EVALUATION OF CONVOLUTION SUMS INVOLVING THE SUM OF DIVISORS FUNCTION FOR LEVELS 48 AND 64

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Received: 8/5/16, Accepted: 9/22/17, Published: 10/16/17

Abstract

The convolution sum $\sum\limits_{\substack{(l,m)\in\mathbb{N}_0^2\\ \alpha\, l+\beta\, m=n}} \sigma(l)\sigma(m)$, where $\alpha\beta = 48$ and $\alpha\beta = 64$, is elementarily evaluated for all natural numbers n. The evaluation of the convolution sums for these levels is achieved using the sum of divisors function, primitive Dirichlet characters and modular forms. The evaluation of these convolution sums is then used to determine formulae for the number of representations of a natural number by the octonary quadratic forms $a\,(x_1^2+x_2^2+x_3^2+x_4^2)+b\,(x_5^2+x_6^2+x_7^2+x_8^2)$ and $c\,(x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2)+d\,(x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2)$, where $(a,b)=(1,12)$, $(1,16)$, $(3,4)$ and $(c,d)=(1,16)$.

1. Introduction

In this paper, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, denote the sets of natural numbers, non-negative integers, integers, rational numbers, real numbers and complex numbers, respectively.

Assume that $d,k,n\in\mathbb{N}$. Then we define the sum $\sigma_k(n)$ of the k^{th} powers of the positive divisors of n by

$$\sigma_k(n) = \sum_{0<d|n} d^k, \tag{1}$$

and we write $\sigma(n)$ as a shorthand for $\sigma_1(n)$ and set $\sigma_k(m)=0$ if $m\notin\mathbb{N}$.

Suppose that $\alpha,\beta\in\mathbb{N}$ are such that $\alpha\le\beta$. We define the convolution sum $W_{(\alpha,\beta)}(n)$ by

$$W_{(\alpha,\beta)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ \alpha\, l+\beta\, m=n}} \sigma(l)\sigma(m). \tag{2}$$

We write $W_\beta(n)$ as a shorthand for $W_{(1,\beta)}(n)$.

The levels $\alpha\beta$ for those convolution sums $W_{(\alpha,\beta)}(n)$ that have been evaluated so far are referenced in Table 1.

Level $\alpha\beta$	**Authors**	**References**
1	M. Besge, J. W. L. Glaisher, S. Ramanujan	[8, 12, 27]
2, 3, 4	J. G. Huard & Z. M. Ou & B. K. Spearman & K. S. Williams	[13]
5, 7	M. Lemire & K. S. Williams, S. Cooper & P. C. Toh	[10, 17]
6	Ş. Alaca & K. S. Williams	[7]
8, 9	K. S. Williams	[31, 30]
10, 11, 13, 14	E. Royer	[28]
12, 16, 18, 24	A. Alaca & Ş. Alaca & K. S. Williams	[2, 3, 4, 5]
15	B. Ramakrishnan & B. Sahu	[26]
10, 20	S. Cooper & D. Ye	[11]
23	H. H. Chan & S. Cooper	[9]
25	E. X. W. Xia & X. L. Tian & O. X. M. Yao	[33]
27, 32	Ş. Alaca & Y. Kesicioğlu	[6]
36	D. Ye	[34]
14, 26, 28, 30	E. Ntienjem	[23]
22, 44, 52	E. Ntienjem	[25]
33, 40, 56 $\alpha\beta = 2^\nu \prod_{j\geq 2}^{\kappa} p_j$, where $\gcd(\alpha,\beta)=1$, $0 \leq \nu \leq 3$, $\kappa \in \mathbb{N}$, $p_j > 2$ distinct primes	E. Ntienjem	[24]

Table 1: Known convolution sums $W_{(\alpha,\beta)}(n)$ of level $\alpha\beta$

In this paper, we evaluate the convolution sums of levels $\alpha\beta = 48$ and $\alpha\beta = 64$. The natural numbers 48 and 64 do not belong to the class of natural numbers for which the evaluation of the convolution sums is discussed by E. Ntienjem [24]. Therefore, according to Table 1, these convolution sums have not been evaluated as yet.

Suppose that $a, b, c, d \in \mathbb{N}$ are such that $\gcd(a, b) = 1$ and $\gcd(c, d) = 1$. The determination of explicit formulae for the number of representations of a positive integer n by the octonary quadratic forms

$$a\,(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b\,(x_5^2 + x_6^2 + x_7^2 + x_8^2), \tag{3}$$

and

$$c\,(x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2) + d\,(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2), \tag{4}$$

is achieved by applying the evaluated convolution sums.

Known explicit formulae for the number of representations of n by the octonary quadratic forms (3) and (4) are referenced in Table 2 and Table 3, respectively.

$(\mathbf{a},\mathbf{b})$	**Authors**	**References**
(1,1),(1,3), (1,9),(2,3)	E. Ntienjem	[24]
(1,2)	K. S. Williams	[31]
(1,4)	A. Alaca & Ş. Alaca & K. S. Williams	[3]
(1,5)	S. Cooper & D. Ye	[11]
(1,6)	B. Ramakrishnan & B. Sahu	[26]
(1,7)	E. Ntienjem	[23]
(1,8)	Ş. Alaca & Y. Kesicioğlu	[6]
(1,11),(1,13)	E. Ntienjem	[25]
(1,10),(1,14), (2,5),(2,7), $a\,b = 2^{\nu}\prod_{j\geq 2}^{\kappa} p_j$, where $\gcd(a,b)=1$, $0\leq\nu\leq 1$, $\kappa\in\mathbb{N}$, $p_j>2$ distinct primes	E. Ntienjem	[24]

Table 2: Known representations of n by the form (3)

$(\mathbf{c},\mathbf{d})$	**Authors**	**References**
(1,1)	G. A. Lomadze	[19]
(1,2)	Ş. Alaca & K. S. Williams	[7]
(1,3)	K. S. Williams	[30]
(1,4),(1,6), (1,8),(2,3)	A. Alaca & Ş. Alaca & K. S. Williams	[2, 3, 4]
(1,5)	B. Ramakrishnan & B. Sahu	[26]
(1,9)	Ş. Alaca & Y. Kesicioğlu	[6]
(1,10), (2,5)	E. Ntienjem	[23]
(1,12),(3,4)	D. Ye	[34]
(1,11), $c\,d = 2^{\nu}\prod_{j\geq 3}^{\kappa} p_j$, where $\gcd(c,d)=1$, $0\leq\nu\leq 3$, $\kappa\in\mathbb{N}$, $p_j>3$ distinct primes	E. Ntienjem	[24]

Table 3: Known representations of n by the form (4)

We apply our evaluation of the convolution sums of levels 48 and 64 and other known convolution sums to determine formulae for the number of representations of a positive integer n by the octonary quadratic forms (3) for which $(a, b) = (1, 12)$, $(1, 16)$, $(3, 4)$ and (4) for which $(c, d) = (1, 16)$. These numbers of representations are also new according to Table 2 and Table 3, respectively.

We have structured this paper as follows. In Section 2 we briefly introduce modular forms, eta functions and convolution sums. We then discuss in Section 3 our main results on the evaluation of the convolution sums; in Sections 4 and 5, we present our main results on the formulae for the number of representations of a positive integer n.

To obtain the results presented in this paper, we used software for symbolic scientific computation which consists of the open source software packages *GiNaC*, *Maxima*, *REDUCE*, *SAGE* and the commercial software package *MAPLE*.

