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ON THE MAXIMAL DENSITY OF INTEGRAL SETS WHOSE DIFFERENCES AVOIDING THE WEIGHTED FIBONACCI NUMBERS

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Abstract

In an unpublished problem collection, Motzkin asks, how dense can a set S of positive integers be, if no two elements of S are allowed to differ by an element of the given set $\mathcal{P}$ of positive integers? The maximal density of such sets, denoted by $\mu(\mathcal{P})$, is known for $|\mathcal{P}| \leq 2$, and several other partial results are also known for the general case. We find some bounds and a few exact values of $\mu(\mathcal{P})$, where the elements P_i of the set $\mathcal{P}$ are defined by $P_i := P_{i-1} + P_{i-2}$, $i \geq 2$ with $P_0 = a, P_1 = b$. Notice that the elements of the sequence $\{P_i\}$ satisfy the same recurrence relation as that satisfied by the well-known Fibonacci numbers F_i with arbitrary initial values. Since $P_i = aF_{i-1} + bF_i$ for all $i \geq 0$, these numbers are also known as weighted Fibonacci numbers. This work generalizes an earlier work of Pandey on Fibonacci numbers.

1. Introduction

Let S be any set of nonnegative integers and let $S(x)$ denote the number of elements $n \in S$ such that $1 \leq n \leq x$, $x \in \mathbb{R}$. The upper density of S, denoted by $\overline{\delta}(S)$, is defined by $\overline{\delta}(S) := \overline{\lim\limits_{x\to\infty}} S(x)/x$. Given the set of positive integers $\mathcal{P}$, S is said to

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be a $\mathcal{P}$-set if $a \in S$, $b \in S$ implies that $a - b \notin \mathcal{P}$. The parameter of interest is the maximal density of a $\mathcal{P}$-set, defined by

$$\mu(\mathcal{P}) := \sup_{S} \overline{\delta}(S),$$

where the supremum is taken over all $\mathcal{P}$-sets S. Cantor and Gordon [2] establish the existence of $\mu(\mathcal{P})$ for any $\mathcal{P}$ and solve the problem for $|\mathcal{P}| \leq 2$. They also prove that

$$\mu(\mathcal{P}) \geq \kappa(\mathcal{P}) := \sup_{\gcd(c,m)=1} \frac{1}{m} \min_{p \in \mathcal{P}} |cp|_m, \tag{1.1}$$

where $|x|_m$ denotes the absolute value of the absolutely least remainder of x modulo m. A remark of Haralambis [6], gives an equivalent formulation for the right-hand expression of the above inequality. Hence, we can write

$$\kappa(\mathcal{P}) = \max_{\substack{m=p+q \\ 1 \leq k \leq \frac{m}{2}}} \frac{1}{m} \min_{p \in \mathcal{P}} |kp|_m, \tag{1.2}$$

where p and q are any two distinct elements of $\mathcal{P}$, with the condition that $\mathcal{P}$ has only finitely many elements.

A result of Cantor and Gordon [2] reduces the calculation of $\mu(\mathcal{P})$ for any general set $\mathcal{P}$ to those sets $\mathcal{P}$ whose elements are relatively prime. Haralambis [6] gives a useful upper bound for $\mu(\mathcal{P})$ and provides an expression for $\mu(\mathcal{P})$ for most members of the families $\{1, j, k\}$, and $\{1, 2, j, k\}$. Liu and Zhu [9] determine the value of $\mu(\mathcal{P})$ for most of the almost difference closed sets $\mathcal{P}$. They [10] further compute $\mu(D_{a,b,m})$ for $1 < a \leq b < m-1$, where $D_{a,b,m} = [1, a-1] \cup [b+1, m-1]$. Gupta and Tripathi [5] determine $\mu(\mathcal{P})$ where elements of $\mathcal{P}$ are in arithmetic progression. Pandey and Tripathi ([12], [13]) discuss this quantity for the families $\mathcal{P} = \{a, b, n(a+b)\}$ and for the sets related to arithmetic progressions.

In this paper, we consider the problem of determining $\mu(\mathcal{P})$ for the set $\mathcal{P} = \{a, b, a+b, \ldots, aF_{k-1} + bF_k\}$ with $\gcd(a, b) = 1$. We write $\mathcal{P} = \{P_0, P_1, P_2, \ldots, P_k\}$, with $P_0 = a$, $P_1 = b$ and $P_i = P_{i-1} + P_{i-2}$ for $i \geq 2$. The well known Fibonacci sequence $\{F_i\}_{i \geq 0}$ and Lucas sequence $\{L_i\}_{i \geq 0}$ are the special cases of the sequence $\{P_i\}_{i \geq 0}$. In Section 2, we evaluate a lower bound for $\mu(\mathcal{P})$ with $|\mathcal{P}| > 5$, by using some identities of the Fibonacci and Lucas sequences and the definition (1.2) of $\kappa(\mathcal{P})$. Whereas, for $|\mathcal{P}| \leq 4$, $\kappa(\mathcal{P})$ and $\mu(\mathcal{P})$ have been studied by Cantor and Gordon [2], and Liu and Zhu [9]. In Section 3, we investigate the values of $\kappa(\mathcal{P})$ and $\mu(\mathcal{P})$ when $|\mathcal{P}| = 5$.

The parameters $\mu(\mathcal{P})$ and $\kappa(\mathcal{P})$ are interesting and useful in the study of some other number theory as well as graph theory problems. The graph-theoretic connection of $\mu(\mathcal{P})$ is the *fractional chromatic number* of the distance graph generated by $\mathcal{P}$. For more detail, one may refer ([3], [9]). The parameter $\kappa(\mathcal{P})$, is related to the well-known conjecture on diophantine approximation due to Wills [15] and independently by Cusick [4], now known as the *lonely runner conjecture* due to Bienia et al. [1].

Due to Cantor and Gordon [2], $\mu(\mathcal{P}) = \kappa(\mathcal{P})$ for all $\mathcal{P}$ with $|\mathcal{P}| \leq 2$. Hence, it is very natural to ask the question of whether $\mu(\mathcal{P}) = \kappa(\mathcal{P})$ when $|\mathcal{P}| = 3$. Haralambis [6] and Liu and Zhu [9] have shown the existence of some infinite families of four-element sets with $\kappa(\mathcal{P}) < \mu(\mathcal{P})$. We give an infinite family of five-element sets $\mathcal{P}$ with $\kappa(\mathcal{P}) < \mu(\mathcal{P})$ in the last section.

2. Main Results

Before we go to our main results we give some identities concerning the Fibonacci and Lucas sequences, denoted respectively by $\{F_i\}_{i\geq 0}$ and $\{L_i\}_{i\geq 0}$, in the lemma given below. Notice that both Fibonacci and Lucas sequences are also defined for negative indices, denoted respectively by $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$. Hence, the identities given below are satisfied for all indices.

Lemma 2.1. *For all integers $m, n, k,$ and i, we have*

1. $F_{n+2} - F_{n-2} = L_n = F_{n-1} + F_{n+1}$.

