ON THE MAXIMAL DENSITY OF INTEGRAL SETS WHOSE DIFFERENCES AVOIDING THE WEIGHTED FIBONACCI NUMBERS

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Abstract

In an unpublished problem collection, Motzkin asks, how dense can a set $S$ of positive integers be, if no two elements of $S$ are allowed to differ by an element of the given set $\mathcal{P}$ of positive integers? The maximal density of such sets, denoted by $\mu(\mathcal{P})$, is known for $|\mathcal{P}| \leq 2$, and several other partial results are also known for the general case. We find some bounds and a few exact values of $\mu(\mathcal{P})$, where the elements $P_i$ of the set $\mathcal{P}$ are defined by $P_i := P_{i-1} + P_{i-2}$, $i \geq 2$ with $P_0 = a, P_1 = b$. Notice that the elements of the sequence $\{P_i\}$ satisfy the same recurrence relation as that satisfied by the well-known Fibonacci numbers $F_i$ with arbitrary initial values. Since $P_i = aF_{i-1} + bF_i$ for all $i \geq 0$, these numbers are also known as weighted Fibonacci numbers. This work generalizes an earlier work of Pandey on Fibonacci numbers.

1. Introduction

Let $S$ be any set of nonnegative integers and let $S(x)$ denote the number of elements $n \in S$ such that $1 \leq n \leq x$, $x \in \mathbb{R}$. The upper density of $S$, denoted by $\delta(S)$, is defined by $\delta(S) := \lim_{x \to \infty} S(x)/x$. Given the set of positive integers $\mathcal{P}$, $S$ is said to
be a $\mathcal{P}$-set if $a \in S$, $b \in S$ implies that $a - b \notin \mathcal{P}$. The parameter of interest is the maximal density of a $\mathcal{P}$-set, defined by

$$
\mu(\mathcal{P}) := \sup_S \delta(S),
$$

where the supremum is taken over all $\mathcal{P}$-sets $S$. Cantor and Gordon [2] establish the existence of $\mu(\mathcal{P})$ for any $\mathcal{P}$ and solve the problem for $|\mathcal{P}| \leq 2$. They also prove that

$$
\mu(\mathcal{P}) \geq \kappa(\mathcal{P}) := \sup_{\text{gcd}(c, m) = 1} \frac{1}{m} \min_{p \in \mathcal{P}} |cp|_m,
$$

(1.1)

where $|x|_m$ denotes the absolute value of the absolutely least remainder of $x$ modulo $m$. A remark of Haralambis [6], gives an equivalent formulation for the right-hand expression of the above inequality. Hence, we can write

$$
\kappa(\mathcal{P}) = \max_{1 \leq k \leq \frac{m}{2}} \frac{1}{m} \min_{p \in \mathcal{P}} |kp|_m,
$$

(1.2)

where $p$ and $q$ are any two distinct elements of $\mathcal{P}$, with the condition that $\mathcal{P}$ has only finitely many elements.

A result of Cantor and Gordon [2] reduces the calculation of $\mu(\mathcal{P})$ for any general set $\mathcal{P}$ to those sets $\mathcal{P}$ whose elements are relatively prime. Haralambis [6] gives a useful upper bound for $\mu(\mathcal{P})$ and provides an expression for $\mu(\mathcal{P})$ for most members of the families $\{1, j, k\}$, and $\{1, 2, j, k\}$. Liu and Zhu [9] determine the value of $\mu(\mathcal{P})$ for most of the almost difference closed sets $\mathcal{P}$. They [10] further compute $\mu(D_{a, b, m})$ for $1 < a \leq b < m - 1$, where $D_{a, b, m} = [1, a - 1] \cup [b + 1, m - 1]$. Gupta and Tripathi [5] determine $\mu(\mathcal{P})$ where elements of $\mathcal{P}$ are in arithmetic progression. Pandey and Tripathi ([12], [13]) discuss this quantity for the families $\mathcal{P} = \{a, b, n(a + b)\}$ and for the sets related to arithmetic progressions.

In this paper, we consider the problem of determining $\mu(\mathcal{P})$ for the set $\mathcal{P} = \{a, b, a + b, \ldots, a F_{k-1} + b F_k\}$ with $\text{gcd}(a, b) = 1$. We write $\mathcal{P} = \{P_0, P_1, P_2, \ldots, P_k\}$, with $P_0 = a$, $P_1 = b$ and $P_i = P_{i-1} + P_{i-2}$ for $i \geq 2$. The well known Fibonacci sequence $\{F_i\}_{i \geq 0}$ and Lucas sequence $\{L_i\}_{i \geq 0}$ are the special cases of the sequence $\{P_i\}_{i \geq 0}$. In Section 2, we evaluate a lower bound for $\mu(\mathcal{P})$ with $|\mathcal{P}| > 5$, by using some identities of the Fibonacci and Lucas sequences and the definition (1.2) of $\kappa(\mathcal{P})$. Whereas, for $|\mathcal{P}| \leq 4$, $\kappa(\mathcal{P})$ and $\mu(\mathcal{P})$ have been studied by Cantor and Gordon [2], and Liu and Zhu [9]. In Section 3, we investigate the values of $\kappa(\mathcal{P})$ and $\mu(\mathcal{P})$ when $|\mathcal{P}| = 5$.

The parameters $\mu(\mathcal{P})$ and $\kappa(\mathcal{P})$ are interesting and useful in the study of some other number theory as well as graph theory problems. The graph-theoretic connection of $\mu(\mathcal{P})$ is the fractional chromatic number of the distance graph generated by $\mathcal{P}$. For more detail, one may refer ([3], [9]). The parameter $\kappa(\mathcal{P})$, is related to
the well-known conjecture on diophantine approximation due to Wills [15] and independently by Cusick [4], now known as the lonely runner conjecture due to Bienia et al. [1].

Due to Cantor and Gordon [2], $\mu(\mathcal{P}) = \kappa(\mathcal{P})$ for all $\mathcal{P}$ with $|\mathcal{P}| \leq 2$. Hence, it is very natural to ask the question of whether $\mu(\mathcal{P}) = \kappa(\mathcal{P})$ when $|\mathcal{P}| = 3$. Haralambis [6] and Liu and Zhu [9] have shown the existence of some infinite families of four-element sets with $\kappa(\mathcal{P}) < \mu(\mathcal{P})$. We give an infinite family of five-element sets $\mathcal{P}$ with $\kappa(\mathcal{P}) < \mu(\mathcal{P})$ in the last section.

2. Main Results

Before we go to our main results we give some identities concerning the Fibonacci and Lucas sequences, denoted respectively by $\{F_i\}_{i \geq 0}$ and $\{L_i\}_{i \geq 0}$, in the lemma given below. Notice that both Fibonacci and Lucas sequences are also defined for negative indices, denoted respectively by $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^nL_n$. Hence, the identities given below are satisfied for all indices.