2. Preliminaries: Modular Forms, Eta Quotients and Convolution Sums

2.1. Modular Forms

Considered are the upper half-plane, $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, and the group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ which is a full modular subgroup of $G = \mathrm{SL}_2(\mathbb{R})$ the group of 2×2-matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Suppose that $N \in \mathbb{N}$. Then

$$\Gamma(N) \quad = \left\{ \ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid \ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod N \ \right\}$$

is a subgroup of Γ. We call the subgroup $\Gamma(N)$ the *principal congruence subgroup of level N*. A subgroup H of G which contains $\Gamma(N)$ is a *congruence subgroup of level N*.

We consider the congruence subgroup

$$\Gamma_0(N) = \left\{ \ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod N \ \right\}.$$

Assume that $k \in \mathbb{Z}, \gamma \in \Gamma$ and $f^{[\gamma]_k} : \mathbb{H} \cup \mathbb{Q} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is the function whose value at z is $f^{[\gamma]_k}(z) = (cz + d)^{-k} f(\gamma(z))$. The definition given below is based on N. Koblitz's textbook [15, p. 108].

Definition 1. Let $N \in \mathbb{N}$, $k \in \mathbb{Z}$, f be a meromorphic function on $\mathbb{H}$ and $\Gamma' \subset \Gamma$ be a congruence subgroup of level N.

(a) f is a *modular function of weight k* for Γ' if

  (a1) $f^{[\gamma]_k} = f$ for all $\gamma \in \Gamma'$,

  (a2) $f^{[\delta]_k}(z)$ can be expressed in the form $\sum\limits_{n \in \mathbb{Z}} a_n e^{\frac{2\pi i z n}{N}}$, wherein $a_n \neq 0$ for finitely many $n \in \mathbb{Z} \setminus \mathbb{N}_0$, for all $\delta \in \Gamma$.

(b) f is a *modular form of weight* k for Γ' if

(b1) f is a modular function of weight k for Γ',

(b2) f is holomorphic on $\mathbb{H}$,

(b3) $a_n = 0$ for all $\delta \in \Gamma$ and for all $n \in \mathbb{Z} \setminus \mathbb{N}_0$.

(c) f is a *cusp form of weight* k *for* Γ' if

(c1) f is a modular form of weight k for Γ',

(c2) $a_0 = 0$ for all $\delta \in \Gamma$.

Let $k, N \in \mathbb{N}$ with $k \geq 4$ even. We denote by $\mathcal{M}_k(\Gamma_0(N))$ the space of modular forms of weight k for $\Gamma_0(N)$, $\mathcal{S}_k(\Gamma_0(N))$ the subspace of cusp forms of weight k for $\Gamma_0(N)$, and $\mathcal{E}_k(\Gamma_0(N))$ the subspace of Eisenstein forms of weight k for $\Gamma_0(N)$. The decomposition of the space of modular forms as a direct sum of the space generated by the Eisenstein series and the space of cusp forms, i.e., $\mathcal{M}_k(\Gamma_0(N)) = \mathcal{E}_k(\Gamma_0(N)) \oplus \mathcal{S}_k(\Gamma_0(N))$, is well-known; see for example W. A. Stein's book (online version) [29, p. 81].

In the sequel, we consider $4 \leq k \in \mathbb{N}$ even and primitive Dirichlet characters χ and ψ with conductors L and R, respectively.

According to Section 5.3 of W. A. Stein's book [29, p. 86] the Eisenstein series $E_{k,\chi,\psi}(q)$ is defined by

$$E_{k,\chi,\psi}(q) = C_0 + \sum_{n=1}^{\infty} \Big(\sum_{d|n} \psi(d)\chi(\frac{n}{d})\, d^{k-1} \Big) q^n, \tag{5}$$

where

$$C_0 = \begin{cases} 0 & \text{if } L > 1 \\ -\frac{B_{k,\chi}}{2k} & \text{if } L = 1 \end{cases}$$

and $B_{k,\chi}$ are the generalized Bernoulli numbers. Based on this consideration Theorems 5.8 and 5.9 in Section 5.3 of [29, p. 86] also hold.

2.2. Eta Quotients

The Dedekind eta function $\eta(z)$ is defined on $\mathbb{H}$ by $\eta(z) = e^{\frac{2\pi i z}{24}} \prod_{n=1}^{\infty}(1 - e^{2\pi i n z})$. Set $q = e^{2\pi i z}$ to deduce that

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n) = q^{\frac{1}{24}} F(q), \qquad \text{where } F(q) = \prod_{n=1}^{\infty}(1 - q^n).$$

Let us use eta function, eta quotient and eta product interchangeably as synonyms.

L. J. P. Kilford's book [14, p. 99] and G. Köhler's book [16, p. 37] contain a proof of the following theorem which will be used to determine eta functions that are elements of $\mathcal{M}_k(\Gamma_0(N))$, and particularly those eta functions that are elements of $\mathcal{S}_k(\Gamma_0(N))$. As remarked by E. Ntienjem [23, 25] and A. Alaca et al. [1], credit for the following result should be given to M. Newman [21, 22] and G. Ligozat [18].

Theorem 2 (M. Newman and G. Ligozat). *Let $N \in \mathbb{N}$, $D(N)$ be the set of all positive divisors of N, $\delta \in D(N)$ and $r_\delta \in \mathbb{Z}$. Let furthermore $f(z) = \prod_{\delta\in D(N)} \eta^{r_\delta}(\delta z)$ be an η-quotient. If the following five conditions are satisfied*

(i) $\sum_{\delta\in D(N)} \delta\, r_\delta \equiv 0 \pmod{24}$, ***(ii)*** $\sum_{\delta\in D(N)} \frac{N}{\delta}\, r_\delta \equiv 0 \pmod{24}$,

(iii) $\prod_{\delta\in D(N)} \delta^{r_\delta}$ *is a square in* $\mathbb{Q}$, ***(iv)*** $0 < \sum_{\delta\in D(N)} r_\delta \equiv 0 \pmod 4$,

(v) $\sum_{\delta\in D(N)} \frac{\gcd(\delta,d)^2}{\delta}\, r_\delta \geq 0$ *for each* $d \in D(N)$,

then $f(z) \in \mathcal{M}_k(\Gamma_0(N))$, where $k = \frac{1}{2}\sum_{\delta\in D(N)} r_\delta$.

*Moreover, the η-quotient $f(z)$ belongs to $\mathcal{S}_k(\Gamma_0(N))$ if **(v)** is replaced by*

(v') $\sum_{\delta\in D(N)} \frac{\gcd(\delta,d)^2}{\delta} r_\delta > 0$ *for each* $d \in D(N)$.

2.3. Evaluating $W_{(\alpha,\beta)}(n)$

Let $\alpha, \beta \in \mathbb{N}$ be such that $\alpha \leq \beta$. The convolution sum $W_{(\alpha,\beta)}(n)$ is defined as in (2).

As proved by E. Ntienjem [23, 24] and according to an observation by A. Alaca et al. [2], we may assume that $\gcd(\alpha,\beta) = 1$. Suppose that $q \in \mathbb{C}$ is such that $|q| < 1$. Let χ and ψ be primitive Dirichlet characters. We assume that $\chi = \psi$ and that χ is a Kronecker symbol in the following. The Eisenstein series

$$L(q) = E_2(q) = 1 - 24\sum_{n=1}^{\infty} \sigma(n)q^n, \quad (6)$$

$$M(q) = E_4(q) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n, \quad (7)$$

$$M_\chi(q) = E_{4,\chi}(q) = C_0 + \sum_{n=1}^{\infty} \chi\,\sigma_3(n)\,q^n, \text{ where } C_0 = \begin{cases} 0 & \text{if } L > 1, \\ -\frac{B_{4,\chi}}{8} & \text{if } L = 1, \end{cases} \quad (8)$$

are vital for the subsequent development of this work.

