ON SHORT ZERO-SUM SEQUENCES OVER ABELIAN p-GROUPS

M. N. Chintamani
School of Mathematics and Statistics, University of Hyderabad, Hyderabad, India
mohansspecial@gmail.com

Prabal Paul
Department of Mathematics, BITS-Pillani, K.K. Birla Goa Campus, Goa, India
prabal.paul@gmail.com

R. Thangadurai
Harish-Chandra Research Institute (HBNI), Jhunsi, Allahabad, India
thanga@hri.res.in

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Abstract

Let G be a finite abelian additive group. In this paper, we deal with a combinatorial constant related to short zero-sum sequences over the abelian p-groups.

1. Introduction

Let G be a finite abelian additive group with exponent $\exp(G)$. A sequence S over G is written as

$$S = \prod_{i=1}^{|S|} g_i = \prod_{g \in G} g^{v_g(S)} \text{ with } v_g(S) \in \mathbb{Z}_{\geq 0}$$

where $v_g(S)$ is called the *multiplicity* of g in S and $|S|$ denotes the length of the sequence S. By the definition of multiplicity, we see that

$$|S| = \sum_{g \in G} v_g(S) \in \mathbb{Z}_{\geq 0}.$$

The sum of all the terms of the sequence S is given by

$$\sigma(S) = \sum_{g \in G} v_g(S) g \in G.$$

A sequence S over G is called a *zero-sum sequence* if $\sigma(S) = 0$. A sequence S is called *a short zero-sum sequence* if $\sigma(S) = 0$ and $|S| \in [1, \exp(G)]$. For all integers

$k \in \mathbb{Z}_{\geq 0}$ and for a sequence S over G, we define

$$N^k(S) = \left| \left\{ I \subset [1, |S|] \ : \ \sum_{i \in I} g_i = 0, |I| = k \right\} \right|,$$

which denotes the number of zero-sum subsequences of S of length k. The Davenport constant, $D(G)$, is the minimal positive integer t such that any given sequence S over G of length $|S| \geq t$ contains a nonempty zero-sum subsequence. The constant $\eta(G)$ is the minimal positive integer t such that any given sequence S over G of length $|S| \geq t$ contains a short zero-sum subsequence. Finally, the EGZ constant $\mathsf{s}(G)$ is the minimal positive integer t such that any given sequence S over G of length $|S| \geq t$ contains a zero-sum subsequence T of length $|T| = \exp(G)$.

These constants are classical invariants attached to a finite abelian group G in combinatorial number theory and have received a lot of attention (see for instance [1, 2, 5, 6, 7, 9, 10, 15, 16]). When G is a cyclic group, we have $\eta(G) = |G|$ and $\mathsf{s}(G) = 2|G| - 1$, by the well-known Erdős-Ginzburg-Ziv theorem [4]. For this contribution, this constant $\mathsf{s}(G)$ is called *EGZ constant*. When $G \cong C_p^2$ for a prime p, Olson [13, 14] proved in 1969 that $\eta(C_p^2) = 3p - 2$ and C. Reiher [15] proved in 2007 that $\mathsf{s}(C_p^2) = 4p - 3$, which was, earlier, conjectured by Kemnitz [11] in 1983. In general, if $G \cong C_m \oplus C_n$ with $m|n$ is the abelian group of rank 2, then it is known that $\mathsf{s}(G) = \eta(G) + n - 1 = 2m + 2n - 3$ as given in [10]. In 1995, Alon and Dubiner [1] proved that $\mathsf{s}(C_n^r) \leq c(r)n$ where $c(r)$ is a computable constant depends only on the rank r.

When G is of rank ≥ 3, nothing more is known. Even when $G \cong C_p^3$, for any prime p, these constants are still unkonwn. Recently, Fan, Gao, Wang and Zhong [7] determined the values $\eta(G)$ and $\mathsf{s}(G)$ for special type of abelian groups of rank 3. Apart from these results, Schmid and Zhuang [16] proved that if G is a finite abelian p-group with $D(G) = 2\exp(G) - 1$, then $\mathsf{s}(G) = 2D(G) - 1 = \eta(G) + \exp(G) - 1$. Moreover, they conjectured the following.