**Lemma 2.1.** For all integers $m, n, k$, and $i$, we have

1. $F_{n+2} - F_{n-2} = L_n = F_{n-1} + F_{n+1}$.
2. $F_{n-2} + F_{n+2} = 3F_n$.
3. $L_{n-1} + L_{n+1} = 5F_n$.
4. $F_mF_{n+1} - F_{m+1}F_n = (-1)^nF_{m-n}$.
5. $F_{n+k} + (-1)^kF_{n-k} = L_kF_n$.
6. $L_{2n-1}L_{i} - 5F_{i-1}F_{2n} = (-1)^{i+1}L_{2n-i}$.

7. $L_i = \begin{cases} 5 \sum_{k=1}^{\frac{i}{2}} (-1)^{k-1}F_{i-(2k-1)} + (-1)^{\frac{i}{2}}L_0, & \text{if } i \text{ is even;} \\ 5 \sum_{k=1}^{i-1} (-1)^{k-1}F_{i-(2k-1)} + (-1)^{\frac{i-1}{2}}L_1, & \text{if } i \text{ is odd.} \end{cases}$

**Proof.** Identities (1), (2), and (3) are simple to observe. Identities (4) and (5) may be found in Koshy [8]. We prove identities (6) and (7) below.

6. We have
\[ L_{2n-1}L_i - 5F_{i-1}F_{2n} \]
\[ = L_{2n-1}L_i - (L_{i-2} + L_i)F_{2n} \text{ (using identity (3))} \]
\[ = (L_{2n-1} - F_{2n})L_i - L_{i-2}F_{2n} \]
\[ = L_iF_{2n-2} - L_{i-2}F_{2n} \]
\[ = (-1)^iF_{2n-i-2} - (-1)^iF_{2n-i+2} \text{ (using identity (5))} \]
\[ = (-1)^iF_{2n-i-2} - (-1)^iF_{2n-i+2} \]
\[ = (-1)^{i+1}(F_{2n-i+2} - F_{2n-i-2}) \]
\[ = (-1)^{i+1}L_{2n-i} \text{ (using identity (1))}. \]

7. Recursively using identity (3), we get
\[ L_i = 5F_{i-1} - L_{i-2} \]
\[ = 5(F_{i-1} - F_{i-3} + \cdots + (-1)^{k-1}F_{i-(2k-1)}) + (-1)^k L_{i-2k}. \]

Thus, for even \( i \),
\[ L_i = 5\left( \sum_{k=1}^{\frac{i}{2}} (-1)^{k-1}F_{i-(2k-1)} \right) + (-1)^\frac{i}{2}L_0; \]

and for odd \( i \),
\[ L_i = 5\left( \sum_{k=1}^{\frac{i-1}{2}} (-1)^{k-1}F_{i-(2k-1)} \right) + (-1)^\frac{i-1}{2}L_1. \]

We write all possible initial values of \( P_0 = a \) and \( P_1 = b \) modulo 5. There are a total of twenty-five choices for the pair \( (a, b) \). But the choice \( (a, b) = (5m, 5l) \), always yields \( \gcd(a, b) \geq 5 \). So, we consider only the remaining twenty-four cases. In the following five lemmas, we compute a lower bound of \( \mu(\mathcal{P}) \) for all possible choices of pairs \( (a, b) \) of initial values.

**Lemma 2.2.** Let \( \mathcal{P} = \{P_0, P_1, P_2, \ldots, P_k\} \), where \( P_i = aF_{i-1} + bF_i \) for all \( i \geq 0 \) and \( 4n + 1 \leq k \leq 4n + 4 \) with \( n \geq 1 \). If \( (a, b) \in \{(5m, 5l+1), (5m+1, 5l+4), (5m+2, 5l+2), (5m+3, 5l), (5m+4, 5l+3) : l, m \in \mathbb{N} \cup \{0\}\} \) with \( \gcd(a, b) = 1 \), then
\[ \kappa(\mathcal{P}) \geq \frac{1}{5} - \frac{2L_{2n-1}}{5(aL_{2n+1} + bL_{2n+2})}. \]
Proof. Clearly, we have $2b - a \equiv 2 \pmod{5}$ and $a + 3b \equiv 3 \pmod{5}$. Set $q = P_{2n+1} + P_{2n+3} = aL_{2n+1} + bL_{2n+2}$. Then
\[
q = aL_{2n+1} + bL_{2n+2} \\
= a(F_{2n} + F_{2n+2}) + b(3F_{2n+2} - 2F_{2n}) \\
= (a - 2b)F_{2n} + (a + 3b)F_{2n+2} \\
\equiv -2F_{2n} + 3F_{2n+2} \\
\equiv 2(4F_{2n+2} - F_{2n}) \\
\equiv 2(F_{2n-2} + F_{2n}) \\
\equiv 2L_{2n-1} \pmod{5}.
\]
Let $p = \frac{q - 2L_{2n-1}}{5}$. We have
\[
2b(p - F_{2n-1}) - a(p + 2F_{2n}) = (2b - a)p - (2bF_{2n-1} + 2aF_{2n}) \\
= (2b - a)\frac{q - 2L_{2n-1}}{5} - (2bF_{2n-1} + 2aF_{2n}) \\
= \frac{(2b - a)q}{5} - \frac{2(a(5F_{2n} - L_{2n-1}) + b(5F_{2n-1} + 2L_{2n-1}))}{5} \\
= \frac{(2b - a)q}{5} - \frac{2(aL_{2n+1} + bL_{2n+2})}{5} \\
= \frac{2b - a - 2}{5}q.
\]
Hence,
\[
a(p + 2F_{2n}) \equiv 2b(p - F_{2n-1}) \pmod{q}. \tag{2.3}
\]
We have $q \equiv -2F_{2n} + 3F_{2n+2} \equiv 2F_{2n+1} + F_{2n+2} \equiv L_{2n+2} \pmod{5}$. Next, let $\gcd(a, q) = d$ and $\gcd(b, q) = d'$. This implies that $d | L_{2n+2}$, which implies $d \not\equiv 0 \pmod{5}$ and $d$ divides $2(p - F_{2n-1}) = \frac{2(q - 2L_{2n-1})}{5}$. Hence, there exists an integer $x$ such that
\[
a x \equiv 2(p - F_{2n-1}) \pmod{q}. \tag{2.4}
\]
Similarly, $d' | L_{2n+1}$, which implies $d' \not\equiv 0 \pmod{5}$ and $d'$ divides $(p + 2F_{2n}) = \frac{(q - 2L_{2n-1})}{5} + 2F_{2n} = \frac{(q + 2L_{2n+1})}{5}$. Hence there exists an integer $y$ such that
\[
b y \equiv (p + 2F_{2n}) \pmod{q}. \tag{2.5}
\]
Moreover, congruence (2.3) implies that there is a common solution $x_\sigma$ of the congruences (2.4) and (2.5), i.e.,
\[
a x_\sigma \equiv 2(p - F_{2n-1}) \pmod{q},
\]
and
\[
b x_\sigma \equiv (p + 2F_{2n}) \pmod{q}.
\]
The common solution is justified as follows: From (2.3), (2.4), and (2.5), we have that
\[ ab(x - y) \equiv 2b(p - F_{2n-1}) - a(p + 2F_{2n}) \equiv 0 \pmod{q}. \]
Moreover, by the definitions of \(d\) and \(d'\), it follows that \(\gcd(ab, q) = dd'\). Therefore, 
\(x - y\) is divisible by \(\frac{q}{dd'}\). Since \(\gcd(d, d') = 1\), we know from Bézout’s identity that there exist integers \(u\) and \(v\) such that
\[
\frac{(x - y)dd'}{q} = ud + vd'.
\]
This leads us to consider the integer \(z\) defined by
\[
z := x - \frac{q}{d} = y + \frac{q}{d}.
\]
Clearly, from the definition of \(d\) and \(d'\), the integer \(z\) verifies \(az \equiv ax\) and \(bz \equiv by\) modulo \(q\).