The following two results which do not depend on a (primitive) Dirichlet character are essential for the sequel of this work.

Lemma 1. *Let $\alpha, \beta \in \mathbb{N}$ be such that $\alpha \leq \beta$. Then*

$$(\alpha\, L(q^\alpha) - \beta\, L(q^\beta))^2 \in \mathcal{M}_4(\Gamma_0(\alpha\beta)).$$

Proof. If $\alpha = \beta$, then trivially $0 = (\alpha\, L(q^\alpha) - \alpha\, L(q^\alpha))^2 \in \mathcal{M}_4(\Gamma_0(\alpha))$ and there is nothing to prove. Therefore, we may suppose that $\alpha \neq \beta > 0$ in the sequel.

We apply the result proved by W. A. Stein [29, Thrms 5.8,5.9, p. 86] to deduce $L(q) - \alpha\, L(q^\alpha) \in \mathcal{M}_2(\Gamma_0(\alpha)) \subseteq \mathcal{M}_2(\Gamma_0(\alpha\beta))$ and $L(q) - \beta\, L(q^\beta) \in \mathcal{M}_2(\Gamma_0(\beta)) \subseteq \mathcal{M}_2(\Gamma_0(\alpha\beta))$. Therefore,

$$\alpha\, L(q^\alpha) - \beta\, L(q^\beta) = (L(q) - \beta\, L(q^\beta)) - (L(q) - \alpha\, L(q^\alpha)) \in \mathcal{M}_2(\Gamma_0(\alpha\beta))$$

and so $(\alpha\, L(q^\alpha) - \beta\, L(q^\beta))^2 \in \mathcal{M}_4(\Gamma_0(\alpha\beta))$. □

Theorem 3. *Let $\alpha, \beta \in \mathbb{N}$ be such that α and β are relatively prime and $\alpha < \beta$. Then*

$$\begin{aligned}(\alpha\, L(q^\alpha) - \beta\, L(q^\beta))^2 =& (\alpha-\beta)^2 + \sum_{n=1}^{\infty}\Big(\ 240\,\alpha^2\,\sigma_3(\frac{n}{\alpha}) + 240\,\beta^2\,\sigma_3(\frac{n}{\beta})\\ &+ 48\,\alpha\,(\beta - 6n)\,\sigma(\frac{n}{\alpha}) + 48\,\beta\,(\alpha - 6n)\,\sigma(\frac{n}{\beta})\\ &- 1152\,\alpha\beta\, W_{(\alpha,\beta)}(n)\ \Big)q^n. \end{aligned} \tag{9}$$

Proof. We first observe that

$$(\alpha\, L(q^\alpha)\ -\ \beta\, L(q^\beta))^2 \ =\ \alpha^2\, L^2(q^\alpha)\ +\ \beta^2\, L^2(q^\beta)\ -\ 2\,\alpha\beta\, L(q^\alpha)L(q^\beta). \tag{10}$$

J. W. L. Glaisher [12] has proved the following identity

$$L^2(q) = 1 + \sum_{n=1}^{\infty}\Big(240\,\sigma_3(n) - 288\,n\,\sigma(n)\Big)q^n \tag{11}$$

which we apply to deduce

$$L^2(q^\alpha) = 1 + \sum_{n=1}^{\infty}\big(240\,\sigma_3(\frac{n}{\alpha}) - 288\,\frac{n}{\alpha}\,\sigma(\frac{n}{\alpha})\big)q^n \tag{12}$$

and

$$L^2(q^\beta) = 1 + \sum_{n=1}^{\infty}\big(240\,\sigma_3(\frac{n}{\beta}) - 288\,\frac{n}{\beta}\,\sigma(\frac{n}{\beta})\big)q^n. \tag{13}$$

Since

$$\big(\sum_{n=1}^{\infty}\sigma(\frac{n}{\alpha})q^n\big)\big(\sum_{n=1}^{\infty}\sigma(\frac{n}{\beta})q^n\big) \ =\ \sum_{n=1}^{\infty}\big(\sum_{\substack{(k,l)\in\mathbb{N}^2\\ \alpha\,k+\beta\,l=n}}\sigma(k)\sigma(l)\,\big)q^n \ =\ \sum_{n=1}^{\infty}W_{(\alpha,\beta)}(n)q^n,$$

we conclude, when using the accordingly modified the Eisenstein series (6), that

$$L(q^\alpha)L(q^\beta) = 1 - 24\sum_{n=1}^{\infty}\sigma(\frac{n}{\alpha})q^n - 24\sum_{n=1}^{\infty}\sigma(\frac{n}{\beta})q^n + 576\sum_{n=1}^{\infty}W_{(\alpha,\beta)}(n)q^n. \tag{14}$$

Therefore,

$$\big(\alpha\,L(q^{\alpha})-\beta\,L(q^{\beta})\big)^2=(\alpha-\beta)^2+\sum_{n=1}^{\infty}\bigg(\;240\,\alpha^2\,\sigma_3(\frac{n}{\alpha})+240\,\beta^2\,\sigma_3(\frac{n}{\beta})$$
$$+\,48\,\alpha\,(\beta-6n)\,\sigma(\frac{n}{\alpha})+48\,\beta\,(\alpha-6n)\,\sigma(\frac{n}{\beta})-1152\,\alpha\beta\,W_{(\alpha,\beta)}(n)\;\bigg)q^n$$

as asserted. □

3. Evaluation of the Convolution Sums $W_{(\alpha,\beta)}(n)$, Where $\alpha\beta=48$ and $\alpha\beta=64$

We explicitly evaluate the convolution sums $W_{(1,48)}(n)$, $W_{(3,16)}(n)$, and $W_{(1,64}(n)$.

3.1. Bases for $\mathcal{E}_4(\Gamma_0(\alpha\beta))$ and $\mathcal{S}_4(\Gamma_0(\alpha\beta))$ With $\alpha\beta=48$ and $\alpha\beta=64$

The dimension formulae for the spaces of Eisenstein forms and cusp forms that we need are given in T. Miyake's book [20, Thrm 2.5.2, p. 60] and W. A. Stein's book [29, Prop. 6.1, p. 91]. They are applied to compute $\dim(\mathcal{E}_4(\Gamma_0(48)))=\dim(\mathcal{E}_4(\Gamma_0(64)))=12$, and $\dim(\mathcal{S}_4(\Gamma_0(48))=\dim(\mathcal{S}_4(\Gamma_0(64))=18$.

Let $D(48)=\{1,2,3,4,6,8,12,16,24,48\}$ and $D(64)=\{1,2,4,8,16,32,64\}$ be the sets of positive divisors of 48 and 64, respectively. It is essential to note that

$$\mathcal{M}_4(\Gamma_0(6))\subset\mathcal{M}_4(\Gamma_0(12))\subset\mathcal{M}_4(\Gamma_0(24))\subset\mathcal{M}_4(\Gamma_0(48)),\tag{15}$$
$$\mathcal{M}_4(\Gamma_0(8))\subset\mathcal{M}_4(\Gamma_0(24))\subset\mathcal{M}_4(\Gamma_0(48)),\tag{16}$$
$$\mathcal{M}_4(\Gamma_0(8))\subset\mathcal{M}_4(\Gamma_0(16))\subset\mathcal{M}_4(\Gamma_0(48)),\tag{17}$$
$$\mathcal{M}_4(\Gamma_0(8))\subset\mathcal{M}_4(\Gamma_0(16))\subset\mathcal{M}_4(\Gamma_0(32))\subset\mathcal{M}_4(\Gamma_0(64)).\tag{18}$$

The inclusion relations in (15) – (17) are graphically illustrated in Figure 1.

