in N, partition $(0,1]$. For each n_1, the numbers in $S_{I(n_1)}$ likewise determine intervals $I(n_1,n_2)$, for every n_2 in N. These intervals partition $(0,1]$ into a much "larger" collection of "shorter" intervals than in the first partition. Indeed, nested within each interval $I(n_1)$ are infinitely many $I(n_1,n_2)$. The procedure continues, so that each x in $(0,1]$ is in exactly one interval J_k of the form $I(n_1,n_2,\ldots,n_k)$, for each k in N. The nested intervals J_k have a limit point, which is x. A more precise treatment follows.

Definition 1. Suppose that $I = (a, b] \subseteq [0, 1]$, and for every $n \in N$, let

$$I(n) = (a + \frac{b-a}{n+1}, a + \frac{b-a}{n}]. \tag{1}$$

For every $n_1 \in N$, let $I(n_1)$ denote the interval (1) obtained from $(a, b] = (0, 1]$. Inductively, for $k \geq 1$, for every $(n_1, n_2, \ldots, n_k) \in N^k$, let $I(n_1, n_2, \ldots, n_k)$ denote the interval (1) obtained from $(a, b] = I(n_1, n_2, \ldots, n_{k-1})$. Clearly,

(i) for each $k \geq 1$, the set of all the intervals $I(n_1, n_2, \ldots, n_k)$ partitions $(0, 1]$;

(ii) if $x \in (0, 1]$, then there exists a unique sequence $(n_1, n_2, \ldots)$ such that

$$\{x\} = \bigcap_{k=1}^{\infty} I(n_1, n_2, \ldots, n_k).$$

We write $NI(x) = (n_1, n_2, \ldots)$ and call this the *nested-interval sequence* of x. Examples include

$$\begin{aligned} NI(1/2) &= (2, 1, 1, 1, \ldots) \\ NI(2/5) &= (2, 2, 2, 2, \ldots) \\ NI(1/e) &= (2, 4, 6, 8, 10, \ldots). \end{aligned}$$

The inverse of NI is written as IN; for example,

$$\begin{aligned} IN(1, 2, 3, 4, 5, \ldots) &= \sum_{k=1}^{\infty} \frac{1}{(k-1)!(k+1)!}, \\ &= 0.688948447698738204055\ldots \end{aligned}$$

By (1), the n_1^{th} subinterval of $(0, 1]$ is

$$I(n_1) = (\frac{1}{n_1+1}, \frac{1}{n_1}], \quad \text{of length } \frac{1}{n_1(n_1+1)},$$

so that the n_2^{th} subinterval of $I(n_1)$ is

$$I(n_1, n_2) = (\frac{1}{n_1+1} + \frac{1}{n_1(n_1+1)(n_2+1)}, \frac{1}{n_1+1} + \frac{1}{n_1 n_2 (n_1+1)}],$$

of length

$$\frac{1}{n_1 n_2 (n_1+1)(n_2+1)}.$$

Inductively, writing $I(n_1, n_2, \ldots, n_{k-1})$ as $(u_{k-1}, v_{k-1}]$, with length L_{k-1}, the n_k^{th} subinterval is

$$I(n_1, n_2, \ldots, n_k) = (u_{k-1} + \frac{L_{k-1}}{n_k+1}, u_{k-1} + \frac{L_{k-1}}{n_k}],$$

of length

$$L_k = \frac{L_{k-1}}{n_k(n_k+1)};$$

this is a rational number with numerator 1. Taking the limit of the left endpoint of $I(n_1, n_2, \ldots, n_k)$ establishes the following theorem.

Theorem 1. *If $NI(x) = (n_1, n_2, \ldots,)$, then*

$$x = \sum_{i=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_{i-1}(n_1+1)(n_2+1)\cdots(n_{i-1}+1)(n_i+1)}, \tag{2}$$

where $n_0 = 1$.

Taking $t_0 = 1$ and writing the k^{th} term of (2) as t_k, we have

$$t_k = \frac{t_{k-1}}{n_{k-1}(n_k+1)},$$

so that $t_k \le t_{k-1}/2$, and regarding the rate of convergence of the series (2), if $x = IN(n_1, n_2, \ldots)$, then

$$0 < x - \sum_{k=1}^{h} t_k = \sum_{k=h+1}^{\infty} t_k \le \sum_{k=h+1}^{\infty} 2^{-k} = 2^{1-h}.$$

Of course, the convergence is much faster if many of the initial numbers n_i exceed 1.

The nested-interval series (2) is an example of a broad class of representations of numbers in $(0, 1]$ called Oppenheim series. Specifically, using the notation of [6], an expansion

$$x = \frac{1}{d_1} + q_1(d_1)\frac{1}{d_2} + q_1(d_1)q_2(d_2)\frac{1}{d_3} + \cdots$$

is an Oppenheim series if

$$d_1 = n_1 + 1, \quad d_{j+1} \ge q_j(d_j)d_j(d_j - 1) + 1, \quad j \ge 1.$$

It is easy to check that those conditions hold for

$$d_1 = n_1 + 1, \quad d_j = n_{j-1}(n_j + 1), \text{ for } j \ge 2, \text{ and } q_j(n) = 1/n \text{ for } j \ge 1,$$

thus establishing that nested-interval series fit into Galambos's organizational scheme. Galambos ([5], 14-19) notes that the family of Oppenheim series includes Engel series, Sylvester series, and Lüroth series, and that it has close connections to Cantor products. Among those subfamilies, the Engel series, of the form

$$\frac{1}{d_1} + \frac{1}{d_1 d_2} + \frac{1}{d_1 d_2 d_3} + \cdots$$

appear to be most closely related to nested-interval series. In Section 7, we consider alternating nested interval representations, which resemble Pierce expansions in the way that nested interval series resemble Engel series. Recent interest in Oppenheim series, especially Engel series, including discussions of number-theoretic properties, associated continued fractions, and metric and probabilisitc properties, is indicated by the references at the end of this article.