Since \(P_t = aF_{i-1} + bF_i\), we have
\[
P_t x_o = F_{i-1} (2(p - F_{2n-1})) + F_i (p + 2F_{2n})
\]
\[
= p(2F_{i-1} + F_i) + 2(F_i F_{2n} - F_{i-1} F_{2n-1})
\]
\[
= \frac{q - 2L_{2n-1}}{5} L_i + 2((-1)^i - 1 F_{2n-i+1} + F_{i-1} F_{2n}) \quad \text{using identity (4)}
\]
\[
= \frac{qL_i - 2(L_{2n-1} L_i - 5F_{i-1} F_{2n})}{5} + (-1)^{i-1} 2F_{2n-i+1}
\]
\[
= \frac{qL_i - (-1)^i 2L_{2n-i}}{5} + (-1)^{i-1} 2F_{2n-i+1} \quad \text{using identity (6)}
\]
\[
= \frac{qL_i - (-1)^{i+1} 2L_{2n-i+2}}{5} \pmod{q}.
\]
Let \(i\) be even. By identity (7), we have
\[
P_t x \equiv (-1)^{\frac{i}{2}} \left(\frac{qL_0 - (-1)^{\frac{i}{2}} 2L_{2n-i+2}}{5}\right) \pmod{q}.
\]
Therefore,
\[
P_t x_o \equiv \begin{cases} 
\frac{qL_0 - 2L_{2n-i+2}}{5} & \text{mod } q, \quad \text{if } i \equiv 0 \pmod{4}; \\
-\frac{qL_0 + 2L_{2n-i+2}}{5} & \text{mod } q, \quad \text{if } i \equiv 2 \pmod{4}.
\end{cases}
\]
Next, let \(i\) be odd. We have
\[
P_t x \equiv (-1)^{\frac{i-1}{2}} \left(\frac{qL_1 + (-1)^{\frac{i-1}{2}} L_{2n-i+2}}{5}\right) \pmod{q}.
\]
Therefore,
\[ P_i x_\sigma \equiv \begin{cases} \frac{qL_i + 2L_{2n-i+2}}{5} & \text{mod } q, \text{ if } i \equiv 1 \text{ (mod 4)}; \\ \frac{-qL_i - 3L_{2n-i+2}}{5} & \text{mod } q, \text{ if } i \equiv 3 \text{ (mod 4)}. \end{cases} \]

Thus, we see that
\[
\min_{0 \leq i \leq (2n+2)} \{|P_i x_\sigma|_q\} = \frac{q - 2L_{2n-1}}{5}.
\]

Using identity (5), for 0 ≤ i ≤ 2n + 2, we have that

(a) \[ F_{4n+3-i} = (-1)^i F_{i-1} + F_{2n+2-i}(F_{2n} + F_{2n+2}), \]

(b) \[ F_{4n+4-i} = (-1)^i F_{i} + F_{2n+2-i}(F_{2n+1} + F_{2n+3}). \]

By a simple manipulation, we get \[ P_{4n+4-i} = (-1)^i P_i + F_{2n+2-i}(P_{2n+1} + P_{2n+3}) = (-1)^i P_i + F_{2n+2-i}q. \] Thus, \[ P_{4n+4-i} x_\sigma \equiv (-1)^i P_i x_\sigma \text{ (mod q).} \] Therefore,
\[
\min_{0 \leq i \leq (2n+4)} \{|P_i x_\sigma|_q\} = \frac{q - 2L_{2n-1}}{5}.
\]

Notice that this absolute minimum is obtained corresponding to the congruences \[ P_{3} x_\sigma \equiv P_{4n+1} x_\sigma \equiv \frac{q - 2L_{2n-1}}{5} \text{ (mod q).} \] Therefore, for 4n + 1 ≤ k ≤ 4n + 4,
\[
\min_{0 \leq i \leq k} \{|P_i x_\sigma|_q\} = \frac{q - 2L_{2n-1}}{5}.
\]

Thus, by definition (1.2) of \( \kappa(\mathcal{P}) \), we get
\[
\kappa(\mathcal{P}) \geq \frac{1}{5} - \frac{2L_{2n-1}}{5(aL_{2n+1} + bL_{2n+2})}.
\]

This completes the proof. \( \square \)

**Lemma 2.3.** Let \( \mathcal{P} = \{P_0, P_1, P_2, \ldots, P_k\} \), where \( P_i = aF_{i-1} + bF_i \) for all \( i \geq 0 \) and \( 4n-1 \leq k \leq 4n+2 \) and \( n \geq 1 \), \( k \neq 3, 4 \). If \( (a, b) \in \{(5m, 5l+2), (5m+1, 5l), (5m+2, 5l+3), (5m+3, 5l+1), (5m+4, 5l+4) : l, m \in \mathbb{N} \cup \{0\}\} \) with \( \gcd(a, b) = 1 \), then
\[
\kappa(\mathcal{P}) \geq \frac{1}{5} - \frac{L_{2n-1}}{5(aL_{2n-1} + bL_{2n})}.
\]

**Proof.** Clearly, we have \( 2b - a \equiv 4 \text{ (mod 5)} \) and \( 3b - 4a \equiv 1 \text{ (mod 5)} \). Set \( q = P_{2n-1} + P_{2n+1} = aL_{2n-1} + bL_{2n} \). Then
\[
q = aL_{2n-1} + bL_{2n} \\
= (4a - 3b)F_{2n} + (-a + 2b)F_{2n+2} \\
\equiv 4F_{2n} - F_{2n+2} \\
\equiv F_{2n-2} + F_{2n} \\
\equiv L_{2n-1} \text{ (mod 5)}.
\]
Let \( p = \frac{q - L_{2n-1}}{5} \). We have
\[
ap \equiv b(2p + F_{2n}) \pmod{q}.
\] (2.6)

Again \( 2q \equiv 3F_{2n} - 2F_{2n+2} = F_{2n-2} - F_{2n+2} = -L_{2n} \pmod{5} \). Next, let \( \gcd(a, q) = d \), and \( \gcd(b, q) = d' \). This implies that \( d | L_{2n} \), which implies \( d \not\equiv 0 \pmod{5} \) and \( d \) divides \( (2p + F_{2n}) = \frac{2(q - L_{2n-1})}{5} + F_{2n} = \frac{2g + L_{2n}}{5} \). Hence, there exists an integer \( x \) such that
\[
ax \equiv (2p + F_{2n}) \pmod{q}.
\] (2.7)

Similarly, \( d' | L_{2n-1} \), which implies \( d' \not\equiv 0 \pmod{5} \) and \( d' \) divides \( p = \frac{(q - L_{2n-1})}{5} \).