As observed in Subsection 2.2, we apply Theorem 2 $(i)-(v')$ to determine as many elements of $\mathcal{S}_4(\Gamma_0(48))$ and $\mathcal{S}_4(\Gamma_0(64))$ as possible. From these elements we then determine the basis elements of $\mathcal{S}_4(\Gamma_0(48))$ and $\mathcal{S}_4(\Gamma_0(64))$, respectively.

Theorem 4. ***(a)*** *The sets* $\mathcal{B}_{E,48}=\{\,M(q^t)\;\mid\;t\in D(48)\,\}\cup\{\,M_{(\frac{-3}{n})}(q^s)\mid s=1,2\}$ *and* $\mathcal{B}_{E,64}=\{\,M(q^t)\;\mid\;t\in D(64)\,\}\cup\{\,M_{(\frac{-3}{n})}(q^s)\mid s=1,2,4,8,16\}$ *are bases of the spaces* $\mathcal{E}_4(\Gamma_0(48))$ *and* $\mathcal{E}_4(\Gamma_0(64))$*, respectively.*

(b) *Let* $i,j\in\mathbb{N}$ *satisfy* $1\le i\le 18$ *and* $1\le j\le 18$*.*
Let $\delta_1\in D(48)$ *and* $(r(i,\delta_1))_{i,\delta_1}$ *be the Table 4 of the powers of* $\eta(\delta_1 z)$*.*
Let $\delta_2\in D(64)$ *and* $(r(j,\delta_2))_{j,\delta_2}$ *be the Table 5 of the powers of* $\eta(\delta_2 z)$*.*
Let furthermore $A_i(q)=\prod_{\delta_1\in D(48)}\eta^{r(i,\delta_1)}(\delta_1 z)$ *and* $B_j(q)=\prod_{\delta_2\in D(64)}\eta^{r(j,\delta_2)}(\delta_2 z)$ *be the selected elements of* $\mathcal{S}_4(\Gamma_0(48))$ *and* $\mathcal{S}_4(\Gamma_0(64))$*, respectively.*

Then the sets $\mathcal{B}_{S,48} = \{ A_i(q) \mid 1 \le i \le 18 \}$ *and* $\mathcal{B}_{S,64} = \{ B_j(q) \mid 1 \le j \le 18 \}$ *are bases of* $\mathcal{S}_4(\Gamma_0(48))$ *and* $\mathcal{S}_4(\Gamma_0(64))$*, respectively.*

(c) *The sets* $\mathcal{B}_{M,48} = \mathcal{B}_{E,48} \cup \mathcal{B}_{S,48}$ *and* $\mathcal{B}_{M,64} = \mathcal{B}_{E,64} \cup \mathcal{B}_{S,64}$ *are bases of the spaces* $\mathcal{M}_4(\Gamma_0(48))$ *and* $\mathcal{M}_4(\Gamma_0(64))$*, respectively.*

For $i, j \in \mathbb{N}$ such that $1 \le i \le 18$ and $1 \le j \le 18$, the eta quotients $A_i(q)$ and $B_j(q)$ can be expressed in the form $\sum_{n=1}^{\infty} a_i(n) q^n$ and $\sum_{n=1}^{\infty} b_j(n) q^n$, respectively.

Proof. We only prove the case $\alpha\beta = 48$. The case $\alpha\beta = 64$ is proved similarly when we use the same primitive Dirichlet character χ as the one for $\alpha\beta = 48$.

(a) Observe that

$$\chi = \left(\frac{-3}{n}\right) = \begin{cases} -1 & \text{if } n \equiv -1 \pmod 3, \\ 0 & \text{if } \gcd(3,n) \neq 1, \\ 1 & \text{if } n \equiv 1 \pmod 3. \end{cases} \tag{19}$$

is a primitive Dirichlet character with conductor 3. Therefore, the Eisenstein series (8) becomes $M_{\left(\frac{-3}{n}\right)}(q) = \sum_{n=1}^{\infty} \left(\frac{-3}{n}\right) \sigma_3(n)\, q^n$.

When we apply Theorem 5.8 in Section 5.3 of W. A. Stein [29, p. 86], it follows that $M(q^t)$ belongs to $\mathcal{M}_4(\Gamma_0(t))$ for each $t \in D(48)$. Since $\mathcal{M}_4(\Gamma_0(t))$ is a vector space, it also holds that $M_{\left(\frac{-3}{n}\right)}(q)$ and $M_{\left(\frac{-3}{n}\right)}(q^2)$ are in $\mathcal{M}_4(\Gamma_0(1))$ and $\mathcal{M}_4(\Gamma_0(2))$, respectively. Since the dimension of the space $\mathcal{E}_4(\Gamma_0(48))$ is finite, it is sufficient to show that the set $\{ M(q^t) \mid t \in D(48) \} \cup \{ M_{\left(\frac{-3}{n}\right)}(q), M_{\left(\frac{-3}{n}\right)}(q^2) \}$ is linearly independent.

Suppose that $x_t, z_1, z_2 \in \mathbb{C}$ with $t \in D(48)$. Then

$$\sum_{t \in D(48)} x_t\, M(q^t) + z_1\, M_{\left(\frac{-3}{n}\right)}(q) + z_2\, M_{\left(\frac{-3}{n}\right)}(q^2) = \sum_{t \in D(48)} x_t + \sum_{n \ge 1} \left(\left(\frac{-3}{n}\right) \sigma_3(n) z_1 + \left(\frac{-3}{n}\right) \sigma_3(\frac{n}{2}) z_2 + 240 \sum_{t \in D(48)} x_t \sigma_3(\frac{n}{t}) \right) q^n = 0.$$

We compare the coefficients of q^n for $n \in D(48)$ to obtain the following homogeneous system of 12 linear equations in 12 unknowns:

$$\left(\frac{-3}{t}\right) \sigma_3(t) z_1 + \left(\frac{-3}{t}\right) \sigma_3(\frac{t}{2}) z_2 + 240 \sum_{u \in D(48)} \sigma_3(\frac{t}{u}) x_u = 0, \qquad t \in D(48).$$

We use a software package for (symbolic) scientific computation to show that the determinant of the matrix of this homogeneous system of 12 linear equations is nonzero. Hence, the solution is $z_1 = z_2 = x_t = 0$ for all $t \in D(48)$. Therefore, the set $\mathcal{B}_{E,48}$ is linearly independent and hence is a basis of $\mathcal{E}_4(\Gamma_0(48))$.

(b) As mentioned above, $A_i(q)$ with $1 \le i \le 18$ are obtained from an exhaustive search using Theorem 2 $(i)-(v')$. Hence, each $A_i(q)$ is in the space $\mathcal{S}_4(\Gamma_0(48))$. Since the dimension of $\mathcal{S}_4(\Gamma_0(48))$ is 18, it is sufficient to show that the set $\{ A_i(q) \mid 1 \le i \le 18\}$ is linearly independent. Suppose that $x_i \in \mathbb{C}$ and $\sum_{i=1}^{18} x_i A_i(q) = 0$. Then

$$\sum_{i=1}^{18} x_i A_i(q) = \sum_{n=1}^{\infty} \Big(\sum_{i=1}^{18} x_i a_i(n) \Big) q^n = 0$$

which gives the following homogeneous system of 18 linear equations in 18 unknowns

$$\sum_{i=1}^{18} a_i(n)\, x_i = 0, \qquad 1 \le n \le 18. \tag{20}$$

The matrix of this homogeneous system of 18 linear equations is quasi triangular with 1 on the diagonal except for the last column which contains nonzero integer values on and above the diagonal. We then use a software package for (symbolic) scientific computation to show that the determinant of the matrix of this homogeneous system of 18 linear equations is nonzero. So, $x_i = 0$ for all $1 \le i \le 18$. Hence, the set $\{ A_i(q) \mid 1 \le i \le 18 \}$ is linearly independent and therefore a basis of $\mathcal{S}_4(\Gamma_0(48))$.