2. $F_{n-2} + F_{n+2} = 3F_n$.

3. $L_{n-1} + L_{n+1} = 5F_n$.

4. $F_m F_{n+1} - F_{m+1} F_n = (-1)^n F_{m-n}$.

5. $F_{n+k} + (-1)^k F_{n-k} = L_k F_n$.

6. $L_{2n-1} L_i - 5F_{i-1} F_{2n} = (-1)^{i+1} L_{2n-i}$.

7. $$L_i = \begin{cases} 5\displaystyle\sum_{k=1}^{\frac{i}{2}} (-1)^{k-1} F_{i-(2k-1)} + (-1)^{\frac{i}{2}} L_0, & \text{if } i \text{ is even;} \\ 5\displaystyle\sum_{k=1}^{\frac{i-1}{2}} (-1)^{k-1} F_{i-(2k-1)} + (-1)^{\frac{i-1}{2}} L_1, & \text{if } i \text{ is odd.} \end{cases}$$

Proof. Identities (1), (2), and (3) are simple to observe. Identities (4) and (5) may be found in Koshy [8]. We prove identities (6) and (7) below.

6. We have

$$
\begin{aligned}
&L_{2n-1}L_i - 5F_{i-1}F_{2n} \\
&\qquad = L_{2n-1}L_i - \big(L_{i-2} + L_i\big)F_{2n} \quad \text{(using identity (3))} \\
&\qquad = \big(L_{2n-1} - F_{2n}\big)L_i - L_{i-2}F_{2n} \\
&\qquad = L_iF_{2n-2} - L_{i-2}F_{2n} \\
&\qquad = (-1)^iF_{2n-i-2} - (-1)^{i-2}F_{2n-i+2} \quad \text{(using identity (5))} \\
&\qquad = (-1)^iF_{2n-i-2} - (-1)^iF_{2n-i+2} \\
&\qquad = (-1)^{i+1}\big(F_{2n-i+2} - F_{2n-i-2}\big) \\
&\qquad = (-1)^{i+1}L_{2n-i} \quad \text{(using identity (1))}.
\end{aligned}
$$

7. Recursively using identity (3), we get

$$
\begin{aligned}
L_i &= 5F_{i-1} - L_{i-2} \\
&= 5\big(F_{i-1} - F_{i-3} + \cdots + (-1)^{k-1}F_{i-(2k-1)}\big) + (-1)^kL_{i-2k}.
\end{aligned}
$$

Thus, for even i,

$$
L_i = 5\big(\sum_{k=1}^{\frac{i}{2}}(-1)^{k-1}F_{i-(2k-1)}\big) + (-1)^{\frac{i}{2}}L_0;
$$

and for odd i,

$$
L_i = 5\big(\sum_{k=1}^{\frac{i-1}{2}}(-1)^{k-1}F_{i-(2k-1)}\big) + (-1)^{\frac{i-1}{2}}L_1.
$$

□

We write all possible initial values of $P_0 = a$ and $P_1 = b$ modulo 5. There are a total of twenty-five choices for the pair (a, b). But the choice $(a, b) = (5m, 5l)$, always yields $\gcd(a, b) \geq 5$. So, we consider only the remaining twenty-four cases. In the following five lemmas, we compute a lower bound of $\mu(\mathcal{P})$ for all possible choices of pairs (a, b) of initial values.

Lemma 2.2. *Let $\mathcal{P} = \{P_0, P_1, P_2, \ldots, P_k\}$, where $P_i = aF_{i-1} + bF_i$ for all $i \geq 0$ and $4n+1 \leq k \leq 4n+4$ with $n \geq 1$. If $(a, b) \in \{(5m, 5l+1), (5m+1, 5l+4), (5m+2, 5l+2), (5m+3, 5l), (5m+4, 5l+3) : l, m \in \mathbb{N} \cup \{0\}\}$ with* $\gcd(a, b) = 1$, *then*

$$
\kappa(\mathcal{P}) \geq \frac{1}{5} - \frac{2L_{2n-1}}{5(aL_{2n+1} + bL_{2n+2})}.
$$

Proof. Clearly, we have $2b - a \equiv 2 \pmod 5$ and $a + 3b \equiv 3 \pmod 5$. Set $q = P_{2n+1} + P_{2n+3} = aL_{2n+1} + bL_{2n+2}$. Then

$$\begin{aligned} q &= aL_{2n+1} + bL_{2n+2} \\ &= a\big(F_{2n} + F_{2n+2}\big) + b\big(3F_{2n+2} - 2F_{2n}\big) \\ &= (a - 2b)F_{2n} + (a + 3b)F_{2n+2} \\ &\equiv -2F_{2n} + 3F_{2n+2} \\ &\equiv 2\big(4F_{2n+2} - F_{2n}\big) \\ &\equiv 2\big(F_{2n-2} + F_{2n}\big) \\ &\equiv 2L_{2n-1} \pmod 5. \end{aligned}$$

Let $p = \frac{q-2L_{2n-1}}{5}$. We have

$$\begin{aligned} 2b\big(p - F_{2n-1}\big) - a\big(p + 2F_{2n}\big) &= \big(2b - a\big)p - \big(2bF_{2n-1} + 2aF_{2n}\big) \\ &= (2b - a)\frac{q - 2L_{2n-1}}{5} - \big(2bF_{2n-1} + 2aF_{2n}\big) \\ &= \frac{(2b - a)q}{5} - \frac{2\big(a(5F_{2n} - L_{2n-1}) + b(5F_{2n-1} + 2L_{2n-1})\big)}{5} \\ &= \frac{(2b - a)q}{5} - \frac{2(aL_{2n+1} + bL_{2n+2})}{5} \\ &= \frac{2b - a - 2}{5}q. \end{aligned}$$

Hence,

$$a(p + 2F_{2n}) \equiv 2b(p - F_{2n-1}) \pmod q. \tag{2.3}$$

We have $q \equiv -2F_{2n} + 3F_{2n+2} \equiv 2F_{2n+1} + F_{2n+2} \equiv L_{2n+2} \pmod 5$. Next, let $\gcd(a, q) = d$ and $\gcd(b, q) = d'$. This implies that $d | L_{2n+2}$, which implies $d \not\equiv 0 \pmod 5$ and d divides $2(p - F_{2n-1}) = \frac{2(q-L_{2n+2})}{5}$. Hence, there exists an integer x such that

$$ax \equiv 2(p - F_{2n-1}) \pmod q. \tag{2.4}$$

Similarly, $d' | L_{2n+1}$, which implies $d' \not\equiv 0 \pmod 5$ and d' divides $(p + 2F_{2n}) = \frac{(q-2L_{2n-1})}{5} + 2F_{2n} = \frac{(q+2L_{2n+1})}{5}$. Hence there exists an integer y such that

$$by \equiv (p + 2F_{2n}) \pmod q. \tag{2.5}$$

Moreover, congruence (2.3) implies that there is a common solution x_σ of the congruences (2.4) and (2.5), i.e.,

$$\begin{aligned} &ax_\sigma \equiv 2(p - F_{2n-1}) \pmod q, \\ \text{and} \quad &bx_\sigma \equiv (p + 2F_{2n}) \pmod q. \end{aligned}$$

The common solution is justified as follows: From (2.3), (2.4), and (2.5), we have that
$$ab(x-y) \equiv 2b(p - F_{2n-1}) - a(p + 2F_{2n}) \equiv 0 \pmod{q}.$$
Moreover, by the definitions of d and d', it follows that $\gcd(ab, q) = dd'$. Therefore, $x - y$ is divisible by $\frac{q}{dd'}$. Since $\gcd(d, d') = 1$, we know from Bézout's identity that there exist integers u and v such that
$$\frac{(x-y)dd'}{q} = ud + vd'.$$
This leads us to consider the integer z defined by
$$z := x - v\frac{q}{d} = y + u\frac{q}{d'}.$$
Clearly, from the definition of d and d', the integer z verifies $az \equiv ax$ and $bz \equiv by$ modulo q.