Hence, there exists an integer \( y \) such that
\[
by \equiv p \pmod{q}.
\] (2.8)

Moreover, as in Lemma 2.2, congruence (2.6) implies that there is a common solution \( x_\sigma \) of the congruences (2.7) and (2.8), i.e.,
\[
ax_\sigma \equiv 2p + F_{2n} \pmod{q},
\]
and
\[
by_\sigma \equiv p \pmod{q}.
\]

Since \( P_i = aF_{i-1} + bF_i \), we have
\[
P_i x_\sigma \equiv F_{i-1}(2p + F_{2n}) + F_i(p)
= p(2F_{i-1} + F_i) + F_{2n}F_{i-1}
= pl_i + F_{2n}F_{i-1}
= \frac{(q - L_{2n-1})l_i + 5F_{2n}F_{i-1}}{5}
= \frac{qL_i - (L_{2n-1}l_i - 5F_{2n}F_{i-1})}{5}
= \frac{qL_i - (-1)^i + 1L_{2n-i}}{5} \pmod{q} \text{ (using identity (6))}.
\]

Let \( i \) be even. By identity (7), we have
\[
P_i x \equiv (-1)^{\frac{i}{2}} \left( qL_0 + (-1)^{\frac{i}{2}}L_{2n-i} \right) \pmod{q}.
\]

Therefore,
\[
P_i x_\sigma \equiv \begin{cases} 
\frac{qL_0 + L_{2n-i}}{5} \pmod{q}, & \text{if } i \equiv 0 \pmod{4}; \\
\frac{-qL_0 - L_{2n-i}}{5} \pmod{q}, & \text{if } i \equiv 2 \pmod{4}.
\end{cases}
\]
Next, let $i$ be odd. We have

$$P_i x \equiv (-1)^{i-1} \left( qL_1 - (-1)^{i-1} \frac{L_{2n-i}}{5} \right) \pmod{q}. $$

Therefore,

$$P_i x \equiv \begin{cases} 
\frac{qL_1 - L_{2n-i}}{5} \pmod{q}, & \text{if } i \equiv 1 \pmod{4}; \\
- \frac{qL_1 + L_{2n-i}}{5} \pmod{q}, & \text{if } i \equiv 3 \pmod{4}. 
\end{cases} $$

Thus, we see that

$$\min_{0 \leq i \leq 2n} \{|P_i x|_q\} = \frac{q - L_{2n-1}}{5}. $$

Using identity (5), for $0 \leq i \leq 2n$, we have that

(a) $F_{4n-i} = (-1)^i F_i + F_{2n-i}(F_{2n-2} + F_{2n}),$

(b) $F_{4n-i} = (-1)^i F_i + F_{2n-i}(F_{2n-1} + F_{2n+1}).$

By a simple manipulation, we get $P_{4n-i} = (-1)^i P_i + F_{2n-i}(P_{2n-1} + P_{2n+1}) = (-1)^i P_i + F_{2n-i}q.$ Thus, $P_{4n-i} \equiv (-1)^i P_i x \pmod{q}.$ Whereas, $P_{4n+1} x \equiv (P_0 - P_1)x \equiv p + F_{2n} \pmod{q}$ and $P_{4n+2} x \equiv (P_{4n+1} + P_4)x \equiv 2p \pmod{q}.$ Therefore, $\min_{0 \leq i \leq 2n} \{|P_i x|_q\} = \frac{q - L_{2n-1}}{5}.$ Notice that this absolute minimum is obtained corresponding to the congruences $P_i x \equiv -P_{4n-i} x \equiv \frac{q - L_{2n-1}}{5} \pmod{q}.$ Therefore, for $4n-1 \leq k \leq 4n+2,$

$$\min_{0 \leq i \leq k} \{|P_i x|_q\} = \frac{q - L_{2n-1}}{5}. $$

Thus, by definition (1.2) of $\kappa(\mathcal{P}),$ we get

$$\kappa(\mathcal{P}) \geq \frac{1}{5} \frac{L_{2n-1}}{5(aL_{2n+1} + bL_{2n})}. $$

This completes the proof.

\textbf{Lemma 2.4.} Let $\mathcal{P} = \{P_0, P_1, P_2, \ldots, P_k\}$, where $P_i = aF_i - 1 + bF_i$ for all $i \geq 0$ and $4n \leq k \leq 4n + 3$ with $n \geq 1$, $k \neq 4$. If $(a,b) \in \{(5m,5l+3), (5m+1,5l+1), (5m+2,5l+4), (5m+3,5l+2), (5m+4,5l) : l, m \in \mathbb{N} \cup \{0\}\}$ with gcd$(a,b) = 1$, then

$$\kappa(\mathcal{P}) \geq \frac{1}{5} \frac{L_{2n-1}}{5(aL_{2n+1} + bL_{2n+2})}. $$
Proof. Clearly, we have $2b - a = 1 \pmod{5}$ and $3b + a = 4 \pmod{5}$. Set $q = P_{2n+1} + P_{2n+3} = aL_{2n+1} + bL_{2n+2}$. Then

$$q = (a - 2b)F_{2n} + (a + 3b)F_{2n+2}$$

$$= 4F_{2n} - F_{2n+2}$$

$$= F_{2n-2} + F_{2n}$$

$$= L_{2n-1} \pmod{5}.$$  

Let $p = \frac{q - L_{2n-1}}{5}$. We have

$$a(p + F_{2n}) \equiv b(2p - F_{2n-1}) \pmod{q}, \quad (2.9)$$

We also have $2q \equiv 2(-F_{2n} + 4F_{2n+2}) \equiv -2F_{2n} + 3F_{2n+2} + 2F_{2n+1} + F_{2n+2} \equiv L_{2n+2} \pmod{5}$. Next, let $\gcd(a, q) = d$ and $\gcd(b, q) = d'$. This implies that $d|L_{2n+2}$, which implies $d \neq 0 \pmod{5}$ and $d$ divides $(2p - F_{2n-1}) = \frac{(2q - L_{2n-1})}{5}$. Hence, there exists an integer $x$ such that