Conjecture 1. ([16]) Let G be a finite abelian p-group with $D(G) \leq 2\exp(G) - 1$. Then

$$\mathsf{s}(G) = 2D(G) - 1 = \eta(G) + \exp(G) - 1.$$

In this article, we prove the following theorems toward Conjecture 1 for a large class of abelian p-groups using the techniques employed in a recent paper of Gao, Han and Zhang [8].

Theorem 1. *Let H be a finite abelian p-group of rank $r(H)$ and $\exp(H) = p^m$ for some positive integer m and for some prime p with $p > 2r(H)$ and $D(H) - 1 =$*

$kp^m + t$ for some positive integer k and a non-negative integer t satisfying $0 \le t \le (p^m - 1)/2$. For all positive integers n with $p^n \ge 2(D(H) - 1)$, let $G = C_{p^n} \oplus H$ be the abelian p-group satisfying $D(G) \le 2p^n - 1 = 2\exp(G) - 1$. Let S be a sequence over G of length $p^n + 2(D(H) - 1)$. If $N^{p^n + j_0 p^m}(S) = 0$ for some integer j_0 with $1 \le j_0 \le k$, then S contains a short zero-sum subsequence.

In [8], Gao, Han and Zhang proved Conjecture 1 for the abelian p-groups G satisfying $k = 1$ or $t \in [p^m/2, p^m)$ (notation is as in Theorem 1). In the following theorem, we deal with the complement of this result.

Theorem 2. *Let H be a finite abelian p-group of rank $r(H)$ and $\exp(H) = p^m$ for some positive integer m and for some prime p with $p > 2r(H)$ and $D(H) - 1 = kp^m + t$ for some positive integer k and a non-negative integer t satisfying $0 \le t \le (p^m - 1)/2$. For all positive integers n with $p^n \ge 2(D(H) - 1) + p^m$, let $G = C_{p^n} \oplus H$ be the abelian p-group satisfying $D(G) \le 2p^n - 1 = 2\exp(G) - 1$. Then, we have,*

$$\eta(G) \le 2D(G) - \exp(G) + (\exp(H) - t - 1) = p^n + 2(D(H) - 1) + p^m - t.$$

Note that when $G = C_{p^n} \oplus H$, then $D(G) = p^n + D(H) - 1$. Therefore, Theorem 2 states that $\eta(G) \le (2D(G) - 1) - \exp(G) + (\exp(H) - t)$ for the case $t \in [0, (p^m - 1)/2]$ and hence $\exp(H) - t - 1$ is the extra term against Conjecture 1.

2. Preliminaries

Throughout this section, we take H to be a finite abelian p-group of rank $r(H)$ and exponent $\exp(H) = p^m$ for some positive integer m. Also, we write $D(H) - 1 = kp^m + t$ for some positive integer k and a non-negative integer t satisfying $0 \le t \le (p^m - 1)/2$. Choose any integer n such that $p^n \ge 2(D(H) - 1)$ and let $G = C_{p^n} \oplus H$.

We have the following lemmas which are needed in the proof of Theorem 1 and Theorem 2.

Lemma 2.1. ([8]) *Let $v = (k+1)p^m - D(H) = p^m - t - 1$. Let S be a sequence over G of length $|S| = p^n + 2(D(H) - 1)$ such that S contains no short zero-sum subsequences. For all integers i with $0 \le i \le k - 1$, let T be a subsequence of S of length $|T| = |S| - ip^m$. Then we have the following;*

$$1 + \sum_{u=0}^{h} \binom{h}{u} \sum_{j=1}^{k} (-1)^{j-1} N^{p^n + jp^m - u}(T) \equiv 0 \pmod{p}, \qquad (2.1)$$

for all $h \in [0, v]$.