Since $P_i = aF_{i-1} + bF_i$, we have
$$\begin{aligned}
P_i x_\sigma &\equiv F_{i-1}\big(2(p - F_{2n-1})\big) + F_i(p + 2F_{2n}) \\
&= p(2F_{i-1} + F_i) + 2(F_iF_{2n} - F_{i-1}F_{2n-1}) \\
&= \frac{q - 2L_{2n-1}}{5} L_i + 2\big((-1)^{i-1}F_{2n-i+1} + F_{i-1}F_{2n}\big) \quad \text{(using identity (4))} \\
&= \frac{qL_i - 2(L_{2n-1}L_i - 5F_{i-1}F_{2n})}{5} + (-1)^{i-1}2F_{2n-i+1} \\
&= \frac{qL_i - (-1)^{i+1}2L_{2n-i}}{5} + (-1)^{i-1}2F_{2n-i+1} \quad \text{(using identity (6))} \\
&= \frac{qL_i - (-1)^{i+1}2(L_{2n-i} - 5F_{2n-i+1})}{5} \\
&= \frac{qL_i + (-1)^{i+1}2L_{2n-i+2}}{5} \pmod{q}.
\end{aligned}$$
Let i be even. By identity (7), we have
$$P_i x \equiv (-1)^{\frac{i}{2}} \left(\frac{qL_0 - (-1)^{\frac{i}{2}} 2L_{2n-i+2}}{5} \right) \pmod{q}.$$
Therefore,
$$P_i x_\sigma \equiv \begin{cases} \frac{qL_0 - 2L_{2n-i+2}}{5} \pmod{q}, & \text{if } i \equiv 0 \pmod{4}; \\ -\frac{qL_0 + 2L_{2n-i+2}}{5} \pmod{q}, & \text{if } i \equiv 2 \pmod{4}. \end{cases}$$
Next, let i be odd. We have
$$P_i x \equiv (-1)^{\frac{i-1}{2}} \left(\frac{qL_1 + (-1)^{\frac{i-1}{2}} L_{2n-i+2}}{5} \right) \pmod{q}.$$

Therefore,

$$P_i x_\sigma \equiv \begin{cases} \frac{qL_1+2L_{2n-i+2}}{5} \pmod{q}, & \text{if } i \equiv 1 \pmod{4}; \\ -\frac{qL_1-2L_{2n-i+2}}{5} \pmod{q}, & \text{if } i \equiv 3 \pmod{4}. \end{cases}$$

Thus, we see that

$$\min_{0\le i\le (2n+2)} \{|P_i x_\sigma|_q\} = \frac{q-2L_{2n-1}}{5}.$$

Using identity (5), for $0 \le i \le 2n+2$, we have that

(a) $F_{4n+3-i} = (-1)^i F_{i-1} + F_{2n+2-i}(F_{2n} + F_{2n+2})$,

(b) $F_{4n+4-i} = (-1)^i F_i + F_{2n+2-i}(F_{2n+1} + F_{2n+3})$.

By a simple manipulation, we get $P_{4n+4-i} = (-1)^i P_i + F_{2n+2-i}(P_{2n+1} + P_{2n+3}) = (-1)^i P_i + F_{2n+2-i}q$. Thus, $P_{4n+4-i}x_\sigma \equiv (-1)^i P_i x_\sigma \pmod{q}$. Therefore,

$$\min_{0\le i\le (4n+4)} \{|P_i x_\sigma|_q\} = \frac{q-2L_{2n-1}}{5}.$$

Notice that this absolute minimum is obtained corresponding to the congruences $P_3 x_\sigma \equiv P_{4n+1}x_\sigma \equiv \frac{q-2L_{2n-1}}{5} \pmod{q}$. Therefore, for $4n+1 \le k \le 4n+4$,

$$\min_{0\le i\le k} \{|P_i x_\sigma|_q\} = \frac{q-2L_{2n-1}}{5}.$$

Thus, by definition (1.2) of $\kappa(\mathcal{P})$, we get

$$\kappa(\mathcal{P}) \ge \frac{1}{5} - \frac{2L_{2n-1}}{5(aL_{2n+1} + bL_{2n+2})}.$$

This completes the proof. $\square$

Lemma 2.3. *Let $\mathcal{P} = \{P_0, P_1, P_2, \ldots, P_k\}$, where $P_i = aF_{i-1} + bF_i$ for all $i \ge 0$ and $4n-1 \le k \le 4n+2$ and $n \ge 1$, $k \ne 3, 4$. If $(a, b) \in \{(5m, 5l+2), (5m+1, 5l), (5m+2, 5l+3), (5m+3, 5l+1), (5m+4, 5l+4) : l, m \in \mathbb{N} \cup \{0\}\}$ with $\gcd(a, b) = 1$, then*

$$\kappa(\mathcal{P}) \ge \frac{1}{5} - \frac{L_{2n-1}}{5(aL_{2n-1} + bL_{2n})}.$$

Proof. Clearly, we have $2b - a \equiv 4 \pmod{5}$ and $3b - 4a \equiv 1 \pmod{5}$. Set $q = P_{2n-1} + P_{2n+1} = aL_{2n-1} + bL_{2n}$. Then

$$\begin{aligned} q &= aL_{2n-1} + bL_{2n} \\ &= (4a-3b)F_{2n} + (-a+2b)F_{2n+2} \\ &\equiv 4F_{2n} - F_{2n+2} \\ &\equiv F_{2n-2} + F_{2n} \\ &\equiv L_{2n-1} \pmod{5}. \end{aligned}$$