$$ax \equiv (2p - F_{2n-1}) \pmod{q} \quad (2.10)$$

Similarly, $d'|L_{2n+1}$, implies $d' \neq 0 \pmod{5}$ and $d'$ divides $(p + F_{2n}) = \frac{(q + L_{2n+1})}{5}$. Hence, there exists an integer $y$ such that

$$by \equiv (p + F_{2n}) \pmod{q} \quad (2.11)$$

Moreover, congruence (2.9) implies that there is a common solution $x_\sigma$ of the congruences (2.10) and (2.11), i.e.,

$$ax_\sigma \equiv 2p - F_{2n-1} \pmod{q},$$

and

$$bx_\sigma \equiv p + F_{2n} \pmod{q}.$$

Since $P_i = aF_{i-1} + bF_i$, we have

$$P_i x_\sigma \equiv F_{i-1}(2p - F_{2n-1}) + F_i(\sigma(p + F_{2n}))$$

$$= p(2F_{i-1} + F_i) + (F_iF_{2n} - F_{i-1}F_{2n-1})$$

$$= \frac{q - L_{2n-1}}{5}L_i + ((-1)^{i-1}F_{2n-i+1} + F_{i-1}F_{2n}) \quad \text{(using identity (4))}$$

$$= \frac{qL_i - (L_{2n-1}L_i - 5F_{i-1}F_{2n})}{5} + (-1)^{i-1}2F_{2n-i+1}$$

$$= \frac{qL_i - (-1)^{i+1}L_{2n-i}}{5} + (-1)^{i-1}F_{2n-i+1} \quad \text{(using identity (6))}$$

$$= \frac{qL_i - (-1)^{i+1}(L_{2n-i} - 5F_{2n-i+1})}{5}$$

$$= \frac{qL_i - (-1)^{i+1}L_{2n-i+2}}{5} \pmod{q}. $$
Let $i$ be even. By identity (5), we have

$$P_i x_\sigma \equiv (-1)^i \left( \frac{2q - (-1)^i L_{2n-i+2}}{5} \right) \pmod{q}.$$ 

Therefore,

$$P_i x_\sigma \equiv \begin{cases} 
\frac{2q - L_{2n-i+2}}{5} \pmod{q}, & \text{if } i \equiv 0 \pmod{4}; \\
\frac{-2q + L_{2n-i+2}}{5} \pmod{q}, & \text{if } i \equiv 2 \pmod{4}.
\end{cases}$$

Next, let $i$ be odd. We have

$$P_i x \equiv (-1)^{i-1} \left( \frac{L_1 q + (-1)^{i-1} L_{2n-i+2}}{5} \right) \pmod{q}.$$ 

Therefore,

$$P_i x_\sigma \equiv \begin{cases} 
\frac{q + L_{2n-i+2}}{5} \pmod{q}, & \text{if } i \equiv 1 \pmod{4}; \\
\frac{-q - L_{2n-i+2}}{5} \pmod{q}, & \text{if } i \equiv 3 \pmod{4}.
\end{cases}$$

Thus, we see that

$$\min_{0 \leq i \leq (2n+2)} \{ |P_i x_\sigma|_q \} = \frac{q - L_{2n-1}}{5}.$$ 

Using identity (5), for $0 \leq i \leq 2n + 2$, we have that

(a) $F_{4n+3-i} = (-1)^i F_{i-1} + F_{2n+2-i}(F_{2n} + F_{2n+2}),$

(b) $F_{4n+4-i} = (-1)^i F_{i} + F_{2n+2-i}(F_{2n+1} + F_{2n+3}).$

By a simple manipulation, we get $P_{4n+4-i} = (-1)^i P_i + F_{2n+2-i}(P_{2n+1} + P_{2n+3}) = (-1)^i P_i + F_{2n+2-i} q$. Thus, $P_{4n+4-i} x_\sigma \equiv (-1)^i P_i x_\sigma \pmod{q}$. Therefore,

$$\min_{0 \leq i \leq (4n+4)} \{ |P_i x_\sigma|_q \} = \frac{q - L_{2n-1}}{5}.$$ 

Notice that this absolute minimum is obtained corresponding to the congruences $P_{3} x_\sigma \equiv P_{4n+1} x_\sigma \equiv \frac{q - 2L_{2n-1}}{5} \pmod{q}$. Therefore, for $4n + 1 \leq k \leq 4n + 4$,

$$\min_{0 \leq i \leq k} \{ |P_i x_\sigma|_q \} = \frac{q - L_{2n-1}}{5}.$$ 

Thus, by definition (1.2) of $\kappa(\mathcal{P})$, we get

$$\kappa(\mathcal{P}) \geq \frac{1}{5} - \frac{L_{2n-1}}{5(aL_{2n+1} + bL_{2n+2})}.$$ 

This completes the proof. □
Lemma 2.5. Let $\mathcal{P} = \{P_0, P_1, P_2, \ldots, P_k\}$, where $P_i = aF_{i-1} + bF_i$ and $4n \leq k \leq 4n+3$ with $n \geq 1$. If $(a, b) \in \{(5m, 5l+4), (5m+1, 5l+2), (5m+2, 5l), (5m+3, 5l+3), (5m+4, 5l+1) : l, m \in \mathbb{N} \cup \{0\}\}$ with $\gcd(a, b) = 1$, then

$$\kappa(\mathcal{P}) \geq \frac{1}{5} \frac{L_{2n-1}}{5(aL_{2n} + bL_{2n+1})}.$$ 

Proof. Clearly, we have $2a + b \equiv -1 \pmod{5}$ and $3a - b \equiv -4 \pmod{5}$. Set $q = P_{2n} + P_{2n+2} = aL_{2n} + bL_{2n+1}$. Then

$$q = aL_{2n} + bL_{2n+1}$$
$$= (2a + b)F_{2n+2} - (3a - b)F_{2n}$$
$$\equiv -F_{2n+2} + 4F_{2n}$$
$$\equiv F_{2n-2} + F_{2n}$$
$$\equiv L_{2n-1} \pmod{5}.$$ 