(c) Since $\mathcal{M}_4(\Gamma_0(48)) = \mathcal{E}_4(\Gamma_0(48)) \oplus \mathcal{S}_4(\Gamma_0(48))$, the result follows from (a) and (b).

□

Remark 5. When we use the inclusion relation (18), we observe that

1. $B_1(q)$ is a basis element of $\mathcal{S}_4(\Gamma_0(8))$.
2. $B_2(q)$ is a basis element of $\mathcal{S}_4(\Gamma_0(16))$.
3. $B_j(q)$ with $j = 3, 4, 5, 7, 9, 16$ belong to $\mathcal{S}_4(\Gamma_0(32))$. However, $B_9(q)$ is not a basis element of $\mathcal{S}_4(\Gamma_0(32))$ since the smallest degree of q^n in $B_9(q)$ is 9 and the dimension of the space $\mathcal{S}_4(\Gamma_0(32))$ is 8.
4. $B_{2j}(q) = B_j(q^2)$ for $j = 1, 2, 3, 4, 7, 9$. Consequently, $b_{2j}(n) = b_j(\frac{n}{2})$ for $j = 1$, 2, 3, 4, 7, 9.

3.2. Evaluation of $W_{(\alpha,\beta)}(n)$ When $\alpha\beta = 48$ and $\alpha\beta = 64$

Lemma 2. *We have*

$$
\begin{aligned}
(L(q) - 48\,L(q^{48}))^2 = 2209 + \sum_{n=1}^{\infty}\Big(&\frac{1164}{5}\,\sigma_3(n) - \frac{20412}{65}\,\sigma_3(\frac{n}{2}) - \frac{324}{5}\,\sigma_3(\frac{n}{3}) \\
- \frac{290736}{65}\,\sigma_3(\frac{n}{4}) + \frac{6372}{65}\,\sigma_3(\frac{n}{6}) + \frac{281664}{65}\,\sigma_3(\frac{n}{8}) &+ \frac{234576}{65}\,\sigma_3(\frac{n}{12}) - \frac{9216}{5}\,\sigma_3(\frac{n}{16}) \\
- \frac{506304}{65}\,\sigma_3(\frac{n}{24}) + \frac{2681856}{5}\,\sigma_3(\frac{n}{48}) &+ \frac{38772}{65}\,a_1(n) + \frac{639792}{65}\,a_2(n) \\
+ \frac{546804}{65}\,a_3(n) + \frac{661824}{65}\,a_4(n) &+ \frac{195264}{65}\,a_5(n) + \frac{3729456}{65}\,a_6(n) \\
+ \frac{67968}{5}\,a_7(n) + \frac{5422464}{65}\,a_8(n) &+ \frac{4414464}{65}\,a_9(n) + \frac{3151872}{65}\,a_{10}(n) \\
+ \frac{1426176}{65}\,a_{11}(n) + \frac{14145408}{65}\,a_{12}(n) &- \frac{3907584}{65}\,a_{13}(n) - \frac{1693440}{13}\,a_{14}(n) \\
+ \frac{313344}{65}\,a_{15}(n) + \frac{7299072}{13}\,a_{16}(n) &+ \frac{1032192}{13}\,a_{17}(n) + \frac{92736}{65}\,a_{18}(n)\Big)q^n, \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
(3\,L(q^3) - 16\,L(q^{16}))^2 = 169 + \sum_{n=1}^{\infty}\Big(&-\frac{36}{5}\,\sigma_3(n) + \frac{43092}{1885}\,\sigma_3(\frac{n}{2}) \\
+ \frac{10476}{5}\,\sigma_3(\frac{n}{3}) + \frac{1094256}{1885}\,\sigma_3(\frac{n}{4}) &- \frac{450252}{1885}\,\sigma_3(\frac{n}{6}) - \frac{1992384}{1885}\,\sigma_3(\frac{n}{8}) \\
- \frac{2722896}{1885}\,\sigma_3(\frac{n}{12}) + \frac{297984}{5}\,\sigma_3(\frac{n}{16}) &- \frac{4522176}{1885}\,\sigma_3(\frac{n}{24}) - \frac{82944}{5}\,\sigma_3(\frac{n}{48}) \\
- \frac{34668}{65}\,a_1(n) + \frac{3135888}{1885}\,a_2(n) &- \frac{140076}{65}\,a_3(n) - \frac{4567104}{1885}\,a_4(n) \\
- \frac{115776}{65}\,a_5(n) + \frac{20304}{1885}\,a_6(n) &+ \frac{91008}{5}\,a_7(n) - \frac{12196224}{1885}\,a_8(n) \\
+ \frac{589824}{65}\,a_9(n) + \frac{10119168}{1885}\,a_{10}(n) &- \frac{140544}{65}\,a_{11}(n) - \frac{118171008}{1885}\,a_{12}(n) \\
- \frac{3032064}{65}\,a_{13}(n) &- \frac{7167744}{377}\,a_{14}(n) - \frac{562176}{65}\,a_{15}(n) \\
- \frac{32182272}{377}\,a_{16}(n) &+ \frac{2313216}{13}\,a_{17}(n) + \frac{35136}{65}\,a_{18}(n)\Big)q^n, \qquad (22)
\end{aligned}
$$

$$\begin{aligned}(L(q)-64\,L(q^{64}))^2 &= 3969+\sum_{n=1}^{\infty}\Big(234\,\sigma_3(n)-18\,\sigma_3(\frac{n}{2})\\ &+\frac{69624}{13}\,\sigma_3(\frac{n}{4})-\frac{74304}{13}\,\sigma_3(\frac{n}{8})-1152\,\sigma_3(\frac{n}{16})-4608\,\sigma_3(\frac{n}{32})+958464\,\sigma_3(\frac{n}{64})\\ &+2790\,b_1(n)+7560\,b_2(n)+20160\,b_3(n)+\frac{112896}{13}\,b_4(n)+96768\,b_5(n)\\ &+48384\,b_6(n)+96768\,b_7(n)+17280\,b_8(n)+73728\,b_9(n)+221184\,b_{10}(n)\\ &+331776\,b_{11}(n)+64512\,b_{12}(n)-221184\,b_{13}(n)-276480\,b_{14}(n)+368640\,b_{15}(n)\\ &-\frac{564480}{13}\,b_{16}(n)+1290240\,b_{17}(n)+110592\,b_{18}(n)\Big)q^n. \quad (23)\end{aligned}$$

Proof. We just prove the case $(3\,L(q^3)-16\,L(q^{16}))^2$. The other cases are shown similarly.