2. Basic Properties

Suppose that $(n_{k+1}, \ldots, n_{k+m}) \in N^m$. In the sequel, the notation

$$\overline{n_{k+1}, \ldots, n_{k+m}}$$

represents the concatenation of infinitely many copies of $(n_{k+1}, \ldots, n_{k+m})$. For example,

$$NI(4/11) = (2, 5, 2, 1, 2, 1, \ldots) = (2, 5, \overline{2, 1}).$$

The longest interval of the form $I(n_1)$ is $(1/2, 1]$, and within it, the longest interval $I(n_1, n_2)$ is $(1/4, 1]$. Inductively, the longest interval $I(n_1, n_2, \ldots, n_m)$ is $[2^{-m}, 1]$, so that as $m \to \infty$, the length of every interval $I(n_1, n_2, \ldots, n_m)$ approaches 0, and also, $NI(1) = (\overline{1})$; indeed,

$$NI(1/n) = (n, \overline{1})$$

for every n in N.

The lexical ordering $\prec$ is defined in N^∞ as follows:

$$(m_1, m_2, \ldots) \prec (n_1, n_2, \ldots)$$

if one of the following two conditions holds: (1) $m_1 < n_1$, or else (2) there exists a greatest k such that $m_k = n_k$ and $m_{k+1} < n_{k+1}$. We state without proof an easy lemma:

Lemma 1. *Suppose that $k \geq 2$, that $(n_1, n_2, \ldots, n_{k-1}) \in N^{k-1}$, and that $(u, v) \in N^2$. Then $v > u$ if and only if $I(n_1, n_2, \ldots, n_{k-1}, v)$ lies to the left of $I(n_1, n_2, \ldots, n_{k-1}, u)$; i.e., every number in the first interval is less than every number in the second.*

Theorem 2. $0 < x < y < 1$ *if and only if* $NI(y) \prec NI(x)$.

Proof. For x and y in $(0, 1]$, let $m_1 = \lfloor 1/x \rfloor$ and $n_1 = \lfloor 1/y \rfloor$, and write

$$NI(x) = (m_1, m_2, \ldots), \ NI(y) = (n_1, n_2, \ldots).$$

First, suppose that $NI(y) \prec NI(x)$. If $n_1 < m_1$, then clearly $x < y$. Suppose then that $n_1 = m_1$. Let k be the greatest index i for which $m_i = n_i$, so that $n_{k+1} > m_{k+1}$. This means that x and y both lie in the interval $I(m_1, m_2, \ldots, m_k)$, but that

$$I(n_1, n_2, \ldots, n_k, n_{k+1}) \text{ lies to the left of } I(m_1, m_2, \ldots, m_k, m_{k+1}),$$

since $1/n_{k+1} < 1/m_{k+1}$. Then $x < y$, by Lemma 1. A proof of the converse is similar and omitted. □

Two informal notes may be useful. First, the notation $NI(x) = (n_1, n_2, \ldots)$ indicates that for every k, the number x lies in the n_k^{th} subinterval of the n_{k-1}^{th} subinterval of the n_{k-2}^{th} subinterval of ... of the n_1^{th} subinterval of $(0, 1]$. Second, a finite tuple $(n_1, n_2, \ldots, n_k)$ indicates not a point but an interval. For every $n \in N$, the interval $(n_1, n_2, \ldots, n_k + n)$ properly contains the interval $(n_1, n_2, \ldots, n_{k+1})$, so that the mapping NI is discontinuous (in a sense whose rigorous definition we skip) at every x. The notation $(n_1, n_2, \ldots)$ can be compared to an odometer having infinitely many places, each of which has infinitely many digits.

3. NI of Rational Numbers

A number x in $(0, 1]$ is *purely periodic* if $NI(x)$ is purely periodic, and *eventually periodic* if $NI(x)$ is eventually periodic. This latter condition means that $NI(x)$ has the form

$$(n_1, n_2, \ldots, n_k, \overline{n_{k+1}, n_{k+2}, \ldots, n_{k+m}}). \tag{3}$$

If m is minimal, we call m the period-length of x, denoted by $L(x)$. For purely periodic numbers x, we list the first two cases with brief examples and then present two theorems that include them. For all $a, b, c, \ldots$ in N,

$$\begin{aligned} IN(\overline{a}) &= \frac{a}{a^2 + a - 1}; \\ IN(\overline{a, b}) &= \frac{b(1 + a + ab)}{ab(a + 1)(b + 1) - 1}. \end{aligned}$$