Let $p = \frac{q-L_{2n-1}}{5}$. We have

$$ap \equiv b(2p + F_{2n}) \pmod{q}. \tag{2.6}$$

Again $2q \equiv 3F_{2n} - 2F_{2n+2} = F_{2n-2} - F_{2n+2} = -L_{2n} \pmod{5}$. Next, let $\gcd(a, q) = d$, and $\gcd(b, q) = d'$. This implies that $d|L_{2n}$, which implies $d \not\equiv 0 \pmod{5}$ and d divides $(2p + F_{2n}) = \frac{2(q-L_{2n-1})}{5} + F_{2n} = \frac{2q+L_{2n}}{5}$. Hence, there exists an integer x such that

$$ax \equiv (2p + F_{2n}) \pmod{q}. \tag{2.7}$$

Similarly, $d'|L_{2n-1}$, which implies $d' \not\equiv 0 \pmod{5}$ and d' divides $p = \frac{(q-L_{2n-1})}{5}$. Hence, there exists an integer y such that

$$by \equiv p \pmod{q}. \tag{2.8}$$

Moreover, as in Lemma 2.2, congruence (2.6) implies that there is a common solution x_σ of the congruences (2.7) and (2.8), i.e.,

$$\begin{aligned} ax_\sigma &\equiv 2p + F_{2n} \pmod{q},\\ \text{and} \qquad bx_\sigma &\equiv p \pmod{q}. \end{aligned}$$

Since $P_i = aF_{i-1} + bF_i$, we have

$$\begin{aligned} P_i x_\sigma &\equiv F_{i-1}(2p + F_{2n}) + F_i(p)\\ &= p(2F_{i-1} + F_i) + F_{2n}F_{i-1}\\ &= pL_i + F_{2n}F_{i-1}\\ &= \frac{(q - L_{2n-1})L_i + 5F_{2n}F_{i-1}}{5}\\ &= \frac{qL_i - (L_{2n-1}L_i - 5F_{2n}F_{i-1})}{5}\\ &= \frac{qL_i - (-1)^{i+1}L_{2n-i}}{5} \pmod{q} \quad \text{(using identity (6))}. \end{aligned}$$

Let i be even. By identity (7), we have

$$P_i x \equiv (-1)^{\frac{i}{2}} \left(\frac{qL_0 + (-1)^{\frac{i}{2}} L_{2n-i}}{5} \right) \pmod{q}.$$

Therefore,

$$P_i x_\sigma \equiv \begin{cases} \frac{qL_0 + L_{2n-i}}{5} \pmod{q}, & \text{if } i \equiv 0 \pmod{4};\\ -\frac{qL_0 - L_{2n-i}}{5} \pmod{q}, & \text{if } i \equiv 2 \pmod{4}. \end{cases}$$

Next, let i be odd. We have

$$P_i x \equiv (-1)^{\frac{i-1}{2}} \left(\frac{qL_1 - (-1)^{\frac{i-1}{2}} L_{2n-i}}{5} \right) \pmod{q}.$$

Therefore,

$$P_i x_\sigma \equiv \begin{cases} \frac{qL_1 - L_{2n-i}}{5} \pmod{q}, & \text{if } i \equiv 1 \pmod{4}; \\ -\frac{qL_1 + L_{2n-i}}{5} \pmod{q}, & \text{if } i \equiv 3 \pmod{4}. \end{cases}$$

Thus, we see that

$$\min_{0 \le i \le (2n)} \{|p_i x_\sigma|_q\} = \frac{q - L_{2n-1}}{5}.$$

Using identity (5), for $0 \le i \le 2n$, we have that

(a) $F_{4n-1-i} = (-1)^i F_{i-1} + F_{2n-i}(F_{2n-2} + F_{2n})$,

(b) $F_{4n-i} = (-1)^i F_i + F_{2n-i}(F_{2n-1} + F_{2n+1})$.

By a simple manipulation, we get $P_{4n-i} = (-1)^i P_i + F_{2n-i}(P_{2n-1} + P_{2n+1}) = (-1)^i P_i + F_{2n-i} q$. Thus, $P_{4n-i} x_\sigma \equiv (-1)^i P_i x_\sigma \pmod{q}$. Whereas, $P_{4n+1}\ x_\sigma \equiv (P_0 - P_1) x_\sigma \equiv p + F_{2n} \pmod{q}$ and $P_{4n+2}\ x_\sigma = (P_{4n+1} + P_{4n}) x_\sigma \equiv P_{4n+1} x_\sigma + P_0 x_\sigma \equiv 3p + 2F_{2n} \equiv -(2p - F_{2n-1}) \pmod{q}$. Therefore, $\min\limits_{0 \le i \le (4n+2)} \{|P_i x_\sigma|_q\} = \frac{q - L_{2n-1}}{5}$. Notice that this absolute minimum is obtained corresponding to the congruences $P_1 x_\sigma \equiv -P_{4n-1} x_\sigma \equiv \frac{q - L_{2n-1}}{5} \pmod{q}$. Therefore, for $4n-1 \le k \le 4n+2$,

$$\min_{0 \le i \le k} \{|P_i x_\sigma|_q\} = \frac{q - L_{2n-1}}{5}.$$

Thus, by definition (1.2) of $\kappa(\mathcal{P})$, we get

$$\kappa(\mathcal{P}) \ge \frac{1}{5} - \frac{L_{2n-1}}{5(aL_{2n-1} + bL_{2n})}.$$

This completes the proof. $\square$

Lemma 2.4. *Let* $\mathcal{P} = \{P_0, P_1, P_2, \ldots, P_k\}$, *where* $P_i = aF_{i-1} + bF_i$ *for all* $i \ge 0$ *and* $4n \le k \le 4n+3$ *with* $n \ge 1$, $k \ne 4$. *If* $(a, b) \in \{(5m, 5l+3), (5m+1, 5l+1), (5m+2, 5l+4), (5m+3, 5l+2), (5m+4, 5l) : l, m \in \mathbb{N} \cup \{0\}\}$ *with* $\gcd(a, b) = 1$, *then*

$$\kappa(\mathcal{P}) \ge \frac{1}{5} - \frac{L_{2n-1}}{5(aL_{2n+1} + bL_{2n+2})}.$$

Proof. Clearly, we have $2b - a \equiv 1 \pmod 5$ and $3b + a \equiv 4 \pmod 5$. Set $q = P_{2n+1} + P_{2n+3} = aL_{2n+1} + bL_{2n+2}$. Then

$$\begin{aligned} q &= (a-2b)F_{2n} + (a+3b)F_{2n+2} \\ &\equiv 4F_{2n} - F_{2n+2} \\ &= F_{2n-2} + F_{2n} \\ &\equiv L_{2n-1} \pmod 5. \end{aligned}$$

Let $p = \frac{q - L_{2n-1}}{5}$. We have

$$a(p + F_{2n}) \equiv b(2p - F_{2n-1}) \pmod q. \tag{2.9}$$

We also have $2q \equiv 2(-F_{2n} + 4F_{2n+2}) \equiv -2F_{2n} + 3F_{2n+2} \equiv 2F_{2n+1} + F_{2n+2} \equiv L_{2n+2} \pmod 5$. Next, let $\gcd(a,q) = d$ and $\gcd(b,q) = d'$. This implies that $d | L_{2n+2}$, which implies $d \not\equiv 0 \pmod 5$ and d divides $(2p - F_{2n-1}) = \frac{(2q - L_{2n+2})}{5}$. Hence, there exists an integer x such that

$$ax \equiv (2p - F_{2n-1}) \pmod q. \tag{2.10}$$

Similarly, $d' | L_{2n+1}$, implies $d' \not\equiv 0 \pmod 5$ and d' divides $(p + F_{2n}) = \frac{(q + L_{2n+1})}{5}$. Hence, there exists an integer y such that

$$by \equiv (p + F_{2n}) \pmod q. \tag{2.11}$$

Moreover, congruence (2.9) implies that there is a common solution x_σ of the congruences (2.10) and (2.11), i.e.,

$$\begin{aligned} ax_\sigma &\equiv 2p - F_{2n-1} \pmod q, \\ \text{and} \qquad bx_\sigma &\equiv p + F_{2n} \pmod q. \end{aligned}$$