Lemma 2.2. *Let* $v = (k+1)p^m - D(H) = p^m - t - 1$. *Let* S *be a sequence over* G *of length* $|S| = p^n + 2(D(H) - 1)$ *such that* S *contains no short zero-sum subsequences. For all integers* i *and* h *satisfying* $0 \le i \le k-1$ *and* $0 \le h \le v-1$, *we have*

$$\binom{|S|}{ip^m} + \sum_{j=1}^{k}(-1)^{j-1}\sum_{u=0}^{h}\binom{h}{u}\binom{|S| - p^n - jp^m + u}{ip^m} N^{p^n + jp^m - u}(S) \equiv 0 \pmod{p}. \tag{2.2}$$

Proof. This lemma is implicitly proved in Lemma 3.1 (3.3) of [8]. In order to get (2.2), we take a subsequence T of S such that $|T| = |S| - ip^m$ for a given integer i with $0 \le i \le k-1$. We can get

$$1 + \sum_{u=0}^{h}\binom{h}{u}\sum_{j=1}^{k}(-1)^{j-1}N^{p^n + jp^m - u}(T) \equiv 0 \pmod{p}.$$

Now we sum over all the subsequences T with $|T| = |S| - ip^m$ and we get

$$\sum_{T, |T| = |S| - ip^m}\left(1 + \sum_{u=0}^{h}\binom{h}{u}\sum_{j=1}^{k}(-1)^{j-1}N^{p^n + jp^m - u}(T)\right) \equiv 0 \pmod{p}. \tag{2.3}$$

Since each subsequence W of S with $|W| \le |S| - ip^m$ can be extended to a subsequence T of length $|T| = |S| - ip^m$ in

$$\binom{|S| - |W|}{|T| - |W|} = \binom{|S| - |W|}{|S| - |T|} = \binom{|S| - |W|}{ip^m}$$

ways, by starting with 0 length subsequence W of S, we see that the number of ways to get subsequences T of S with $|T| = |S| - ip^m$ is $\binom{|S|}{ip^m}$. Then, using this and expanding the sum in (2.3), we arrive at (2.2). ∎

Corollary 2.2.1. *Let* S *be a sequence over* G *as defined in Lemma 2.2. For all integers* i *with* $0 \le i \le k-1$, *we have*

$$\binom{|S|}{ip^m} + \sum_{j=1}^{k}(-1)^{j-1}\binom{|S| - p^n - jp^m}{ip^m} N^{p^n + jp^m}(S) \equiv 0 \pmod{p}. \tag{2.4}$$

Proof. Put $h = 0$ in Lemma 2.2 to get the result. ∎

Theorem 2.3. ([12]) *Let p be a prime number. Let a and b be positive integers with $a = a_n p^n + a_{n-1}p^{n-1} + \cdots + a_0$ with $a_i \in \{0, 1, \ldots, p-1\}$ and $b = b_n p^n + b_{n-1}p^{n-1} + \cdots + b_0$ with $b_i \in \{0, 1, \ldots, p-1\}$. Then*

$$\binom{a}{b} \equiv \binom{a_n}{b_n}\binom{a_{n-1}}{b_{n-1}} \cdots \binom{a_0}{b_0} \pmod{p}.$$

Theorem 2.4. ([8]) *Let n and k be positive integers with $1 \le 2k \le n$. Let A be the following $(k+1) \times (k+1)$ matrix with positive integers*

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \binom{n}{1} & \binom{n-1}{1} & \cdots & \binom{n-k}{1} \\ \binom{n}{2} & \binom{n-1}{2} & \cdots & \binom{n-k}{2} \\ \cdots & & \cdots & \\ \binom{n}{k} & \binom{n-1}{k} & \cdots & \binom{n-k}{k} \end{pmatrix}.$$

Then, the determinant of A is

$$det(A) = \left(\prod_{t=1}^{k} t!\right)^{-1} \prod_{1 \le i < j \le k} (j - i).$$