From Lemma 1 it follows that $(3\,L(q^3)-16\,L(q^{16}))^2\in\mathcal{M}_4(\Gamma_0(48))$. Hence, by Theorem 4 (c), there exist $X_\delta, Z_1, Z_2, Y_j\in\mathbb{C}$ with $1\le j\le 18$ such that

$$\begin{aligned}(3\,L(q^3)-16\,L(q^{16}))^2 &= \sum_{\delta|48}X_\delta\,M(q^\delta)+Z_1\,M_{(\frac{-3}{n})}(q)+Z_2\,M_{(\frac{-3}{n})}(q^2)+\sum_{j=1}^{18}Y_j\,A_j(q) \quad (24)\\ &= \sum_{\delta|48}X_\delta+\sum_{n=1}^{\infty}\Big(\Big(\frac{-3}{n}\Big)\sigma_3(n)\,Z_1+\Big(\frac{-3}{n}\Big)\sigma_3(\frac{n}{2})\,Z_2\\ &+240\sum_{\delta|48}\sigma_3(\frac{n}{\delta})\,X_\delta+\sum_{j=1}^{18}a_j(n)\,Y_j\Big)q^n.\end{aligned}$$

We compare the right hand side of (24) with that of (9) when we have set $(\alpha,\beta)=(3,16)$ in (9). We then obtain

$$\begin{aligned}&\sum_{n=1}^{\infty}\Big(\Big(\frac{-3}{n}\Big)\sigma_3(n)\,Z_1+\Big(\frac{-3}{n}\Big)\sigma_3(\frac{n}{2})\,Z_2+240\sum_{\delta|48}\sigma_3(\frac{n}{\delta})\,X_\delta+\sum_{j=1}^{18}a_j(n)\,Y_j\Big)q^n\\ &=\sum_{n=1}^{\infty}\Big(2160\,\sigma_3(\frac{n}{3})+61440\,\sigma_3(\frac{n}{16})+144\,(16-6\,n)\,\sigma(\frac{n}{3})+768\,(3-6\,n)\,\sigma(\frac{n}{16})\\ &\qquad-55296\,W_{(3,16)}(n)\Big)q^n.\end{aligned}$$

When we then take the coefficients of q^n for which n is in

$$\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,\\ 22,24,25,26,27,28,30,32,48\},$$

we obtain a system of 30 linear equations whose resolution using a software package for symbolic scientific computation yields the unique solution which determines the values of the unknowns Z_1, Z_2, X_δ for all $\delta \in D(48)$ and the values of the unkowns Y_j for all $1 \le j \le 18$. Therefore, we get the stated result. $\square$

Our main result of this section will now be stated and proved.

Theorem 6. *Let n be a positive integer. Then*

$$\begin{aligned}
W_{(1,48)}(n) =& \frac{1}{7680}\,\sigma_3(n) + \frac{189}{33280}\,\sigma_3(\frac{n}{2}) + \frac{3}{2560}\,\sigma_3(\frac{n}{3}) + \frac{673}{8320}\,\sigma_3(\frac{n}{4}) \\
&- \frac{59}{33280}\,\sigma_3(\frac{n}{6}) - \frac{163}{2080}\,\sigma_3(\frac{n}{8}) - \frac{543}{8320}\,\sigma_3(\frac{n}{12}) + \frac{1}{30}\,\sigma_3(\frac{n}{16}) + \frac{293}{2080}\,\sigma_3(\frac{n}{24}) \\
&+ \frac{3}{10}\,\sigma_3(\frac{n}{48}) + (\frac{1}{24} - \frac{1}{192}n)\sigma(n) + (\frac{1}{24} - \frac{1}{4}n)\sigma(\frac{n}{48}) \\
&- \frac{359}{33280}\,a_1(n) - \frac{1481}{8320}\,a_2(n) - \frac{5063}{33280}\,a_3(n) - \frac{383}{2080}\,a_4(n) - \frac{113}{2080}\,a_5(n) \\
&- \frac{8633}{8320}\,a_6(n) - \frac{59}{240}\,a_7(n) - \frac{1569}{1040}\,a_8(n) - \frac{479}{390}\,a_9(n) - \frac{57}{65}\,a_{10}(n) \\
&- \frac{619}{1560}\,a_{11}(n) - \frac{4093}{1040}\,a_{12}(n) + \frac{212}{195}\,a_{13}(n) + \frac{245}{104}\,a_{14}(n) - \frac{17}{195}\,a_{15}(n) \\
&- \frac{132}{13}\,a_{16}(n) - \frac{56}{39}\,a_{17}(n) - \frac{161}{6240}\,a_{18}(n), \qquad (25)
\end{aligned}$$

$$\begin{aligned}
W_{(3,16)}(n) =& \frac{1}{7680}\,\sigma_3(n) - \frac{399}{965120}\,\sigma_3(\frac{n}{2}) + \frac{3}{2560}\,\sigma_3(\frac{n}{3}) - \frac{2533}{241280}\,\sigma_3(\frac{n}{4}) \\
&+ \frac{4169}{965120}\,\sigma_3(\frac{n}{6}) + \frac{1153}{60320}\,\sigma_3(\frac{n}{8}) + \frac{6303}{241280}\,\sigma_3(\frac{n}{12}) + \frac{1}{30}\,\sigma_3(\frac{n}{16}) \\
&+ \frac{2617}{60320}\,\sigma_3(\frac{n}{24}) + \frac{3}{10}\,\sigma_3(\frac{n}{48}) + (\frac{1}{24} - \frac{1}{64}n)\sigma(\frac{n}{3}) \\
&+ (\frac{1}{24} - \frac{1}{12}n)\sigma(\frac{n}{16}) + \frac{321}{33280}\,a_1(n) - \frac{7259}{241280}\,a_2(n) \\
&+ \frac{1297}{33280}\,a_3(n) + \frac{2643}{60320}\,a_4(n) + \frac{67}{2080}\,a_5(n) - \frac{47}{241280}\,a_6(n) \\
&- \frac{79}{240}\,a_7(n) + \frac{3529}{30160}\,a_8(n) - \frac{32}{195}\,a_9(n) - \frac{183}{1885}\,a_{10}(n) + \frac{61}{1560}\,a_{11}(n) \\
&+ \frac{34193}{30160}\,a_{12}(n) + \frac{329}{390}\,a_{13}(n) + \frac{1037}{3016}\,a_{14}(n) + \frac{61}{390}\,a_{15}(n) + \frac{582}{377}\,a_{16}(n) \\
&- \frac{251}{78}\,a_{17}(n) - \frac{61}{6240}\,a_{18}(n), \qquad (26)
\end{aligned}$$

$$\begin{aligned}W_{(1,64)}(n) =&\frac{1}{12288}\,\sigma_3(n)+\frac{1}{4096}\,\sigma_3(\frac{n}{2})-\frac{967}{13312}\,\sigma_3(\frac{n}{4})+\frac{129}{1664}\,\sigma_3(\frac{n}{8})\\&+\frac{1}{64}\,\sigma_3(\frac{n}{16})+\frac{1}{16}\,\sigma_3(\frac{n}{32})+\frac{1}{3}\,\sigma_3(\frac{n}{64})+(\frac{1}{24}-\frac{1}{208}n)\sigma(n)\\&+(\frac{1}{24}-\frac{1}{4}n)\sigma(\frac{n}{64})-\frac{155}{4096}\,b_1(n)-\frac{105}{1024}\,b_2(n)-\frac{35}{128}\,b_3(n)-\frac{49}{416}\,b_4(n)\\&-\frac{21}{16}\,b_5(n)-\frac{21}{32}\,b_6(n)-\frac{21}{16}\,b_7(n)-\frac{15}{64}\,b_8(n)-b_9(n)-3\,b_{10}(n)\\&-\frac{9}{2}\,b_{11}(n)-\frac{7}{8}\,b_{12}(n)+3\,b_{13}(n)+\frac{15}{4}\,b_{14}(n)-5\,b_{15}(n)\\&+\frac{245}{416}\,b_{16}(n)-\frac{35}{2}\,b_{17}(n)-\frac{3}{2}\,b_{18}(n).\end{aligned}\tag{27}$$

Proof. We prove the case $W_{(3,16)}(n)$ as the other cases are proved similarly.