Since $P_i = aF_{i-1} + bF_i$, we have

$$\begin{aligned} P_i x_\sigma &\equiv F_{i-1}(2p - F_{2n-1}) + F_i\big(\sigma(p + F_{2n})\big) \\ &= p(2F_{i-1} + F_i) + (F_i F_{2n} - F_{i-1}F_{2n-1}) \\ &= \frac{q - L_{2n-1}}{5} L_i + ((-1)^{i-1} F_{2n-i+1} + F_{i-1}F_{2n}) \quad \text{(using identity (4))} \\ &= \frac{qL_i - (L_{2n-1}L_i - 5F_{i-1}F_{2n})}{5} + (-1)^{i-1} 2F_{2n-i+1} \\ &= \frac{qL_i - (-1)^{i+1} L_{2n-i}}{5} + (-1)^{i-1} F_{2n-i+1} \quad \text{(using identity (6))} \\ &= \frac{qL_i - (-1)^{i+1}(L_{2n-i} - 5F_{2n-i+1})}{5} \\ &= \frac{qL_i + (-1)^{i+1} L_{2n-i+2}}{5} \pmod q. \end{aligned}$$

Let i be even. By identity (5), we have

$$P_i x_\sigma \equiv (-1)^{\frac{i}{2}} \left(\frac{2q - (-1)^{\frac{i}{2}} L_{2n-i+2}}{5} \right) \pmod{q}.$$

Therefore,

$$P_i x_\sigma \equiv \begin{cases} \frac{2q - L_{2n-i+2}}{5} \pmod{q}, & \text{if } i \equiv 0 \pmod{4}; \\ -\frac{2q + L_{2n-i+2}}{5} \pmod{q}, & \text{if } i \equiv 2 \pmod{4}. \end{cases}$$

Next, let i be odd. We have

$$P_i x \equiv (-1)^{\frac{i-1}{2}} \left(\frac{L_1\ q + (-1)^{\frac{i-1}{2}} L_{2n-i+2}}{5} \right) \pmod{q}.$$

Therefore,

$$P_i x_\sigma \equiv \begin{cases} \frac{q + L_{2n-i+2}}{5} \pmod{q}, & \text{if } i \equiv 1 \pmod{4}; \\ -\frac{q - L_{2n-i+2}}{5} \pmod{q}, & \text{if } i \equiv 3 \pmod{4}. \end{cases}$$

Thus, we see that

$$\min_{0 \le i \le (2n+2)} \{ |P_i x_\sigma|_q \} = \frac{q - L_{2n-1}}{5}.$$

Using identity (5), for $0 \le i \le 2n+2$, we have that

(a) $F_{4n+3-i} = (-1)^i F_{i-1} + F_{2n+2-i}(F_{2n} + F_{2n+2})$,

(b) $F_{4n+4-i} = (-1)^i F_i + F_{2n+2-i}(F_{2n+1} + F_{2n+3})$.

By a simple manipulation, we get $P_{4n+4-i} = (-1)^i P_i + F_{2n+2-i}(P_{2n+1} + P_{2n+3}) = (-1)^i P_i + F_{2n+2-i} q$. Thus, $P_{4n+4-i} x_\sigma \equiv (-1)^i P_i x_\sigma \pmod{q}$. Therefore,

$$\min_{0 \le i \le (4n+4)} \{ |P_i x_\sigma|_q \} = \frac{q - L_{2n-1}}{5}.$$

Notice that this absolute minimum is obtained corresponding to the congruences $P_3 x_\sigma \equiv P_{4n+1} x_\sigma \equiv \frac{q - 2L_{2n-1}}{5} \pmod{q}$. Therefore, for $4n+1 \le k \le 4n+4$,

$$\min_{0 \le i \le k} \{ |P_i x_\sigma|_q \} = \frac{q - L_{2n-1}}{5}.$$

Thus, by definition (1.2) of $\kappa(\mathcal{P})$, we get

$$\kappa(\mathcal{P}) \ge \frac{1}{5} - \frac{L_{2n-1}}{5(aL_{2n+1} + bL_{2n+2})}.$$

This completes the proof. $\square$

Lemma 2.5. *Let $\mathcal{P} = \{P_0, P_1, P_2, \ldots, P_k\}$, where $P_i = aF_{i-1} + bF_i$ and $4n \le k \le 4n+3$ with $n \ge 1$. If $(a,b) \in \{(5m, 5l+4), (5m+1, 5l+2), (5m+2, 5l), (5m+3, 5l+3), (5m+4, 5l+1) : l, m \in \mathbb{N} \cup \{0\}\}$ with $\gcd(a,b) = 1$, then*

$$\kappa(\mathcal{P}) \ge \frac{1}{5} - \frac{L_{2n-1}}{5(aL_{2n} + bL_{2n+1})}.$$

Proof. Clearly, we have $2a + b \equiv -1 \pmod 5$ and $3a - b \equiv -4 \pmod 5$. Set $q = P_{2n} + P_{2n+2} = aL_{2n} + bL_{2n+1}$. Then

$$\begin{aligned} q &= aL_{2n} + bL_{2n+1} \\ &= (2a+b)F_{2n+2} - (3a-b)F_{2n} \\ &\equiv -F_{2n+2} + 4F_{2n} \\ &\equiv F_{2n-2} + F_{2n} \\ &\equiv L_{2n-1} \pmod 5. \end{aligned}$$

Let $p = \frac{q - L_{2n-1}}{5}$. We have

$$a(2p + F_{2n}) \equiv -b(p + F_{2n}) \pmod q. \tag{2.12}$$

We also have $q \equiv L_{2n-1} - 5F_{2n} \equiv -L_{2n+1} \pmod 5$. Next, let $\gcd(a,q) = d$ and $\gcd(b,q) = d'$. This implies that $d | L_{2n+1}$, which implies $d \not\equiv 0 \pmod 5$ and d divides $(p + F_{2n}) = \frac{(q + L_{2n+1})}{5}$. Hence, there exists an integer x such that

$$ax \equiv -(p + F_{2n}) \pmod q. \tag{2.13}$$

Similarly, $d' | L_{2n}$, implies $d' \not\equiv 0 \pmod 5$ and d' divides $(2p + F_{2n}) = \frac{2(q - L_{2n-1})}{5} + F_{2n} = \frac{(2q + 2L_{2n})}{5}$. Hence, there exists an integer y such that

$$by \equiv (2p + F_{2n}) \pmod q. \tag{2.14}$$

Moreover, congruence (2.12) implies that there is a common solution x_σ of the congruences (2.13) and (2.14), i.e.,

$$\begin{aligned} & ax_\sigma \equiv -(p + F_{2n}) \pmod q, \\ \text{and} \quad & bx_\sigma \equiv 2p + F_{2n} \pmod q. \end{aligned}$$