3. Proof of Theorem 1

Proof of Theorem 1. Let S be a sequence over G of length $|S| = p^n + 2(D(H) - 1)$. By the assumption, for some integer j_0 with $0 \le j_0 \le k$, we have $N^{p^n + j_0 p^m}(S) = 0$. Without loss of generality, we assume that $j_0 = k$ and hence $N^{p^n + kp^m}(S) = 0$, as the proofs of the other cases are similar. We need to prove that S contains a short zero-sum subsequence. On the contrary, we assume that S contains no short zero-sum subsequences. Hence, by Corollary 2.2.1, for all integers i with $0 \le i \le k-1$ and by the assumption with $j_0 = k$, we get

$$\binom{|S|}{ip^m} + \sum_{j=1}^{k-1} (-1)^{j-1} \binom{|S| - p^n - jp^m}{ip^m} N^{p^n + jp^m}(S) \equiv 0 \pmod{p}. \tag{3.1}$$

Note that for all integers j with $1 \le j \le k-1$, we have

$$|S| - p^n - jp^m = p^n + 2(kp^m + t) - p^n - jp^m = (2k - j)p^m + 2t.$$

Since $D(H) - 1 = kp^m + t$ for some integer t with $0 \le t \le (p^m - 1)/2$ and $p > 2r(H)$, we see that

$$D(H) - 1 \le r(H)\exp(H) - r(H) < r(H)p^m \le \left(\frac{p}{2} - 1\right) p^m \implies k \le \frac{p}{2} - 1.$$

Therefore, for all integers j with $1 \leq j \leq k-1$, we see that $2k - j < 2k < p$ and every integer $i < k < p$. Since $2t \leq p^m - 1 < p^m$, by Theorem 2.3, we get

$$\binom{|S| - p^n - jp^m}{ip^m} \equiv \binom{2k-j}{i}\binom{2t}{0} \equiv \binom{2k-j}{i} \pmod{p} \tag{3.2}$$

for all integers j with $1 \leq j \leq k-1$ and for all integers i with $0 \leq i \leq k-1$. Also, since $|S| = p^n + 2kp^m + 2t$ and $2t < p^m < p^n$, by Theorem 2.3, we get

$$\binom{|S|}{ip^m} \equiv \binom{2k}{i} \pmod{p} \tag{3.3}$$

for all integers i with $0 \leq i \leq k-1$. Therefore, by (3.1), (3.2) and (3.3), we get

$$\binom{2k}{i} + \sum_{j=1}^{k-1} \binom{2k-j}{i} (-1)^{j-1} N^{p^n + jp^m}(S) \equiv 0 \pmod{p} \tag{3.4}$$

for all $i = 0, 1, \ldots, k-1$. Now, put

$$X_j = (-1)^{j-1} N^{p^n + jp^m}(S)$$

for all $j = 1, 2, \ldots, k-1$ and $X_0 = 1$ as variables modulo p. Then, by putting $i = 0, 1, 2, \ldots, k-1$ in (3.4), we get a system of k linear equations in k variables modulo p as follows.

$$X_0 + X_1 + X_2 + \cdots + X_{k-1} = 0;$$
$$\binom{2k}{1}X_0 + \binom{2k-1}{1}X_1 + \binom{2k-2}{1}X_2 + \cdots + \binom{k+1}{1}X_{k-1} = 0;$$
$$\cdots\cdots\cdots$$
$$\cdots\cdots\cdots$$
$$\binom{2k}{k-1}X_0 + \binom{2k-1}{k-1}X_1 + \binom{2k-2}{k-1}X_2 + \cdots + \binom{k+1}{k-1}X_{k-1} = 0.$$

Note that the coefficient matrix of the above system of linear equations is nothing but A in Theorem 2.4 with $n = 2k$ and $k = k - 1$. Therefore, by Theorem 2.4, the determinant of the coefficient matrix is non-zero modulo p which forces the system to have only the trivial solution modulo p. That is,

$$X_0 \equiv X_1 \equiv \cdots \equiv X_{k-1} \equiv 0 \pmod{p},$$

which is a contradiction to $X_0 = 1 \not\equiv 0 \pmod{p}$. This proves the result. $\square$

4. Proof of Theorem 2

To prove Theorem 2, first we need to extend Lemma 2.2 so that it holds true for all $i \in [0, k]$ and for all $h \in [0, v)$, when $|S| = p^n + 2(D(H) - 1) + p^m - t$ for some integer t satisfying $0 \le t \le (p^m - 1)/2$ (notation is as in Section 2). Recall that H is an abelian p-group of rank $r(H)$ and exponent p^m. Let n be a positive integer such that $p^n \ge 2(D(H) - 1) + p^m$ and $G = C_{p^n} \oplus H$.