We compare the right hand side of (22) with that of (9) when we have set $(\alpha,\beta)=(3,16)$ in (9). That then yield

$$\begin{aligned}\sum_{n=1}^{\infty}\Big(&\,2160\,\sigma_3(\frac{n}{3})+61440\,\sigma_3(\frac{n}{16})+144\,(16-6\,n)\,\sigma(\frac{n}{3})+768\,(3-6\,n)\,\sigma(\frac{n}{16})\\&-55296\,W_{(3,16)}(n)\Big)q^n=\sum_{n=1}^{\infty}\Big(-\frac{36}{5}\,\sigma_3(n)+\frac{43092}{1885}\,\sigma_3(\frac{n}{2})\\&+\frac{10476}{5}\,\sigma_3(\frac{n}{3})+\frac{1094256}{1885}\,\sigma_3(\frac{n}{4})-\frac{450252}{1885}\,\sigma_3(\frac{n}{6})-\frac{1992384}{1885}\,\sigma_3(\frac{n}{8})\\&-\frac{2722896}{1885}\,\sigma_3(\frac{n}{12})+\frac{297984}{5}\,\sigma_3(\frac{n}{16})-\frac{4522176}{1885}\,\sigma_3(\frac{n}{24})-\frac{82944}{5}\,\sigma_3(\frac{n}{48})\\&-\frac{34668}{65}\,a_1(n)+\frac{3135888}{1885}\,a_2(n)-\frac{140076}{65}\,a_3(n)-\frac{4567104}{1885}\,a_4(n)\\&-\frac{115776}{65}\,a_5(n)+\frac{20304}{1885}\,a_6(n)+\frac{91008}{5}\,a_7(n)-\frac{12196224}{1885}\,a_8(n)\\&+\frac{589824}{65}\,a_9(n)+\frac{10119168}{1885}\,a_{10}(n)-\frac{140544}{65}\,a_{11}(n)-\frac{118171008}{1885}\,a_{12}(n)\\&-\frac{3032064}{65}\,a_{13}(n)-\frac{7167744}{377}\,a_{14}(n)-\frac{562176}{65}\,a_{15}(n)\\&-\frac{32182272}{377}\,a_{16}(n)+\frac{2313216}{13}\,a_{17}(n)+\frac{35136}{65}\,a_{18}(n)\Big)q^n.\end{aligned}$$

When we solve for $W_{(3,16)}(n)$, we obtain the stated result. □

4. Number of Representations of a Positive Integer n by the Octonary Quadratic Form $a\,(x_1^2+x_2^2+x_3^2+x_4^2)+b\,(x_5^2+x_6^2+x_7^2+x_8^2)$

Let $n\in\mathbb{N}_0$. The number of representations of n by the quaternary quadratic form $x_1^2+x_2^2+x_3^2+x_4^2$ is defined by

$$r_4(n)=\mathrm{card}(\{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4\mid n=x_1^2+x_2^2+x_3^2+x_4^2\}).$$

Clearly $r_4(0)=1$. For $n\in\mathbb{N}$ the Jacobi's formula $r_4(n)$ is

$$r_4(n) = 8\,\sigma(n) - 32\,\sigma(\frac{n}{4}). \tag{28}$$

A proof of the Jacobi's formula $r_4(n)$ is given in K. S. Williams' book [32, Thm 9.5, p. 83]. This Jacobi's formula will be very useful in the following.

Let $a,b\in\mathbb{N}$ be such that $a\leq b$ and $\gcd(a,b)=1$. Let $N_{(a,b)}(n)$ denote the number of representations of n by the octonary quadratic form (3). Then

$$\begin{aligned}N_{(a,b)}(n) = \operatorname{card}(\{(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\in\mathbb{Z}^8 \mid n = a\,(x_1^2+x_2^2 \\ +x_3^2+x_4^2)+b\,(x_5^2+x_6^2+x_7^2+x_8^2)\}).\end{aligned}$$

The following result is then deduced.

Theorem 7. *Let $n\in\mathbb{N}$. Then*

$$\begin{aligned}
N_{(1,12)}(n) =& 8\sigma(n) - 32\sigma(\frac{n}{4}) + 8\sigma(\frac{n}{12}) - 32\sigma(\frac{n}{48}) \\
&+ 64\,W_{(1,12)}(n) + 1024\,W_{(1,12)}(\frac{n}{4}) - 256\left(W_{(1,3)}(\frac{n}{4}) + W_{(1,48)}(n)\right),\\
N_{(1,16)}(n) =& 8\sigma(n) - 32\sigma(\frac{n}{4}) + 8\sigma(\frac{n}{16}) - 32\sigma(\frac{n}{64}) \\
&+ 64\,W_{(1,16)}(n) + 1024\,W_{(1,16)}(\frac{n}{4}) - 256\left(W_{(1,4)}(\frac{n}{4}) + W_{(1,64)}(n)\right),\\
N_{(3,4)}(n) =& 8\sigma(\frac{n}{3}) - 32\sigma(\frac{n}{12}) + 8\sigma(\frac{n}{4}) - 32\sigma(\frac{n}{16}) \\
&+ 64\,W_{(3,4)}(n) + 1024\,W_{(3,4)}(\frac{n}{4}) - 256\left(W_{(1,3)}(\frac{n}{4}) + W_{(3,16)}(n)\right).
\end{aligned}$$

Proof. We only prove the formula for $N_{(1,12)}(n)$ since those for $N_{(1,16)}(n)$ and $N_{(3,4)}(n)$ are done similarly.

From the definition of $N_{(a,b)}(n)$ we set $(a,b)=(1,12)$ and it then follows that

$$N_{(1,12)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ l+12\,m=n}} r_4(l)r_4(m) = r_4(n)r_4(0) + r_4(0)r_4(\frac{n}{12}) + \sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+12\,m=n}} r_4(l)r_4(m).$$

We apply Jacobi's formula (28) to derive

$$\begin{aligned}
N_{(1,12)}(n) =& 8\sigma(n) - 32\sigma(\frac{n}{4}) + 8\sigma(\frac{n}{12}) - 32\sigma(\frac{n}{48}) \\
&+ \sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+12\,m=n}} (8\sigma(l) - 32\sigma(\frac{l}{4}))(8\sigma(m) - 32\sigma(\frac{m}{4})).
\end{aligned}$$

We observe from the previous identity that

$$(8\sigma(l)-32\sigma(\frac{l}{4}))(8\sigma(m)-32\sigma(\frac{m}{4})) = 64\sigma(l)\sigma(m)-256\sigma(\frac{l}{4})\sigma(m) - 256\sigma(l)\sigma(\frac{m}{4})+1024\sigma(\frac{l}{4})\sigma(\frac{m}{4}).$$

A. Alaca et al. [2] have evaluated

$$W_{(1,12)}(n)=\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+12\,m=n}}\sigma(l)\sigma(m).$$

When we apply the injection which sends l to $4l$, then we infer

$$W_{(1,3)}(\frac{n}{4})=\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+12\,m=n}}\sigma(\frac{l}{4})\sigma(m)=\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+3\,m=\frac{n}{4}}}\sigma(l)\sigma(m).$$

The evaluation of $W_{(1,3)}(n)$ is given by J. G. Huard et al. [13, Thrm 3, p. 20]. When we use the injective function which sends m to $4m$, we conclude that

$$W_{(1,48)}(n)=\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+12\,m=n}}\sigma(l)\sigma(\frac{m}{4})=\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+48\,m=n}}\sigma(l)\sigma(m).$$

The evaluation of $W_{(1,48)}(n)$ is given in (25). When we simultaneously make use of the injective functions which send l to $4l$ and m to $4m$, we deduce that

$$\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+12\,m=n}}\sigma(\frac{l}{4})\sigma(\frac{m}{4})=\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+12\,m=\frac{n}{4}}}\sigma(l)\sigma(m)=W_{(1,12)}(\frac{n}{4}).$$

Again, A. Alaca et al. [2] have given the evaluation of $W_{(1,12)}(n)$.