Since $P_i = aF_{i-1} + bF_i$, we have

$$\begin{aligned}
P_i x_\sigma &\equiv F_{i-1}\left(-(p+F_{2n})\right) + F_i(2p+F_{2n}) \\
&= p(2F_i - F_{i-1}) + F_{2n}F_{i-2} \\
&= p(L_{i-1}) + F_{2n}F_{i-2} \\
&= \frac{(q-L_{2n-1})(L_{i-1}) + 5F_{2n}F_{i-2}}{5} \\
&= \frac{qL_{i-1} - (L_{2n-1}L_{i-1} - 5F_{2n}F_{i-2})}{5} \\
&= \frac{qL_{i-1} - (-1)^i L_{2n-i+1}}{5} \pmod{q} \quad \text{(using identity (6))}.
\end{aligned}$$

Let i be even. By identity (7), we have

$$P_i x \equiv (-1)^{\frac{i-2}{2}} \left(\frac{L_1\ q - (-1)^{\frac{i+2}{2}} L_{2n-i+1}}{5} \right) \pmod{q}.$$

Therefore,

$$P_i x_\sigma \equiv \begin{cases} -\frac{q+L_{2n-i+1}}{5} \pmod{q}, & \text{if } i \equiv 0 \pmod{4}; \\ \frac{q-L_{2n-i+1}}{5} \pmod{q}, & \text{if } i \equiv 2 \pmod{4}. \end{cases}$$

Next, let i be odd. Then, we have

$$P_i x_\sigma \equiv (-1)^{\frac{i-1}{2}} \left(\frac{2q - (-1)^{\frac{i+1}{2}} L_{2n-i+1}}{5} \right) \pmod{q}.$$

Therefore,

$$P_i x_\sigma \equiv \begin{cases} \frac{2q+L_{2n-i+1}}{5} \pmod{q}, & \text{if } i \equiv 1 \pmod{4}; \\ -\frac{2q-L_{2n-i+1}}{5} \pmod{q}, & \text{if } i \equiv 3 \pmod{4}. \end{cases}$$

Thus, we see that

$$\min_{0 \le i \le (2n+1)} \{|P_i x_\sigma|_q\} = \frac{q - L_{2n-1}}{5}. \tag{2.15}$$

Using identity (5), for $0 \le i \le 2n+1$, we have that

(a) $F_{4n+1-i} = (-1)^{i-1}F_{i-1} + F_{2n+1-i}(F_{2n-1} + F_{2n+1})$,

(b) $F_{4n+2-i} = (-1)^{i-1}F_i + F_{2n+1-i}(F_{2n} + F_{2n+2})$.

By a simple manipulation, we get $P_{4n+2-i} = (-1)^{i-1}P_i + F_{2n+1-i}(P_{2n} + P_{2n+2}) = (-1)^{i-1}P_i + F_{2n+1-i}q$. Thus, $P_{4n+2-i}x_\sigma \equiv (-1)^{i-1}P_i x_\sigma \pmod{q}$ whereas $P_{4n+3}x_\sigma = P_{4n+2}x_\sigma + P_{4n+1}x_\sigma \equiv (-P_0 + P_1)x_\sigma \equiv 3p + 2F_{2n} \equiv -(2p - F_{2n-1}) \pmod{q}$.

Therefore, $\min\limits_{0\le i\le (4n+3)}\{|P_i x_\sigma|_q\} = \frac{q-L_{2n-1}}{5}$. Notice that this absolute minimum is obtained corresponding to the congruences $P_2 x_\sigma \equiv -P_{4n}x_\sigma \equiv \frac{q-2L_{2n-1}}{5} \pmod q$. Therefore, for $4n \le k \le 4n+3$,

$$\min_{0\le i\le k}\{|P_i x_\sigma|_q\} = \frac{q-L_{2n-1}}{5}.$$

Thus, by definition (1.2) of $\kappa(\mathcal{P})$, we get

$$\kappa(\mathcal{P}) \ge \frac{1}{5} - \frac{L_{2n-1}}{5(aL_{2n}+bL_{2n+1})}.$$

This completes the proof. □

The following corollary due to Pandey [11] may be obtained as a special case of the above lemma.

Corollary 2.1. *Let $\mathcal{P} = \{F_2, F_3, \ldots, F_t\}$ and $n \ge 1$ be an integer such that $4n+2 \le t \le 4n+5$, then*

$$\kappa(\mathcal{P}) \ge \frac{F_{2n+1}}{F_{2n+2}+F_{2n+4}}.$$

Proof. If $a = 1$ and $b = 2$, then $P_i = F_{i+2}$. So, by the above lemma

$$\begin{aligned}
\kappa(\mathcal{P}) &\ge \frac{1}{5} - \frac{L_{2n-1}}{5(L_{2n}+2L_{2n+1})} \\
&= \frac{(L_{2n}+2L_{2n+1}) - L_{2n-1}}{5(L_{2n+1}+L_{2n+2})} \\
&= \frac{L_{2n}+L_{2n+2}}{5L_{2n+3}} \\
&= \frac{5F_{2n+1}}{5L_{2n+3}} \\
&= \frac{F_{2n+1}}{F_{2n+2}+F_{2n+4}}.
\end{aligned}$$ □

Lemma 2.6. *Let $\mathcal{P} = \{P_0, P_1, P_2, \ldots, P_k\}$, where $P_i = aF_{i-1} + bF_i$ for all $i \ge 0$ with $k \ge 5$. If $(a,b) \in \{(5m+1, 5l+3), (5m+2, 5l+1), (5m+3, 5l+4), (5m+4, 5l+2) : l, m \in \mathbb{N}^*\}$ with $\gcd(a,b) = 1$, then*

$$\kappa(\mathcal{P}) \ge \frac{1}{5}.$$

Proof. Using identity (7), we may show that 5 does not divide $P_i = aF_{i-1} + bF_i$ for any i and for any $(a,b) \in \{(5m+1, 5l+3), (5m+2, 5l+1), (5m+3, 5l+4), (5m+4, 5l+2) : l, m \in \mathbb{N} \cup \{0\}\}$. Set $c = 1$ and $m = 5$. Then, by definition (1.1) of $\kappa(\mathcal{P})$, we have $\kappa(\mathcal{P}) \ge \frac{1}{5}$. This completes the proof. □

Corollary 2.2. *Let $L = \{L_0, L_1, \ldots, L_k\}$ with $k \geq 3$. Then $\mu(L) = \frac{1}{5}$.*

Proof. If $a = 2$ and $b = 1$, then $P_i = 2F_{i-1} + F_i = L_i$. So $\mathcal{P} = L$. Hence, from the above lemma $\mu(L) \geq \kappa(L) \geq \frac{1}{5}$. On the other hand, by a result of Liu and Zhu [9], $\mu(\{L_0, L_1, L_2, L_3\}) = \mu\{2, 1, 3, 4\} = \frac{1}{5}$, which gives $\mu(L) \leq \mu(\{L_0, L_1, L_2, L_3\}) = \frac{1}{5}$. Hence, $\mu(L) = \frac{1}{5}$. □

3. The Values of $\kappa(\mathcal{P})$ and $\mu(\mathcal{P})$ When $|\mathcal{P}| \leq 5$

Consider the set $\mathcal{P} = \{a, b, a+b, \ldots, aF_{k-1} + bF_k\}$ with $\gcd(a, b) = 1$. Cantor and Gordon [2] completely determined both $\kappa(\mathcal{P})$ and $\mu(\mathcal{P})$ when $k = 0$ or $k = 1$.