In particular, for $L(x) = 1$,

$$NI(1) = (\overline{1}), \quad NI(2/5) = (\overline{2}), \quad NI(3/11) = (\overline{3}),$$

and for $L(x) = 2$,

$$NI(8/11) = (\overline{12}), \quad NI(6/11) = (\overline{21}).$$

Theorem 3. *Suppose that* $(n_1, n_2, \ldots, n_k) \in N^k$. *Let*

$$\begin{aligned} P &= \prod_{i=1}^{k} n_i\,(n_i+1)\,; \\ u_i &= n_i(n_{i+1}+1) \text{ for } i = 1, 2, \ldots, k-1. \end{aligned}$$

Then

$$IN(\overline{n_1, n_2, \ldots, n_k}) = \frac{n_k(1 + u_{k-1} + u_{k-1}u_{k-2} + \cdots + u_{k-1}u_{k-2}\cdots u_1)}{P-1}. \tag{4}$$

Proof. Let $u_i = n_i(n_{i+1}+1)$ for all i in N, and assume that $n_{k+j} = n_j$ for $j = 1, 2, \ldots, k-1$ and all k in N. Let $P = u_1 u_2 \cdots u_k$, so that

$$\begin{aligned} P &= n_1 n_2 \cdots n_k (n_1+1)(n_2+1)\cdots(n_k+1), \\ u_k &= n_k(n_{k+1}+1) = n_k(n_1+1). \end{aligned}$$

Let $x = IN(\overline{n_1, n_2, \ldots, n_k})$. By (2),

$$\begin{aligned} x &= \frac{1}{n_1+1} + \frac{1}{n_1(n_1+1)(n_2+1)} + \frac{1}{n_1 n_2(n_1+1)(n_2+1)(n_3+1)} + \cdots \\ &= \frac{1}{n_1+1} + \frac{1}{(n_1+1)u_1} + \frac{1}{(n_1+1)u_1u_2} + \frac{1}{(n_1+1)u_1u_2u_3} + \cdots \\ &= \frac{1}{n_1+1}\left(\frac{1}{u_1} + \frac{1}{u_1u_2} + \frac{1}{u_1u_2u_3} + \cdots + \frac{1}{P}\right) \\ &\quad + \frac{1}{n_1+1}\left(\frac{1}{u_1P} + \frac{1}{u_1u_2P} + \frac{1}{u_1u_2u_3P} + \cdots + \frac{1}{P^2}\right) \\ &\quad + \frac{1}{n_1+1}\left(\frac{1}{u_1P^2} + \frac{1}{u_1u_2P^2} + \frac{1}{u_1u_2u_3P^2} + \cdots + \frac{1}{P^3}\right) + \cdots \end{aligned}$$

$$\begin{aligned} x &= \frac{1}{n_1+1}\left(\frac{1}{u_1} + \frac{1}{u_1P} + \frac{1}{u_1P^2} + \cdots\right) \\ &\quad + \frac{1}{n_1+1}\left(\frac{1}{u_1u_2} + \frac{1}{u_1u_2P} + \frac{1}{u_1u_2P^2} + \cdots\right) + \cdots \\ &= \frac{1}{n_1+1}\left(1 + \frac{P}{u_1(P-1)} + \frac{P}{u_1u_2(P-1)} + \cdots + \frac{P}{P(P-1)}\right) \\ &= \frac{1}{n_1+1}\,\frac{u_1u_2\cdots u_k + u_2\cdots u_k + \cdots + u_{k-1}u_k + u_k}{P-1} \\ &= \frac{u_k}{n_1+1}\,\frac{u_1u_2\cdots u_{k-1} + u_2\cdots u_{k-1} + \cdots + u_{k-1} + 1}{P-1} \\ &= \frac{n_k(u_1u_2\cdots u_{k-1} + u_2\cdots u_{k-1} + \cdots + u_{k-1} + 1)}{P-1}. \end{aligned}$$

□

Theorem 4. *Let T_k be the set of all k-tuples t over $\{0, 1, 2\}$ that satisfying the following three conditions:*
(i) $t \neq (0, 0, \ldots, 0)$;
(ii) the first positive term of t is not 2;
(iii) in t, 0 follows only 0.
Then

$$IN(\overline{n_1, n_2, \ldots, n_k}) = \frac{\sum\limits_{(p_1, p_2, \ldots, p_k) \in T_k} n_1^{p_1} n_2^{p_2} \cdots n_k^{p_k}}{-1 + \prod\limits_{i=1}^{k} n_i(n_i + 1)}. \quad (5)$$

Proof. It suffices to show that the numerators of (2) and (4) are equal. The numerator of (2) is

$$\begin{aligned} & n_k + n_k n_{k-1}(n_k + 1) + n_k n_{k-1} n_{k-2}(n_k + 1)(n_{k-1} + 1) \\ & + \cdots + n_k n_{k-1} n_{k-2} \cdots n_1 (n_k + 1)(n_{k-1} + 1) \cdots (n_2 + 1). \end{aligned} \quad (6)$$

The first product in this sum, namely n_k, corresponds to the k-tuple $(0, \ldots, 0, 1)$, and the second product corresponds to the two k-tuples

$$(0, \ldots, 1, 1) \text{ and } (0, \ldots, 1, 2).$$

In general, the i^{th} product in (6) comes from the $(i-1)^{th}$ product by adding the following two k-tuples to each k-tuple already determined:

$$\begin{aligned} & (0, \ldots, 0, 1, 0, \ldots, 0), \text{ with 1 in place } k - i - 1 \\ & (0, \ldots, 1, 1, 0, \ldots, 0), \text{ with 1 in places } k - i - 1 \text{and } k - i, \end{aligned}$$

for $i = 2, \ldots, k - 1$. In this manner, all the products in (6) are accounted for, and the corresponding set of k-tuples are clearly characterized by conditions (i)-(iii). □