Lemma 4.1. *Let S be a given sequence over G of length $|S| = p^n + 2(D(H) - 1) + p^m - t$ where t is an integer satisfying $D(H) - 1 = kp^m + t$ with $0 \le t \le (p^m - 1)/2$. Let $v = (k+1)p^m - D(H) = p^m - t - 1$. If S contains no short zero-sum subsequences, then for all integers h and i satisfying $0 \le h < v$ and $0 \le i \le k$, we have*

$$\binom{|S|}{ip^m} + \sum_{j=1}^{k} (-1)^{j-1} \sum_{u=0}^{h} \binom{h}{u} \binom{|S| - p^n - jp^m + u}{ip^m} N^{p^n + jp^m - u}(S) \equiv 0 \pmod{p}. \tag{4.1}$$

Proof. The proof of this lemma is similar to the proof of Lemma 3.1 of [8]. Let S be the given sequence over G of length $|S| = p^n + 2(D(H) - 1) + p^m - t$ and S contains no short zero-sum subsequences.

Claim 1. $N^a(S) = 0$ for all integers a satisfying $1 \le a \le p^n$ or $p^n + D(H) \le a \le |S|$.

Since S contains no short zero-sum subsequences, we see that $N^a(S) = 0$ for all integers a satisfying $1 \le a \le p^n$. By the argument of Lemma 3.1 of [8], we see that $N^a(S) = 0$ for all integers a satisfying $p^n + D(H) \le a \le |S| - p^m + t$. Thus, to prove Claim 1, we need to prove that $N^a(S) = 0$ for all integers a satisfying $|S| - p^m + t + 1 \le a \le |S|$.

Let W be a zero-sum subsequence of S of length $|W| = a$ for some integer a satisfying $|S| - p^m + t + 1 \le a \le |S|$. Since $D(G) = p^n + D(H) - 1$ (by [13]) and $|W| = a \ge |S| - p^m + t + 1 = p^n + 2(D(H) - 1) + 1$, we see that W contains at least two disjoint zero-sum subsequences W_1 and W_2 such that $W = W_1W_2$. Since $N^b(S) = 0$ for all integers $1 \le b \le p^n$, we see that $|W_1| \ge p^n + 1$ and $|W_2| \ge p^n + 1$ and hence $|W| \ge 2p^n + 2$, which is a contradiction because $|S| < 2p^n + 2$ (as by hypothesis, $p^n \ge 2(D(H) - 1) + p^m$). This proves Claim 1.

Using Claim 1, the proof of Lemma 3.1 (3.1) of [8] yields Lemma 2.1 in Section 2 for the sequence S which in turn produces the congruence (4.1) for all integers i and h satisfying $0 \le i \le k - 1$ and $0 \le h < v$. Hence, it is enough to prove the congruence (4.1) for $i = k$ and for all integers h with $0 \le h < v$. Let T be any subsequence of S of length $|T| = |S| - kp^m$. Then consider the sequence $T0^h$ for a given integer h with $0 \leq h < v$. Note that

$$\begin{aligned}|T0^h| &= |T| + h = |S| - kp^m + h \\ &= p^n + 2(D(H)-1) + p^m - t - kp^m + h \\ &= p^n + D(H) - 1 + p^m + D(H) - 1 - kp^m - t + h \\ &\geq D(G) + p^m - 1.\end{aligned}$$

Then the rest of the proof is just the same as that of Lemma 3.1 in [8]. □

Proof of Theorem 2. Let H be a finite abelian p-group of rank $r = r(H)$ and the exponent p^m. Let $D(H) - 1 = kp^m + t$ for some positive integer k and for some integer t with $0 \leq t \leq (p^m - 1)/2$. Let n be an integer such that $p^n \geq 2(D(H) - 1) + p^m$ and let $G = C_{p^n} \oplus H$.