Let $p = \frac{q - L_{2n-1}}{5}$. We have

$$a(2p + F_{2n}) \equiv -b(p + F_{2n}) \pmod{q}. \quad (2.12)$$

We also have $q \equiv L_{2n-1} - 5F_{2n} \equiv -L_{2n+1} \pmod{5}$. Next, let $\gcd(a, q) = d$ and $\gcd(b, q) = d'$. This implies that $d|L_{2n+1}$, which implies $d \neq 0 \pmod{5}$ and $d$ divides $(p + F_{2n}) = \frac{(q + L_{2n+1})}{5}$. Hence, there exists an integer $x$ such that

$$ax \equiv -(p + F_{2n}) \pmod{q}. \quad (2.13)$$

Similarly, $d'|L_{2n}$, implies $d' \neq 0 \pmod{5}$ and $d'$ divides $(2p + F_{2n}) = \frac{2(q - L_{2n-1})}{5} + F_{2n} = \frac{(2q + 2L_{2n})}{5}$. Hence, there exists an integer $y$ such that

$$by \equiv (2p + F_{2n}) \pmod{q}. \quad (2.14)$$

Moreover, congruence (2.12) implies that there is a common solution $x_\sigma$ of the congruences (2.13) and (2.14), i.e.,

$$ax_\sigma \equiv -(p + F_{2n}) \pmod{q},$$
and
$$bx_\sigma \equiv 2p + F_{2n} \pmod{q}.$$
Since \( P_i = aF_{i-1} + bF_i \), we have

\[
P_i x_\sigma \equiv F_{i-1}(-(p + F_{2n})) + F_i(2p + F_{2n})
\]
\[
= p(2F_i - F_{i-1}) + F_{2n}F_{i-2}
\]
\[
= p(L_{i-1}) + F_{2n}F_{i-2}
\]
\[
= \frac{(q - L_{2n-1})(L_{i-1}) + 5F_{2n}F_{i-2}}{5}
\]
\[
= \frac{qL_{i-1} - (L_{2n-1}L_{i-1} - 5F_{2n}F_{i-2})}{5}
\]
\[
= \frac{qL_{i-1} - (-1)^iL_{2n-i+1}}{5} \pmod q \quad \text{(using identity (6)).}
\]

Let \( i \) be even. By identity (7), we have

\[
P_i x \equiv (-1)^{\frac{i+2}{2}} \left( L_1 q - (-1)^{\frac{i+2}{2}}L_{2n-i+1} \right) \pmod q.
\]

Therefore,

\[
P_i x_\sigma \equiv \begin{cases} 
-\frac{q+L_{2n-i+1}}{5} & \text{if } i \equiv 0 \pmod 4; \\
\frac{L_{2n-i+1}}{5} & \text{if } i \equiv 2 \pmod 4.
\end{cases}
\]

Next, let \( i \) be odd. Then, we have

\[
P_i x_\sigma \equiv \begin{cases} 
\frac{2q-(-1)^{\frac{i+1}{2}}L_{2n-i+1}}{5} & \text{if } i \equiv 1 \pmod 4; \\
\frac{-2q+L_{2n-i+1}}{5} & \text{if } i \equiv 3 \pmod 4.
\end{cases}
\]

Thus, we see that

\[
\min_{0 \leq i \leq (2n+1)} \{|P_i x_\sigma|_q\} = \frac{q - L_{2n-1}}{5}. \quad (2.15)
\]

Using identity (5), for \( 0 \leq i \leq 2n + 1 \), we have that

(a) \( F_{4n+1-i} = (-1)^{i-1}F_{i-1} + F_{2n+1-i}(F_{2n-1} + F_{2n+1}) \),

(b) \( F_{4n+2-i} = (-1)^{i-1}F_{i} + F_{2n+1-i}(F_{2n} + F_{2n+2}) \).

By a simple manipulation, we get \( P_{4n+2-i} = (-1)^{i-1}P_i + F_{2n+1-i}(P_{2n} + P_{2n+2}) = (-1)^{i-1}P_i + F_{2n+1-i}q \). Thus, \( P_{4n+2-i} x_\sigma \equiv (-1)^{i-1}P_i x_\sigma \pmod q \) whereas \( P_{4n+3} x_\sigma = P_{4n+2} x_\sigma + P_{4n+1} x_\sigma \equiv (-P_0 + P_1) x_\sigma \equiv 3p + 2F_{2n} \equiv -(2p - F_{2n-1}) \pmod q \).
Therefore, \( \min_{0 \leq i \leq (4n+3)} \{|P_i x_{\sigma}|_q\} = \frac{q-L_{2n-1}}{5} \). Notice that this absolute minimum is obtained corresponding to the congruences \( P_2 x_{\sigma} \equiv -P_{4n} x_{\sigma} \equiv \frac{q-2L_{2n-1}}{5} \pmod{q} \). Therefore, for \( 4n \leq k \leq 4n+3 \),

\[
\min_{0 \leq i \leq k} \{|P_i x_{\sigma}|_q\} = \frac{q-L_{2n-1}}{5}.
\]

Thus, by definition (1.2) of \( \kappa(\mathcal{P}) \), we get

\[
\kappa(\mathcal{P}) \geq \frac{1}{5} - \frac{L_{2n-1}}{5(aL_{2n} + bL_{2n+1})}.
\]

This completes the proof. \( \Box \)

The following corollary due to Pandey [11] may be obtained as a special case of the above lemma.

**Corollary 2.1.** Let \( \mathcal{P} = \{F_2, F_3, \ldots, F_t\} \) and \( n \geq 1 \) be an integer such that \( 4n+2 \leq t \leq 4n+5 \), then

\[
\kappa(\mathcal{P}) \geq \frac{F_{2n+1}}{F_{2n+2} + F_{2n+4}}.
\]

**Proof.** If \( a = 1 \) and \( b = 2 \), then \( P_t = F_{t+2} \). So, by the above lemma

\[
\kappa(\mathcal{P}) \geq \frac{1}{5} - \frac{L_{2n-1}}{5(L_{2n} + 2L_{2n+1})}
= \left(\frac{L_{2n} + 2L_{2n+1} - L_{2n-1}}{5(L_{2n+1} + L_{2n+2})}\right)
= \frac{L_{2n} + L_{2n+2}}{5L_{2n+3}}
= \frac{5F_{2n+1}}{5L_{2n+3}}
= \frac{F_{2n+1}}{F_{2n+2} + F_{2n+4}}.
\]

**Lemma 2.6.** Let \( \mathcal{P} = \{P_0, P_1, P_2, \ldots, P_k\} \), where \( P_i = aF_{i-1} + bF_i \) for all \( i \geq 0 \) with \( k \geq 5 \). If \( (a, b) \in \{(5m+1, 5l+3), (5m+2, 5l+1), (5m+3, 5l+4), (5m+4, 5l+2) : l, m \in \mathbb{N}^+\} \) with \( \gcd(a, b) = 1 \), then

\[
\kappa(\mathcal{P}) \geq \frac{1}{5}.
\]

**Proof.** Using identity (7), we may show that 5 does not divide \( P_i = aF_{i-1} + bF_i \) for any \( i \) and for any \( (a, b) \in \{(5m+1, 5l+3), (5m+2, 5l+1), (5m+3, 5l+4), (5m+4, 5l+2) : l, m \in \mathbb{N} \cup \{0\}\} \). Set \( c = 1 \) and \( m = 5 \). Then, by definition (1.1) of \( \kappa(\mathcal{P}) \), we have \( \kappa(\mathcal{P}) \geq \frac{1}{5} \). This completes the proof. \( \Box \)
Corollary 2.2. Let \( L = \{L_0, L_1, \ldots, L_k\} \) with \( k \geq 3 \). Then \( \mu(L) = \frac{1}{5} \).