The sets T_k can also be determined inductively: Starting with $T_1 = \{(1)\}$, we have $T_k = U_k \cup V_k \cup W_k$ for $k \geq 2$, where

$$\begin{aligned} U_k &= \{(0, p_1, \ldots, p_{k-1}) : (p_1, \ldots, p_{k-1}) \in T_{k-1}\}; \\ V_k &= \{(p_1, \ldots, p_{k-1}, 1) : (p_1, \ldots, p_{k-1}) \in T_{k-1} \text{and } p_1 \neq 0\}; \\ W_k &= \{(p_1, \ldots, p_{k-1}, 2) : (p_1, \ldots, p_{k-1}) \in T_{k-1} \text{and } p_1 \neq 0\}. \end{aligned}$$

Note that $|T_k| = 2^k - 1$. For example,

$$\begin{aligned} T_4 = \; & \{(0,0,0,1), (0,0,1,1), (0,0,1,2), (0,1,1,1), (0,1,1,2), (0,1,2,1), (0,1,2,2), \\ & (1,1,1,1), (1,1,1,2), (1,1,2,1), (1,1,2,2), (1,2,1,1), (1,2,1,2), (1,2,2,1), (1,2,2,2)\}, \end{aligned}$$

so that the numerator for $IN(\overline{n_1, n_2, n_3, n_4})$ is a sum of 15 terms.

Theorem 5. *Suppose that* $\sigma = (n_1, n_2, \ldots, n_k, \overline{n_{k+1}, n_{k+2}, \ldots, n_{k+m}})$ *is an eventually periodic sequence. Let*

$$u = IN(\overline{n_{k+1}, n_{k+2}, \ldots, n_{k+m}}). \tag{7}$$

Then

$$\begin{aligned} IN(\sigma) \quad = \quad & \frac{1}{n_1+1} + \frac{1}{n_1(n_1+1)\,(n_2+1)} + \cdots \\ & + \frac{1}{n_1 n_2 \cdots n_{k-1}(n_1+1)\cdots(n_k+1)} \\ & + \frac{u}{n_1 n_2 \cdots n_{k-1} n_k (n_1+1)\cdots(n_k+1)}. \end{aligned} \tag{8}$$

Proof. By (2),

$$\begin{aligned} & x - \frac{1}{n_1+1} - \frac{1}{n_1(n_1+1)\,(n_2+1)} - \cdots - \frac{1}{n_1 n_2 \cdots n_{k-1}(n_1+1)\cdots(n_k+1)} \\ = \quad & \sum_{i=k+1}^{\infty} \frac{1}{n_1 n_2 \cdots n_{i-1}(n_1+1)(n_2+1)\cdots(n_{i-1}+1)(n_i+1)}, \end{aligned} \tag{9}$$

so that it suffices to show that the last term in (8) equals the series in (9). This equality follows by dividing both sides of the equation (7) by

$$n_1 n_2 \cdots n_{k-1} n_k (n_1+1)\cdots(n_k+1).$$

□

Theorem 6. x *is rational if and only if* $IN(x)$ *is eventually periodic.*

Proof. Suppose that $IN(x)$ is eventually periodic. Then (8) shows $IN(x)$ as a sum of $k+1$ rational numbers, where the $(k+1)^{st}$ term is rational by Theorem 3. For the converse, it will be helpful to define the *placement* of any x in an interval (a, b) by

$$p(x) = \frac{x-a}{b-a}.$$

For x in $(0, 1]$ with $NI(x) = (n_1, n_2, \ldots, n_k)$, let p_i be the placement of x in the interval $I(n_1, n_2, \ldots, n_i)$, and call $(x, p_1, p_2, \ldots)$ the *placement sequence* of x. We shall prove that if x is rational, then its placement sequence is eventually periodic, which implies that $IN(x)$ is eventually periodic.

Beginning with $p_0 = m/n$, assume for arbitrary $k \geq 1$ that if $i \leq k-1$, then

$p_i = h/n$ for some $h \in \{1, 2, \ldots, n\}$. Using notation given above,

$$\begin{aligned} p_k &= \frac{\frac{m}{n} - u_k}{v_k - u_k} \\ &= \frac{\frac{m}{n} - u_{k-1} - \frac{L_{k-1}}{n_k+1}}{\frac{L_{k-1}}{n_k} - \frac{L_{k-1}}{n_k+1}} \\ &= \frac{n_k n_{k+1}(\frac{m}{n} - u_{k-1} - \frac{L_{k-1}}{n_k+1})}{L_{k-1}} \\ &= \frac{n_k n_{k+1}}{L_{k-1}}\frac{m}{n} - \frac{n_k n_{k+1} u_{k-1}}{L_{k-1}} - n_k. \end{aligned}$$