Theorem 3.1 ([2], Theorem 3)**.** *Let $\mathcal{P} = \{a\}$ or $\mathcal{P} = \{a, b\}$ with $\gcd(a, b) = 1$ and both a and b are odd. Then $\mu(\mathcal{P}) = \kappa(\mathcal{P}) = \frac{1}{2}$.*

Theorem 3.2 ([2], Theorem 4)**.** *Let $\mathcal{P} = \{a, b\}$ with $\gcd(a, b) = 1$ and a and b are of opposite parity. Then $\mu(\mathcal{P}) = \kappa(\mathcal{P}) = \frac{a+b-1}{2(a+b)}$.*

For $k = 2$ or $\mathcal{P} = \{a, b, a+b\}$, Rabinowitz and Proulx [14] gave a lower bound for $\mu(\mathcal{P})$ and conjectured that the bound is the exact value. Liu and Zhu [9] confirmed their conjecture and completely determined the values of $\kappa(\mathcal{P})$ and $\mu(\mathcal{P})$.

Theorem 3.3 ([9], Theorem 3.1)**.** *Let $\mathcal{P} = \{a, b, a+b\}$, where $0 < a < b$ and $gcd(a, b) = 1$. Then*

$$\mu(\mathcal{P}) = \kappa(\mathcal{P}) = \begin{cases} \frac{1}{3}, & \textit{if } b \equiv a \pmod 3; \\ \frac{2a+b-1}{3(2a+b)}, & \textit{if } b \equiv a+1 \pmod 3; \\ \frac{a+2b-1}{3(a+2b)}, & \textit{if } b \equiv a+2 \pmod 3. \end{cases}$$

They [9] further computed the value of $\kappa(\mathcal{P})$ for the four-element set $\mathcal{P} = \{x, y, y-x, x+y\}, y > x$ and gave a better lower bound than $\kappa(\mathcal{P})$ for $\mu(\mathcal{P})$ when both x and y are odd. The case when x and y are of opposite parity, has been settled by Kemnitz and Kolberg [7].

Theorem 3.4 ([9], Lemma 4.1)**.** *Let $\mathcal{P} = \{x, y, y-x, y+x\}, y > x$, where $gcd(x, y) = 1$. If $x = 2k+1$ and $y = 2m+1$, then $\mu(\mathcal{P}) \geq \frac{(k+1)m}{4(k+1)m+1}$. If x, y are of opposite parity, then $\mu(\mathcal{P}) = 1/4$.*

Theorem 3.5 ([9], Corollary 5.3)**.** *Let $\mathcal{P} = \{x, y, y-x, y+x\}$ with $gcd(x, y) = 1$.*

Let $\phi_4(n)$ *denote* $\lfloor \frac{n}{4} \rfloor / n$*. Then*

$$\kappa(\mathcal{P}) = \begin{cases} \phi_4(2y+x), & \text{if } x \equiv 0 \pmod 4 \text{ and } y \equiv 3 \pmod 4, \text{ or} \\ & \text{if } x \equiv 1 \pmod 4 \text{ and } y \equiv 0 \pmod 4, \text{ or} \\ & \text{if } x \equiv 3 \pmod 4 \text{ and } y \equiv 1,3 \pmod 4; \\ \phi_4(2y+x), & \text{if } x \equiv 0 \pmod 4 \text{ and } y \equiv 1 \pmod 4, \text{ or} \\ & \text{if } x \equiv 1 \pmod 4 \text{ and } y \equiv 3 \pmod 4, \text{ and } y < 3x \\ \phi_4(2y-x), & \text{if } x \equiv 3 \pmod 4 \text{ and } y \equiv 0 \pmod 4, \text{ or} \\ & \text{if } x \equiv y \equiv 1 \pmod 4, \text{ or} \\ & \text{if } x \equiv 1 \pmod 4 \text{ and } y \equiv 3 \pmod 4, \text{ and } y \geq 3x. \end{cases}$$

Liu and Zhu [9] also gave an infinite family of sets $\mathcal{P}$ satisfying $\kappa(\mathcal{P}) < \mu(\mathcal{P})$.

For $k = 3$ or $\mathcal{P} = \{a, b, a+b, a+2b\} = \{b, a+b, a, a+2b\}$, using the results discussed in [9] for $\mathcal{P} = \{x, y, y-x, y+x\}$, we can estimate the value of $\mu(\mathcal{P})$ and achieve $\kappa(\mathcal{P})$.

The rest of this section deals with the case $k = 4$, i.e., when $\mathcal{P} = \{a, b, a+b, a+2b, 2a+3b\}$.

Lemma 3.1. *Let* $\mathcal{P} = \{a, b, a+b, a+2b, 2a+3b\}$ *with* $\gcd(a, b) = 1$*. Then* $\kappa(\mathcal{P}) \leq \mu(\mathcal{P}) \leq 1/4$.

Proof. Since both the sets $\{b, a+b, a, a+2b\}$ and $\{a+b, a+2b, b, 2a+3b\}$ are proper subsets of $\mathcal{P}$, we have, $\mu(\mathcal{P}) \leq \mu(\{b, a+b, a, a+2b\})$ and $\mu(\mathcal{P}) \leq \mu(\{a+b, a+2b, b, 2a+3b\})$. One can easily verify that for all pairs of values of a, b with $\gcd(a, b) = 1$, either b and $a+b$ or $a+b$ and $a+2b$ are of opposite parity. Therefore, using Theorem 3.4, $\kappa(\mathcal{P}) \leq \mu(\mathcal{P}) \leq 1/4$. □

Theorem 3.6. *Let* $\mathcal{P} = \{a, b, a+b, a+2b, 2a+3b\}$ *with* $\gcd(a, b) = 1$*. If* a *and* b *both are odd, then* $\mu(\mathcal{P}) = 1/4$*. In particular,* $\kappa(\mathcal{P}) = \mu(\mathcal{P}) = 1/4$*, when both* a *and* b *are* $1 \pmod 4$ *or both* a *and* b *are* $3 \pmod 4$.