We then gather these evaluations together to obtain the stated result for $N_{(1,12)}(n)$. □

Instead of using the results of the evaluation of $W_{(1,12)}(n)$ and $W_{(3,4)}(n)$ obtained by A. Alaca et al. [2], one should apply those improved results obtained by E. Ntienjem [24].

5. Number of Representations of a positive Integer n by the Octonary Quadratic Form $c\,(x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2)+d\,(x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2)$

We apply (26) and (25) as well as other evaluated convolution sums to determine a formula for the number of representations of a positive integer n by the octonary quadratic form (4) for $(c,d)=(1,16)$.

Let $n \in \mathbb{N}_0$ and let $s_4(n)$ denote the number of representations of n by the quaternary quadratic form $x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2$, that is,

$$s_4(n) = \operatorname{card}(\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2\}).$$

It is clear that $s_4(0) = 1$. J. G. Huard et al. [13], G. A. Lomadze [19] and K. S. Williams [32, Thm 17.3, p. 225] have proved that for all $n \in \mathbb{N}$

$$s_4(n) = 12\,\sigma(n) - 36\,\sigma(\frac{n}{3}). \tag{29}$$

The number of representations of n by the octonary quadratic form (4) is

$$\begin{aligned} R_{(1,16)}(n) = \operatorname{card}(\{&(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8 \mid n = (x_1^2 + x_1x_2 \\ &+ x_2^2 + x_3^2 + x_3x_4 + x_4^2) + 16\,(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2)\}). \end{aligned}$$

We then infer the following result:

Theorem 8. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} R_{(1,16)}(n) =&12\sigma(n) - 36\sigma(\frac{n}{3}) + 12\sigma(\frac{n}{16}) - 36\sigma(\frac{n}{48}) + 144\,W_{(1,16)}(n) \\ &+ 1296\,W_{(1,16)}(\frac{n}{3}) - 432\left(W_{(3,16)}(n) + W_{(1,48)}(n)\right). \end{aligned}$$

Proof. We have

$$R_{(1,16)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2 \\ l+16\,m=n}} s_4(l)s_4(m) = s_4(n)s_4(0) + s_4(0)s_4(\frac{n}{16}) + \sum_{\substack{(l,m)\in\mathbb{N}^2 \\ l+16\,m=n}} s_4(l)s_4(m).$$

We apply (29) to derive

$$\begin{aligned} R_{(1,16)}(n) = 12\sigma(n) - 36\sigma(\frac{n}{3}) + 12\sigma(\frac{n}{16}) - 36\sigma(\frac{n}{48}) \\ + \sum_{\substack{(l,m)\in\mathbb{N}^2 \\ l+16\,m=n}} (12\sigma(l) - 36\sigma(\frac{l}{3}))(12\sigma(m) - 36\sigma(\frac{m}{3})). \end{aligned}$$

We know that

$$\begin{aligned} (12\sigma(l) - 36\sigma(\frac{l}{3}))(12\sigma(m) - 36\sigma(\frac{m}{3})) = 144\sigma(l)\sigma(m) - 432\sigma(\frac{l}{3})\sigma(m) \\ - 432\sigma(l)\sigma(\frac{m}{3}) + 1296\sigma(\frac{l}{3})\sigma(\frac{m}{3}). \end{aligned}$$

The evaluation of

$$W_{(1,16)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}^2 \\ l+16\,m=n}} \sigma(l)\sigma(m)$$

is shown by A. Alaca et al. [5]. We apply the injective function which sends m to $3m$ to derive

$$\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+16\,m=n}} \sigma(l)\sigma(\frac{m}{3}) = \sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+48\,m=n}} \sigma(l)\sigma(m) = W_{(1,48)}(n)$$

which is given in (25). We make use of the injective function which sends l to $3l$ to conclude

$$\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+16\,m=n}} \sigma(m)\sigma(\frac{l}{3}) = \sum_{\substack{(l,m)\in\mathbb{N}^2\\ 3\,l+16\,m=n}} \sigma(l)\sigma(m) = W_{(3,16)}(n)$$

which is given in (26). We simultaneously apply the injective functions which send l to $3l$ and m to $3m$ to infer

$$\sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+16\,m=n}} \sigma(\frac{m}{3})\sigma(\frac{l}{3}) = \sum_{\substack{(l,m)\in\mathbb{N}^2\\ l+16\,m=\frac{n}{3}}} \sigma(l)\sigma(m) = W_{(1,16)}(\frac{n}{3}).$$

which, again, is evaluated by A. Alaca et al. [5].

Finally, all these evaluations are put together to obtain the stated result for $R_{(1,16)}(n)$. □

Acknowledgments We express our gratitude to the anonymous referee and to Prof. Dr. K. S. Williams for fruitful comments and suggestions on a draft of this paper.

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Tables

	1	**2**	**3**	**4**	**6**	**8**	**12**	**16**	**24**	**48**
1	0	4	0	4	0	0	0	0	0	0
2	0	0	0	4	0	4	0	0	0	0
3	0	0	0	0	4	0	4	0	0	0
4	0	0	0	2	0	2	2	0	2	0
5	0	2	0	-2	-2	2	6	0	2	0

6	0	0	0	0	0	0	4	0	4	0
7	0	2	0	0	-2	-2	4	0	6	0
8	0	0	0	0	0	2	0	2	2	2
9	0	0	0	0	0	0	6	0	-2	4
10	0	0	0	0	0	1	4	1	-3	5
11	0	0	0	0	0	3	2	-3	1	5
12	0	0	0	0	0	4	0	-2	0	6
13	0	0	0	-1	0	7	1	-4	-3	8
14	0	0	0	0	0	3	4	-3	-5	9
15	0	0	0	0	0	4	2	-2	-6	10
16	0	-1	0	-1	3	3	-1	-3	-1	9
17	0	0	0	2	0	0	0	-2	-2	10
18	3	0	1	0	-1	0	0	3	5	-3

Table 4: Power of η-functions being basis elements of $\mathcal{S}_4(\Gamma_0(48))$

	1	**2**	**4**	**8**	**16**	**32**	**64**
1	0	4	4	0	0	0	0
2	0	0	4	4	0	0	0
3	0	4	0	0	4	0	0
4	0	0	0	4	4	0	0
5	0	2	1	0	3	2	0
6	0	0	4	0	0	4	0
7	0	0	2	0	2	4	0
8	0	0	0	0	4	4	0
9	0	0	2	2	-4	8	0
10	0	0	0	2	-2	8	0
11	0	0	0	1	2	3	2
12	0	0	0	0	6	-2	4
13	0	-4	10	-1	0	-3	6
14	0	0	0	2	0	2	4
15	0	0	0	1	4	-3	6
16	0	0	-4	6	2	4	0
17	0	0	-2	8	-2	-4	8
18	0	0	0	2	2	-4	8

Table 5: Power of η-functions being basis elements of $\mathcal{S}_4(\Gamma_0(64))$

Figures

Figure 1: Inclusion relation of the modular space of weight 4 for $\Gamma_0(48)$.