Here, $\frac{n_k n_{k+1}}{L_{k-1}}$ and $\frac{n_k n_{k+1} u_{k-1}}{L_{k-1}}$ are integers. Writing $\frac{n_k n_{k+1}}{L_{k-1}}$ as H and using the fact that $0 < p_k \leq 1$, we have

$$p_k = \frac{Hm}{n} - \left\lfloor \frac{Hm}{n} \right\rfloor,$$

so that $p_k = \frac{hm}{n}$ where $1 \leq h \leq n$. By the pigeon-hole principle, the first $n+1$ numbers p_i are not distinct. Let j be the least index such that $p_j = p_i$ for some $i < j$. By the geometric similarity of the successive nested intervals, we have $p_{j+q} = p_{i+q}$ for all $q \geq 0$. That is, the placement sequence of x is eventually periodic. If $i+q$ is the index such that x is in the n_{i+q}^{th} subinterval of $(u_i, v_i]$, then x is also in the n_{i+q}^{th} subinterval of $(u_j, v_j]$, so that $n_{j+q} = n_{i+q}$ for all $q \geq 0$. That is, $IN(x)$ is eventually periodic. □

Note that the last term on the right-hand side of (8) is expressible in closed form, by both (4) and (5). Thus, by Theorem 6, a closed-form representation is established for $NI(m/n)$ for every positive rational number $m/n < 1$.

4. Fractility

Call numbers x and x' in $(0, 1]$ *equivalent* if the sequences $NI(x) = (n_1, n_2, \ldots)$ and $NI(x') = (n'_1, n'_2, \ldots)$ are eventually identical; i.e., if there exist h and k such that $n_{h+i} = n'_{k+i}$ for all i. For $n > 1$, the number of equivalence classes of the set

$$S(n) = \{m/n : 0 < m < n\}$$

is here introduced as the *fractility of* n. denoted by $F(n)$. By Theorem 6, each $NI(m/n)$ has a representation $(n_1, n_2, \ldots, n_k, \overline{w})$, where for convenience we take

$k = 0$ in case $NI(m/n)$ is purely periodic. Let $\overline{w}^*$ be the tuple $\overline{w} = (n_{k+1}, n_{k+2}, \ldots, n_{k+L})$ of minimal length L such that $(n_k, n_{k+1}, \ldots, n_{k+L-1})$ is not a period of $NI(m/n)$. We call $L = L(m/n)$ the *period-length* of m/n and note that m_1/n and m_2/n_2 are equivalent if $(n_{k_2+1}, n_{k_2+2}, \ldots, n_{k_2+L})$ is a cyclic permutation of $(n_{k_1+1}, n_{k_1+2}, \ldots, n_{k_1+L})$. For example $3/25$ and $4/25$ are equivalent:

$$NI(3/25) = (8, \overline{1, 3, 2, 6, 1, 2}) \quad \text{and} \quad NI(4/25) = (\overline{6, 1, 2, 1, 3, 2}).$$

There are 10 sequences $NI(m/25)$ for which $(\overline{w}^*) = (\overline{1})$, and 13 for which $(\overline{w}^*) = (\overline{6, 1, 2, 1, 3, 2})$. and 1 for which $(\overline{w}^*) = (\overline{2})$. For selected values of n, Table 1 shows fractility, class representatives $\overline{w}^*$, and the cardinalities of these classes:

n	$F(n)$	$(\overline{w}^*), \#$
2	1	$(\overline{1}), 1$
3	1	$(\overline{1}), 2$
4	1	$(\overline{1}), 3$
5	2	$(\overline{1}), 3;\ (\overline{2}), 1$
11	3	$(\overline{1}), 2;\ (\overline{1, 2}), 4;\ (\overline{3}), 4$
13	3	$(\overline{1}), 5;\ (\overline{1, 4}), 2;\ (\overline{1, 2, 3}), 5$
25	3	$(\overline{1}), 10;\ (\overline{1, 2, 1, 3, 2, 6}), 13;\ (\overline{2}), 1$
41	4	$(\overline{1}), 6;\ (\overline{1, 1, 1, 2, 2}), 25;\ (\overline{1, 1, 8}), 8;\ (\overline{6}), 1$
55	5	$(\overline{1}), 20;\ (\overline{7}), 14;\ (\overline{1, 2}), 13;\ (\overline{3}), 6;\ (\overline{2}), 1$

Table 1. Fractility and equivalence classes

We conjecture that for every n' in N, there are infinitely many n for which $F(n) = n'$. In particular the first 20 values of n for which $F(n) = 1$ (and $(\overline{w}^*) = (\overline{1})$) are as follows:

$$2, 3, 4, 6, 7, 8, 9, 12, 14, 16, 17, 18, 21, 24, 27, 28, 31, 32, 34.$$

For more terms of this sequence, see A269804 in the Online Encyclopedia of Integer Sequences [1].

For certain values of n, the sequence $NI(m/n)$ belongs to the class $(\overline{1})$ for exactly two values of m. The first 14 of these n are

$$3, 11, 23, 47, 59, 83, 107, 131, 167, 179, 227, 251, 263, 347,$$

all of which are primes.

Of particular interest are numbers n for which $F(n) = 1$. In that case, the number u in Theorem 5 is 1, so that the sum in (8) is a representation by Egyptian fractions such that if $1/s$ is followed immediately by $1/t$, then t/s is an integer (as in the Engel representation of m/n).