Proof. By Lemma 3.1, we have $\mu(\mathcal{P}) \leq 1/4$. It suffices to show that $\mu(\mathcal{P}) \geq 1/4$. Note that $\mathcal{P}$ contains only one even element and that is $a+b$. We consider the set

$$S = \bigcup_{i \geq 0} \{2i(a+b), 2i(a+b)+2, \ldots, (2i+1)(a+b)-2\}.$$

Clearly S is a periodic set with period $2(a+b)$. It is not difficult to show that S is a $\mathcal{P}$-set and $|S \cap \{0, 1, \ldots 2(a+b)-1\}| = \frac{a+b}{2}$. Therefore, S has density $\frac{a+b}{2(2(a+b))} = \frac{1}{4}$. This implies that $\mu(\mathcal{P}) \geq 1/4$.

However, when both a and b are $1 \pmod 4$, or both a and b are $3 \pmod 4$, none of the elements in $\mathcal{P}$ is a multiple of 4. Let $c = 1$ and $m = 4$. Then, $\min\{|ca|_m, |cb|_m, |c(a+b)|_m, |c(a+2b)|_m, |c(2a+3b)|_m\} = 1$. Therefore, by definition (1.1) of $\kappa(\mathcal{P})$, we have $\kappa(\mathcal{P}) \geq \frac{1}{4}$. Hence, $\kappa(\mathcal{P}) = \mu(\mathcal{P}) = 1/4$. □

In the next theorem, we evaluate $\kappa(\mathcal{P})$, where $\mathcal{P} = \{a, b, a+b, a+2b, 2a+3b\}$ for the remaining possible pairs (a, b).

Theorem 3.7. *Let $\mathcal{P} = \{a, b, a+b, a+2b, 2a+3b\}$ with $\gcd(a, b) = 1$. Let $\phi_4(n)$ denote $\lfloor \frac{n}{4} \rfloor / n$. Then*

$$\kappa(\mathcal{P}) = \begin{cases} \phi_4(2a+2b), & \text{if } b-a \equiv 1 \pmod 4; \\ \phi_4(a+3b), & \text{if } b-a \equiv 2 \pmod 4; \\ \phi_4(a+3b), & \text{if } b-a \equiv 3 \pmod 4. \end{cases}$$

Proof. Let $\beta(\mathcal{P})$ be the corresponding value on the right-hand side of the equality. First we show that $\beta(\mathcal{P}) \le \kappa(\mathcal{P})$. Let σ be either $+1$ or -1. We consider three cases for three different values of $\beta(\mathcal{P})$.

`Case 1.` $(\beta(\mathcal{P}) = \phi_4(2a+2b))$.

In this case, $b - a \equiv 1 \pmod 4$. Set $q = 2a+2b$. We have $q \equiv 2 \pmod 4$. Choose x_σ such that

$$\begin{aligned} ax_\sigma &\equiv \sigma\left(\frac{a+b-1}{2}\right) \pmod q, \\ \text{and} \qquad bx_\sigma &\equiv \sigma\left(\frac{a+b+1}{2}\right) \pmod q. \end{aligned}$$

Then

$$(a+b)x_\sigma \equiv \sigma\,(a+b) \pmod q.$$

Whereas,

$$\begin{aligned} (a+2b) &\equiv -a \pmod q, \\ \text{and} \qquad (2a+3b) &\equiv b \pmod q. \end{aligned}$$

Therefore, $\min\{|ax_\sigma|_q, |bx_\sigma|_q, |(a+b)x_\sigma|_q, |(a+2b)x_\sigma|_q, |(2a+3b)x_\sigma|_q\} = \frac{a+b-1}{2}$. This implies that $\kappa(\mathcal{P}) \ge \frac{a+b-1}{2(2a+2b)} = \phi_4(2a+2b) = \beta(\mathcal{P})$.

`Case 2.` $(\beta(\mathcal{P}) = \phi_4(a+3b))$.

In this case, $b - a \equiv 2 \pmod 4$. Set $q = a+3b$. We have $q \equiv 2 \pmod 4$. Choose x_σ such that

$$\begin{aligned} ax_\sigma &\equiv \sigma\left(\frac{a+3b+6}{2}\right) \pmod q, \\ \text{and} \qquad bx_\sigma &\equiv \sigma\left(\frac{a+3b-2}{2}\right) \pmod q. \end{aligned}$$

Since,

$$
\begin{aligned}
(a+b) &\equiv -2b \pmod{q},\\
(a+2b) &\equiv -b \pmod{q},\\
\text{and} \qquad (2a+3b) &\equiv a \pmod{q},
\end{aligned}
$$

we have, $\min\{|ax_\sigma|_q, |bx_\sigma|_q, |(a+b)x_\sigma|_q, |(a+2b)x_\sigma|_q, |(2a+3b)x_\sigma|_q\} = \frac{a+3b-2}{4}$. This implies that $\kappa(\mathcal{P}) \geq \frac{a+3b-2}{4(a+3b)} = \phi_4(a+3b) = \beta(\mathcal{P})$.

`Case 3.` $(\beta(\mathcal{P}) = \phi_4(a+3b))$.

In this case, $b-a \equiv 3 \pmod{4}$. Set $q = a+3b$. We have $q \equiv 1 \pmod{4}$. Choose x_σ such that

$$
ax_\sigma \equiv \sigma\left(\frac{a+3b+3}{4}\right) \pmod{q} \quad \text{and} \quad bx_\sigma \equiv \sigma\left(\frac{a+3b-1}{4}\right) \pmod{q}.
$$

Since,

$$
\begin{aligned}
(a+b) &\equiv -2b \pmod{q},\\
(a+2b) &\equiv -b \pmod{q},\\
\text{and} \qquad (2a+3b) &\equiv a \pmod{q},
\end{aligned}
$$

we have, $\min\{|ax_\sigma|_q, |bx_\sigma|_q, |(a+b)x_\sigma|_q, |(a+2b)x_\sigma|_q, |(2a+3b)x_\sigma|_q\} = \frac{a+3b-1}{4}$. This implies that $\kappa(\mathcal{P}) \geq \frac{a+3b-1}{4(a+3b)} = \phi_4(a+3b) = \beta(\mathcal{P})$.

To show that the equality holds, we observe that in all above cases, $\beta(\mathcal{P})$ values are equal to

$$
\max\left\{\frac{p}{q} : \frac{p}{q} < \frac{1}{4} \text{ and q divides the sum of two elements of } \mathcal{P}\right\}.
$$

It is known [6] that $\kappa(\mathcal{P})$ is a fraction whose denominator always divides the sum of some pair of elements in $\mathcal{P}$. Using this fact and Theorem 3.5, we may verify that for all pairs of the values of (a,b), $\kappa(\mathcal{P}) < \frac{1}{4}$. Thus, we have $\kappa(\mathcal{P}) \leq \beta(\mathcal{P})$. This completes the proof. □

Remark 3.1. *If one of a or b is* 1 *modulo* 4 *and other one is* 3 *modulo* 4*, then by Theorems 3.6 and 3.7, we get* $\kappa(\mathcal{P}) < \mu(\mathcal{P}) = 1/4$.

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