Proof. If \( a = 2 \) and \( b = 1 \), then \( P_i = 2F_{i-1} + F_i = L_i \). So \( P = L \). Hence, from the above lemma \( \mu(L) \geq \kappa(L) \geq \frac{1}{5} \). On the other hand, by a result of Liu and Zhu [9], \( \mu(\{L_0, L_1, L_2, L_3\}) = \mu(2, 1, 3, 4) = \frac{1}{5} \), which gives \( \mu(L) \leq \mu(\{L_0, L_1, L_2, L_3\}) = \frac{1}{5} \).

Hence, \( \mu(L) = \frac{1}{5} \). \( \square \)

3. The Values of \( \kappa(\mathcal{P}) \) and \( \mu(\mathcal{P}) \) When \( |\mathcal{P}| \leq 5 \)

Consider the set \( \mathcal{P} = \{a, b, a + b, \ldots, aF_{k-1} + bF_k\} \) with \( \gcd(a, b) = 1 \). Cantor and Gordon [2] completely determined both \( \kappa(\mathcal{P}) \) and \( \mu(\mathcal{P}) \) when \( k = 0 \) or \( k = 1 \).

Theorem 3.1 ([2], Theorem 3). Let \( \mathcal{P} = \{a\} \) or \( \mathcal{P} = \{a, b\} \) with \( \gcd(a, b) = 1 \) and both \( a \) and \( b \) are odd. Then \( \mu(\mathcal{P}) = \kappa(\mathcal{P}) = \frac{1}{2} \).

Theorem 3.2 ([2], Theorem 4). Let \( \mathcal{P} = \{a, b\} \) with \( \gcd(a, b) = 1 \) and \( a \) and \( b \) are of opposite parity. Then \( \mu(\mathcal{P}) = \kappa(\mathcal{P}) = \frac{a + b - 1}{2(a+b)} \).

For \( k = 2 \) or \( \mathcal{P} = \{a, b, a+b\} \), Rabinowitz and Proulx [14] gave a lower bound for \( \mu(\mathcal{P}) \) and conjectured that the bound is the exact value. Liu and Zhu [9] confirmed their conjecture and completely determined the values of \( \kappa(\mathcal{P}) \) and \( \mu(\mathcal{P}) \).

Theorem 3.3 ([9], Theorem 3.1). Let \( \mathcal{P} = \{a, b, a + b\} \), where \( 0 < a < b \) and \( \gcd(a, b) = 1 \). Then

\[
\mu(\mathcal{P}) = \kappa(\mathcal{P}) = \begin{cases} 
\frac{1}{3}, & \text{if } b \equiv a \pmod{3}; \\
\frac{2a+b-1}{3(a+b)}, & \text{if } b \equiv a + 1 \pmod{3}; \\
\frac{a+2b-1}{3(a+2b)}, & \text{if } b \equiv a + 2 \pmod{3}.
\end{cases}
\]

They [9] further computed the value of \( \kappa(\mathcal{P}) \) for the four-element set \( \mathcal{P} = \{x, y, y - x, x + y\} \), \( y > x \) and gave a better lower bound than \( \kappa(\mathcal{P}) \) for \( \mu(\mathcal{P}) \) when both \( x \) and \( y \) are odd. The case when \( x \) and \( y \) are of opposite parity, has been settled by Kennitz and Kolberg [7].

Theorem 3.4 ([9], Lemma 4.1). Let \( \mathcal{P} = \{x, y, y - x, y + x\} \), \( y > x \), where \( \gcd(x, y) = 1 \). If \( x = 2k + 1 \) and \( y = 2m + 1 \), then \( \mu(\mathcal{P}) \geq \frac{(k+1)m}{4(k+1)m+1} \). If \( x \), \( y \) are of opposite parity, then \( \mu(\mathcal{P}) = 1/4 \).

Theorem 3.5 ([9], Corollary 5.3). Let \( \mathcal{P} = \{x, y, y - x, y + x\} \) with \( \gcd(x, y) = 1 \).
Let $\phi_4(n)$ denote $[\frac{n}{4}] / n$. Then

$$
\kappa(P) = \begin{cases} 
\phi_4(2y + x), & \text{if } x \equiv 0 \pmod{4} \text{ and } y \equiv 3 \pmod{4}, \text{ or} \\
\phi_4(2y + x), & \text{if } x \equiv 1 \pmod{4} \text{ and } y \equiv 0 \pmod{4}, \text{ or} \\
\phi_4(2y - x), & \text{if } x \equiv 3 \pmod{4} \text{ and } y \equiv 1,3 \pmod{4}; \\
\phi_4(2y + x), & \text{if } x \equiv 0 \pmod{4} \text{ and } y \equiv 1 \pmod{4}, \text{ or} \\
\phi_4(2y - x), & \text{if } x \equiv 3 \pmod{4} \text{ and } y \equiv 0 \pmod{4}, \text{ or} \\
\phi_4(2y + x), & \text{if } x \equiv 1 \pmod{4} \text{ and } y \equiv 3 \pmod{4}, \text{ and } y < 3x \\
\phi_4(2y - x), & \text{if } x \equiv 1 \pmod{4} \text{ and } y \equiv 3 \pmod{4}, \text{ and } y \geq 3x.
\end{cases}
$$

Liu and Zhu [9] also gave an infinite family of sets $P$ satisfying $\kappa(P) < \mu(P)$.

For $k = 3$ or $P = \{a, b, a + b, a + 2b\} = \{b, a + b, a, a + 2b\}$, using the results discussed in [9] for $P = \{x, y, y - x, y + x\}$, we can estimate the value of $\mu(P)$ and achieve $\kappa(P)$.

The rest of this section deals with the case $k = 4$, i.e., when $P = \{a, b, a + b, a + 2b, 2a + 3b\}$.

**Lemma 3.1.** Let $P = \{a, b, a + b, a + 2b, 2a + 3b\}$ with $\gcd(a, b) = 1$. Then $\kappa(P) \leq \mu(P) \leq 1/4$.

**Proof.** Since both the sets $\{b, a + b, a + 2b\}$ and $\{a + b, a + 2b, b, 2a + 3b\}$ are proper subsets of $P$, we have, $\mu(P) \leq \mu(\{b, a + b, a + 2b\})$ and $\mu(P) \leq \mu(\{a + b, a + 2b, b, 2a + 3b\})$. One can easily verify that for all pairs of values of $a, b$ with $\gcd(a, b) = 1$, either $b$ and $a + b$ or $a + b$ and $a + 2b$ are of opposite parity. Therefore, using Theorem 3.4, $\kappa(P) \leq \mu(P) \leq 1/4$. \)

**Theorem 3.6.** Let $P = \{a, b, a + b, a + 2b, 2a + 3b\}$ with $\gcd(a, b) = 1$. If $a$ and $b$ both are odd, then $\mu(P) = 1/4$. In particular, $\kappa(P) = \mu(P) = 1/4$, when both $a$ and $b$ are 1 (mod 4) or both $a$ and $b$ are 3 (mod 4).