The first eight primes p for which $F(p) = 1$ are these:

$$2, 3, 7, 17, 31, 113, 151, 241.$$

For many products q of these primes, we have $F(q) = 1$, but not in all cases; e.g., for $n = 7 \cdot 17 = 119$,

$$(\frac{2}{119}) = (59, 2, 1, 1, 1, 1, 1, 5, 1, 1, 5, 1, 1, 5, 1, 1, \ldots),$$

and $F(119) = 3$ counts the classes $(\overline{1}), (\overline{2, 4}), (\overline{1, 1, 5})$. In general, it appears that if $n_1 | n_2$, then $F(n_2) \geq F(n_1)$.

5. IN of Selected Sequences

Generalized hypergeometric functions ${}_pF_q$ can be used to identify $IN(\sigma)$ for many classes of sequences σ. For example, if σ is an arithmetic sequence with first term a and common difference d, then

$$IN(a, a+d, a+2d, a+3d, \ldots) = \frac{{}_1F_2(1; \frac{a}{d}, \frac{a+d+1}{d}; \frac{1}{d^2})}{a+1},$$

which includes the following special cases (where B^* denotes the modified Bessel function of the 1^{st} kind):

$$\begin{aligned} IN(1, 2, 3, 4, 5, \ldots) &= B^*(2, 2) = 0.68894844769873820405\ldots \\ IN(1, 3, 5, 7, 9, \ldots) &= 1 - 1/e = 0.63212055882855767840\ldots \\ IN(2, 4, 6, 8, 10, \ldots) &= 1/e = 0.36787944117144232160\ldots \\ IN(n, n+1, n+2, \ldots) &= (n-1)!n! \sum_{k=n}^{\infty} \frac{1}{(k-1)!(k+1)!}. \end{aligned}$$

Let

$$s = \frac{3n-4}{4n-4} \quad \text{and} \quad t = \frac{\sqrt{n^2 - 16n + 32}}{4n-4}.$$

Then

$$IN(n\text{-gonal sequence}) = \frac{1}{2}\ {}_0F_3(; \frac{2}{n-2}, s-t, s+t; \frac{4}{(n-2)^2}); \tag{10}$$

in particular, for the triangular numbers and squares,

$$\begin{aligned} IN(1, 3, 6, 10, 15, \ldots) &= (1/2)\ {}_0F_3(; 2, \frac{5 - i\sqrt{7}}{2}, \frac{5 + i\sqrt{7}}{2}; 4) \qquad (11) \\ &= 0.63104313392591298206\ldots \\ IN(1, 4, 9, 16, 25, \ldots) &= (1/2)\ {}_0F_3(; 1, 2-i, 2+i; 1) \qquad (12) \\ &= 0.84220525264467028346\ldots \end{aligned}$$

Also, for cubes,

$$IN(1,8,27,64,125,\ldots) = \frac{1}{2}\ {}_0F_5(;1,1,3,\frac{3-i\sqrt{3}}{2},\frac{3+i\sqrt{3}}{2};1) \quad (13)$$
$$0.5558037127656825889 0\ldots$$

Next, we sample $IN(\sigma)$ for sequences σ containing equally spaced $1'$s:

σ	$IN(\sigma)$
$(1,1,2,1,1,4,1,1,6,1,1,\ldots)$	$(e^{1/2}+3e^{-1/2})/4$
$(1,2,1,1,4,1,1,6,1,1,\ldots)$	$(e^{1/2}+3e^{-1/2})/2$
$(2,1,1,4,1,1,6,1,1,\ldots)$	$e^{1/2}+3e^{-1/2}$
$(1,1,3,1,1,5,1,1,7,1,1,\ldots)$	$1+e^{1/2}-3e^{-1/2}$
$(1^3,2,1^3,4,1^3,6,1^3,8,\ldots)$	$\sqrt{1/2}e^{-\sqrt{1/8}}$
$(1,3,1,5,1,7,1,9,1,11,\ldots)$	$1+(\sqrt{2}-1)e^{\sqrt{1/2}}-(\sqrt{2}+1)e^{-\sqrt{1/2}}$
$(1,2,1,4,1,6,1,8,1,10,\ldots)$	$((1-\sqrt{2})e^{\sqrt{1/2}}+(1+\sqrt{1/2})e^{-\sqrt{1/2}})/2$

Table 2. $IN(\sigma)$ for selected sequences σ

In the sequel, 1^k represents the k-tuple consisting of k 1s. There are many identities involving equally spaced blocks 1^k, such as this one:

$$IN(1^k,2,1^k,4,1^k,6,1^k,8,\ldots) = 1-2^{k+1}+(2^{k+1}-2)_0F_1(2,2^{-k})+\ {}_0F_1(3,2^{-k}).$$

We turn next to a generalization of arithmetic sequences. Suppose that $F=\{a_i+nd_i\}$ is a family of arithmetic sequences given by i^{th}term a_i+nd_i for $i=1,2,\ldots,m$, where $m\geq 2$. The *riffle* of F, denoted by $\mathbb{R}(F)$, is the sequence given by

$$\begin{aligned}\mathbb{R}(F) &= \mathbb{R}(a_1,a_2\ldots,a_m;d_1,d_2\ldots,d_m)\\ &= (a_1,a_2\ldots,a_m,\\ &\quad a_1+d_1,a_2+d_2\ldots,a_m+d_m,\\ &\quad a_1+2d_1,a_2+2d_2\ldots,a_m+2d_m,\\ &\quad a_1+3d_1,a_2+3d_2\ldots,a_m+3d_m,\\ &\quad \ldots\end{aligned}$$