To prove $\eta(G) \leq p^n + 2(D(H) - 1) + p^m - t$, we let S be a sequence over G of length $|S| = p^n + 2(D(H) - 1) + p^m - t$ and we prove that S contains a short zero-sum subsequence.

Suppose S contains no short zero-sum subsequences. Hence, by Lemma 4.1 with $h = 0$, we get

$$\binom{|S|}{ip^m} + \sum_{j=1}^{k} (-1)^{j-1} \binom{|S| - p^n - jp^m}{ip^m} N^{p^n + jp^m}(S) \equiv 0 \pmod{p}, \tag{4.2}$$

for all integers i with $0 \leq i \leq k$. Note that for all integers j satisfying $0 \leq j \leq k$, we have

$$|S| - p^n - jp^m = p^n + 2(kp^m + t) + p^m - t - p^n - jp^m = (2k + 1 - j)p^m + t.$$

Since $D(H) - 1 = kp^m + t$ for some integer t with $0 \leq t \leq (p^m - 1)/2$ and $p > 2r(H)$, we see that

$$D(H) - 1 \leq r(H)\exp(H) - r(H) < r(H)p^m \leq \left(\frac{p}{2} - 1\right)p^m \text{ implies } k \leq \frac{p}{2} - 1.$$

Therefore, for all integers j with $1 \leq j \leq k$, we see that $2k + 1 - j < 2k + 1 < p$ and every integer $i \leq k < p/2$. Also, since $|S| = p^n + 2(D(H) - 1) + p^m - t = p^n + 2(kp^m + t) + p^m - t = p^n + (2k + 1)p^m + t$, and $2t \leq p^m - 1 < p^m$, by Theorem 2.3, we get

$$\binom{|S| - p^n - jp^m}{ip^m} \equiv \binom{2k + 1 - j}{i}\binom{t}{0} \equiv \binom{2k + 1 - j}{i} \pmod{p}$$

for all integers j satisfying $1 \leq j \leq k$. Also, since $|S| = p^n + (2k + 1)p^m + t$ and $2t < p^m < p^n$, by Theorem 2.3, we get

$$\binom{|S|}{ip^m} \equiv \binom{2k + 1}{i} \pmod{p}$$

for all integers i with $0 \leq i \leq k$. Therefore, by (4.2), we get

$$\binom{2k+1}{i} + \sum_{j=1}^{k} \binom{2k+1-j}{i} (-1)^{j-1} N^{p^n+jp^m}(S) \equiv 0 \pmod{p} \tag{4.3}$$

for all integers i satisfying $i = 0, 1, \ldots, k$.

Now, put

$$X_j = (-1)^{j-1} N^{p^n+jp^m}(S)$$

for all $j = 1, 2, \ldots, k$ and $X_0 = 1$ as variables modulo p. By (4.3), we have the following system of linear equations in $k+1$ variables modulo p.

$$X_0 + X_1 + X_2 + \cdots + X_k = 0;$$

$$\binom{2k+1}{1} X_0 + \binom{2k+1-1}{1} X_1 + \binom{2k+1-2}{1} X_2 + \cdots + \binom{2k+1-k}{1} X_k = 0;$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

$$\binom{2k+1}{k} X_0 + \binom{2k+1-1}{k} X_1 + \binom{2k+1-2}{k} X_2 + \cdots + \binom{2k+1-k}{k} X_k = 0.$$

Note that the coefficient matrix of the above system of linear equations is nothing but A in Theorem 2.4 with $n = 2k+1$ and $k = k$. Therefore, by Theorem 2.4, the determinant of the coefficient matrix is non-zero modulo p which forces the system to have only the trivial solution modulo p. That is,

$$X_0 \equiv X_1 \equiv \cdots \equiv X_k \equiv 0 \pmod{p},$$

which is a contradiction as $X_0 = 1 \not\equiv 0 \pmod{p}$. This proves the theorem. □

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