**Proof.** By Lemma 3.1, we have $\mu(P) \leq 1/4$. It suffices to show that $\mu(P) \geq 1/4$. Note that $P$ contains only one even element and that is $a + b$. We consider the set

$$
S = \bigcup_{i \geq 0} \{2i(a + b), 2i(a + b) + 2, \ldots, (2i + 1)(a + b) - 2\}.
$$

Clearly $S$ is a periodic set with period $2(a + b)$. It is not difficult to show that $S$ is a $P$-set and $|S \cap \{0, 1, \ldots, 2(a + b) - 1\}| = \frac{a + b}{2}$. Therefore, $S$ has density $\frac{a + b}{2(a + 2b)} = \frac{1}{4}$. This implies that $\mu(P) \geq 1/4$.

However, when both $a$ and $b$ are 1 (mod 4), or both $a$ and $b$ are 3 (mod 4), none of the elements in $P$ is a multiple of 4. Let $c = 1$ and $m = 4$. Then, $\min\{|ca|_m, |cb|_m, |c(a + b)|_m, |c(a + 2b)|_m, |c(2a + 3b)|_m\} = 1$. Therefore, by definition (1.1) of $\kappa(P)$, we have $\kappa(P) \geq \frac{1}{4}$. Hence, $\kappa(P) = \mu(P) = 1/4$. \)
In the next theorem, we evaluate $\kappa(\mathcal{P})$, where $\mathcal{P} = \{a, b, a + b, a + 2b, 2a + 3b\}$ for the remaining possible pairs $(a, b)$.

**Theorem 3.7.** Let $\mathcal{P} = \{a, b, a + b, a + 2b, 2a + 3b\}$ with $\gcd(a, b) = 1$. Let $\phi_4(n)$ denote $\left\lceil \frac{n}{4} \right\rceil / n$. Then

$$
k(\mathcal{P}) = \begin{cases} 
\phi_4(2a + 2b), & \text{if } b - a \equiv 1 \pmod{4}; \\
\phi_4(a + 3b), & \text{if } b - a \equiv 2 \pmod{4}; \\
\phi_4(a + 3b), & \text{if } b - a \equiv 3 \pmod{4}.
\end{cases}
$$

**Proof.** Let $\beta(\mathcal{P})$ be the corresponding value on the right-hand side of the equality. First we show that $\beta(\mathcal{P}) \leq \kappa(\mathcal{P})$. Let $\sigma$ be either +1 or −1. We consider three cases for three different values of $\beta(\mathcal{P})$.

**Case 1.** ($\beta(\mathcal{P}) = \phi_4(2a + 2b)$).

In this case, $b - a \equiv 1 \pmod{4}$. Set $q = 2a + 2b$. We have $q \equiv 2 \pmod{4}$. Choose $x_\sigma$ such that

$$
ax_\sigma \equiv \sigma \left( \frac{a + b - 1}{2} \right) \pmod{q},
$$

and

$$
bx_\sigma \equiv \sigma \left( \frac{a + b + 1}{2} \right) \pmod{q}.
$$

Then

$$(a + b)x_\sigma \equiv \sigma (a + b) \pmod{q}.
$$

Whereas,

$$(a + 2b) \equiv -a \pmod{q},
$$

and

$$(2a + 3b) \equiv b \pmod{q}.
$$

Therefore, $\min \{|ax_\sigma|_q, |bx_\sigma|_q, |(a + b)x_\sigma|_q, |(a + 2b)x_\sigma|_q, |(2a + 3b)x_\sigma|_q\} = \frac{a + b - 1}{2}$. This implies that $\kappa(\mathcal{P}) \geq \frac{a + b - 1}{2(2a + 2b)} = \phi_4(2a + 2b) = \beta(\mathcal{P})$.

**Case 2.** ($\beta(\mathcal{P}) = \phi_4(a + 3b)$).

In this case, $b - a \equiv 2 \pmod{4}$. Set $q = a + 3b$. We have $q \equiv 2 \pmod{4}$. Choose $x_\sigma$ such that

$$
ax_\sigma \equiv \sigma \left( \frac{a + 3b + 6}{2} \right) \pmod{q},
$$

and

$$
bx_\sigma \equiv \sigma \left( \frac{a + 3b - 2}{2} \right) \pmod{q}.
$$
Since,

\[(a + b) \equiv -2b \pmod{q},\]
\[(a + 2b) \equiv -b \pmod{q},\]
and \[(2a + 3b) \equiv a \pmod{q},\]

we have, \[\min\{|ax|_q, |bx|_q, |(a + b)x|_q, |(a + 2b)x|_q, |(2a + 3b)x|_q\} = \frac{a + 3b - 2}{4}.\]

This implies that \[\kappa(P) \geq \frac{a + 3b - 2}{4(a + 3b)} = \phi_4(a + 3b) = \beta(P).\]

Case 3. \((\beta(P) = \phi_4(a + 3b)).\)

In this case, \(b - a \equiv 3 \pmod{4}\). Set \(q = a + 3b\). We have \(q \equiv 1 \pmod{4}\). Choose \(x\sigma\) such that

\[ax\sigma \equiv \sigma \left( \frac{a + 3b + 3}{4} \right) \pmod{q} \quad \text{and} \quad bx\sigma \equiv \sigma \left( \frac{a + 3b - 1}{4} \right) \pmod{q}.\]

Since,

\[(a + b) \equiv -2b \pmod{q},\]
\[(a + 2b) \equiv -b \pmod{q},\]
and \[(2a + 3b) \equiv a \pmod{q},\]

we have, \[\min\{|ax\sigma|_q, |bx\sigma|_q, |(a + b)x\sigma|_q, |(a + 2b)x\sigma|_q, |(2a + 3b)x\sigma|_q\} = \frac{a + 3b - 1}{4}.\]

This implies that \[\kappa(P) \geq \frac{a + 3b - 1}{4(a + 3b)} = \phi_4(a + 3b) = \beta(P).\]

To show that the equality holds, we observe that in all above cases, \(\beta(P)\) values are equal to

\[\max \left\{ \frac{p}{q} : \frac{p}{q} < \frac{1}{4} \text{ and } q \text{ divides the sum of two elements of } P \right\}.\]

It is known [6] that \(\kappa(P)\) is a fraction whose denominator always divides the sum of some pair of elements in \(P\). Using this fact and Theorem 3.5, we may verify that for all pairs of the values of \((a, b)\), \(\kappa(P) < \frac{1}{4}\). Thus, we have \(\kappa(P) \leq \beta(P)\). This completes the proof.

\[\square\]

**Remark 3.1.** If one of \(a\) or \(b\) is 1 modulo 4 and other one is 3 modulo 4, then by Theorems 3.6 and 3.7, we get \(\kappa(P) < \mu(P) = 1/4\).

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