Riffles for $m=1,2,3$ correspond to sums of hypergeometric functions as follows:

$$IN(\mathbb{R}(a;d)) = \frac{1}{a+1}\ {}_1F_2(1;\frac{a}{d};\frac{a+d+1}{d};\frac{1}{d^2})$$

$$\begin{aligned}IN(\mathbb{R}(a_1,a_2;d_1,d_2)) &= \frac{{}_1F_4(1;\frac{a_1+d_1+1}{d_1},\frac{a_1}{d_1},\frac{a_2+1}{d_2},\frac{a_2}{d_2};\frac{1}{d_1^2d_2^2})}{a_1+1}\\ &+\frac{{}_1F_4(1;\frac{a_1+d_1}{d_1},\frac{a_1+d_1+1}{d_1},\frac{a_2+d_2+1}{d_2},\frac{a_2}{d_2};\frac{1}{d_1^2d_2^2})}{a_1(a_1+1)(a_2+1)}\end{aligned}$$

$$\begin{aligned} IN(\mathbb{R}(a_1,a_2,a_3;d_1,d_2,d_3)) &= \frac{T_1}{a_1+1}+\frac{T_2}{a_1(a_1+1)(a_2+1)} \\ &\quad +\frac{T_3}{a_1a_2(a_1+1)(a_2+1)(a_3+1)}, \end{aligned}$$

where

$$\begin{aligned} T_1 &= {}_1F_6(1;\frac{a_1+d_1+1}{d_1},\frac{a_1}{d_1},\frac{a_2+1}{d_2},\frac{a_2}{d_2},\frac{a_3+1}{d_3},\frac{a_3}{d_3};\frac{1}{d_1^2d_2^2d_3^2}) \\ T_2 &= {}_1F_6(1;\frac{a_1+1}{d_1},\frac{a_1+d_1+1}{d_1},\frac{a_2+d_2+1}{d_2},\frac{a_2}{d_2},\frac{a_3+d_3}{d_3},\frac{a_3}{d_3};\frac{1}{d_1^2d_2^2d_3^2}) \\ T_3 &= {}_1F_6(1;\frac{a_1+d_1}{d_1},\frac{a_1+d_1+1}{d_1},\frac{a_2+d_2}{d_2},\frac{a_2+d_2+1}{d_2},\frac{a_3+d_3+1}{d_3},\frac{a_3}{d_3};\frac{1}{d_1^2d_2^2d_3^2}). \end{aligned}$$

Next, let B denote a Bessel number of the 1st kind, and B^*, a modified Bessel number of the 1st kind:

$$\begin{aligned} IN(1,3,1,5,1,7,1,9,1,11,1,13,\ldots) &= 1-2\ \cosh\sqrt{1/2}+2\sqrt{2}\sinh\sqrt{1/2} \\ &= 0.6496996095025017647\ldots \\ IN(1,1,3,1,1,5,1,1,7,1,1,9,1,1,\ldots) &= 1-2\ \cosh\sqrt{1/2}+4\sinh\sqrt{1/2} \\ &= 0.829129291562227876\ldots \\ IN(1,1,2,2,3,3,4,4,5,5,6,6,7,7,\ldots) &= (1/4)({}_0F_3(;1,2,3;1)+\ {}_0F_3(;2,3,3;1)) \\ &= 0.84911315735735096381\ldots \\ IN(1,1,3,3,5,5,7,7,9,9,\ldots) &= 1-B^*(0,2)+(3/2)B^*(1,2)-(1/2)B(1,2) \\ &= 0.81800757574148963403\ldots \\ IN(2,2,4,4,6,6,8,8,10,10,\ldots) &= -B^*(0,2)+(3/2)B^*(1,2)+(1/2)B(1,2) \\ &= 0.39473238349836302l2\ldots \\ IN(2,1,4,3,6,5,8,7,10,9,\ldots) &= 1+2B^*(0,2)-3B^*(1,2)-B(2,2) \\ &= 0.43442601214450962558\ldots \end{aligned}$$

Let σ be the sequence given by $\sigma(n)=a+bn+cn^2$, for $n\geq 0$. Let

$$s=\sqrt{b^2-4ac}\ \text{ and }\ t=\sqrt{b^2-4ac-4c}.$$

Then

$$IN(\sigma)=\frac{1}{a+1}\ {}_1F_4(1;\frac{b-s}{2c},\frac{b+s}{2c},\frac{b+2c-t}{2c},\frac{b+2c+t}{2c};\frac{1}{c^2});$$

for example, for $\sigma(n)=2-2n+n^2$, with first terms $2,1,2,5,10,17,26,37$, we have

$$IN(\sigma)=\frac{1}{3}\ {}_1F_4(1;-1-i,-1+i,-i\sqrt{2},i\sqrt{2};1).$$

6. Connections with Chebyshev Polynomials

The Chebyshev polynomials of the 1st kind are defined by $T_0(x) = 1$, $T_1(x) = x$, and

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

Examples follow for $k \geq 2$:

$$\begin{aligned}
IN(1^2, 2, 1^2, 7, 1^2, 97, 1^2, 11817, \ldots) &= \sqrt{3}/2 = 0.8660\ldots \\
IN(1^2, 3, 1^2, 17, 1^2, 577, 1^2, 665957, \ldots) &= 2\sqrt{2} - 2 = 0,8284\ldots \\
IN(1^2, 4, 1^2, 31, 1^2, 1921, 1^2, 7380481, \ldots) &= (3\sqrt{15} - 10)/2 = 0.8094\ldots \\
IN(1^2, 5, 1^2, 49, 1^2, 4801, 1^2, 46099201, \ldots) &= 4\sqrt{6} - 9 = 0.7979\ldots \\
IN(1^2, T_1(k), 1^2, T_2(k), 1^2, T_4(k), 1^2, T_8(k), \ldots) &= (k-1)\sqrt{k^2-1} - \frac{(k-2)(k+1)}{2}.
\end{aligned}$$

Deleting the initial 1^2 gives

$$\begin{aligned}
IN(2, 1^2, 7, 1^2, 97, 1^2, 11817, \ldots) &= 2\sqrt{3} - 3 = 0.4641\ldots \\
IN(3, 1^2, 17, 1^2, 577, 1^2, 665957, \ldots) &= 8\sqrt{2} - 11 = 0,3137\ldots \\
IN(4, 1^2, 31, 1^2, 1921, 1^2, 7380481, \ldots) &= 6\sqrt{15} - 23 = 0.2379\ldots \\
IN(5, 1^2, 49, 1^2, 4801, 1^2, 46099201, \ldots) &= 16\sqrt{6} - 39 = 0.1918\ldots \\
IN(T_1(k), 1^2, T_2(k), 1^2, T_4(k), 1^2, T_8(k), \ldots) &= (2k-2)\sqrt{k^2-1} - k^2 + k - 1.
\end{aligned}$$

7. Alternating Nested Interval Sequences

The alternating form from (5), denoted by $AIN(n_1, n_2, \ldots)$, is given by

$$AIN(n_1, n_2, \ldots) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{n_1 n_2 \cdots n_{k-1}(n_1+1)(n_2+1)\cdots(n_{k-1}+1)(n_k+1)}. \quad (14)$$

This sum represents the limit point of a sequence of nested intervals obtained from $(n_1, n_2, \ldots)$ as follows. Let t_i be the i^{th} term of the series in (14). Then

$$\begin{aligned}
t_1 - t_2 &\in (0, t_1) \\
t_1 - t_2 + t_3 &\in (t_1 - t_2, t_1) \\
&\vdots \\
t_1 - t_2 + t_3 - \cdots - t_{2k} &\in (t_1 - t_2 + t_3 - \cdots - t_{2k-2}, t_1 - t_2 + t_3 - \cdots + t_{2k-1}) \\
t_1 - t_2 + t_3 - \cdots - t_{2k} + t_{2k+1} &\in (t_1 - t_2 + t_3 - \cdots - t_{2k}, t_1 - t_2 + t_3 - \cdots + t_{2k-1}).
\end{aligned}$$

The successive intervals on the right-side of these containments are clearly nested and approach 0 in length. The next list identifies $AIN(n_1, n_2, \ldots)$ for selected sequences:

$$\begin{aligned}
AIN(1,2,3,4,5,6,7,8,9,10,\ldots) &= B(2,2)\\
AIN(1,2,1,3,1,4,1,5,1,6,\ldots) &= -1+2\sqrt{2}B^*(1,\sqrt{2})-4B^*(2,\sqrt{2})\\
AIN(1,1,2,1,1,3,1,1,4,1,1,5,\ldots) &= 3-4B(1,1)-8B(2,1)\\
AIN(1,3,5,7,9,11,13,15,17,\ldots) &= -1+\cos 1+\sin 1\\
AIN(1,3,1,5,1,7,1,9,1,11,\ldots) &= -5+6\cosh(\sqrt{1/2})-2\sqrt{2}\sinh(\sqrt{1/2})\\
AIN(1,1,3,1,1,5,1,1,7,1,1,9,\ldots) &= 11-10\cos(1/2)-4\sin(1/2)\\
AIN(2,4,6,8,10,12,14,16,18,\ldots) &= \sin 1-\cos 1.
\end{aligned}$$

Corresponding to the sums (10)-(13),

$$\begin{aligned}
AIN(n\text{-gonal sequence}) &= \frac{1}{2}\ {}_0F_3(;\frac{2}{n-2},s-t,s+t;\frac{-4}{(n-2)^2});\\
AIN(1,3,6,10,15,\ldots) &= (1/2)\ {}_0F_3(;2,\frac{5-i\sqrt{7}}{2},\frac{5+i\sqrt{7}}{2};-4)\\
&= 0.38086275532933578907 4\ldots\\
AIN(1,4,9,16,25,\ldots) &= (1/2)\ {}_0F_3(;1,2-i,2+i;-1)\\
& \ 0.40248369936681118538 6\\
AIN(1,8,27,64,125,\ldots) &= \frac{1}{2}\ {}_0F_5(;1,1,3,\frac{3-i\sqrt{3}}{2},\frac{3+i\sqrt{3}}{2};-1)\\
&= 0.44469231901539865758\ldots
\end{aligned}$$

8. Mathematica Programs

Results in this paper can be obtained and modified using Mathematica code accessible as follows: return to the page from which this paper was downloaded, and click the "Select" box and choose to retrieve "Code".

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