A DIGIT REVERSAL PROPERTY FOR STERN POLYNOMIALS

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Abstract

We consider the following polynomial generalization of Stern's diatomic series: let $s_1(x,y)=1$, and for $n\geq 1$ set $s_{2n}(x,y)=s_n(x,y)$ and $s_{2n+1}(x,y)=x\,s_n(x,y)+y\,s_{n+1}(x,y)$. The coefficient $[x^iy^j]s_n(x,y)$ is the number of hyperbinary expansions of $n-1$ with exactly i occurrences of the digit $\mathtt{2}$ and j occurrences of $\mathtt{0}$. We prove that the polynomials s_n are invariant under *digit reversal*, that is, $s_n=s_{n^R}$, where n^R is obtained from n by reversing the binary expansion of n.

1. Introduction

The Stern sequence (also called Stern's diatomic sequence or Stern–Brocot sequence) s is defined by the recurrence $s_1=1$, $s_{2n}=s_n$ and $s_{2n+1}=s_n+s_{n+1}$. It was pointed out by Dijkstra [4], [3, pp.230–232] that this sequence satisfies a symmetry property with respect to reversal of the binary expansion of n. More precisely, for a positive integer n having the proper base-2 expansion $n=\left(\varepsilon_\nu\varepsilon_{\nu-1}\ldots\varepsilon_0\right)_2$ we define

$$n^R=\sum_{0\leq i\leq\nu}\varepsilon_{\nu-i}2^i=\left(\varepsilon_0\varepsilon_1\ldots\varepsilon_\nu\right)_2.$$

Theorem A (Dijkstra). $s_n=s_{n^R}$.

Stern's diatomic sequence is closely related to continued fractions (see Stern [14], Lehmer [10], Lind [11], Graham, Knuth, Patashnik [8, Exercise 6.50]):
if $n=(1^{k_0}0^{k_1}\cdots 1^{k_{r-2}}0^{k_{r-1}}1^{k_r})_2$, then s_n is the numerator of the continued fraction $[k_0;k_1,\ldots,k_r]$. Theorem A is therefore the same as the statement that $[k_0;k_1,\ldots,k_r]$ and $[k_r;k_{r-1},\ldots,k_0]$ have the same numerator, which can be proved via continuants.

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In [12], Morgenbesser and the author proved a digit reversal property for the correlation

$$\gamma_t(\alpha) = \lim_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \mathrm{e}\bigl(\alpha\sigma_q(n+t) - \alpha\sigma_q(n)\bigr),$$

where $\mathrm{e}(x) = \exp(2\pi i x)$ and $\sigma_q(n)$ is the sum of digits of n in base q ($q \ge 2$ an integer). That is, we proved that $\gamma_t(\alpha) = \gamma_{t^R}(\alpha)$, where the digit reversal is in base q. We note that for the case $q = 2$ this statement is a special case of Theorem 1 below.

In this paper, we wish to give a refinement of Theorem A, concerning *hyperbinary expansions*. A hyperbinary expansion [5] of a nonnegative integer n is a sequence $(\varepsilon_{\nu-1}, \ldots, \varepsilon_0) \in \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}^\nu$ such that $\sum_{0\le i<\nu} \varepsilon_i 2^i = n$. We call such an expansion *proper* if either $\nu = 0$ or $\nu > 0$ and $\varepsilon_{\nu-1} \ne \mathtt{0}$. Since we only work with proper expansions, we will usually omit the prefix "proper". By induction, using the defining recurrence relation, it is not difficult to see that s_n is the number of hyperbinary expansions of $n-1$ (see also Proposition 1, which generalizes this property). This property is stated as Theorem 5.2 in [13], which seems to be the earliest (correct) appearance in the literature — we note that the statement on the bottom of page 275 of [2] is erroneous, which can be seen considering the case $n = 5$. The same statement can be found on page 57 of [11].

2. Main Results

Our main theorem generalizes Theorem A. We define bivariate polynomials $s_n(x, y)$ by

$$\begin{aligned} s_1(x,y) &= 1,\\ s_{2n}(x,y) &= s_n(x,y),\\ s_{2n+1}(x,y) &= x\, s_n(x,y) + y\, s_{n+1}(x,y). \end{aligned}$$

Note that this definition of Stern polynomials differs from the one given in [6] and also from the definition in [9]. However, the univariate polynomials $s_n(x, 1)$ appear, up to a shift of the index n, in the article [1] by Bates and Mansour: the authors write "[...]the nth term $f(n; q)$ of the q-analogue of the Calkin–Wilf sequence is the generating function for the number of hyperbinary expansions of n according to the number of powers that are used exactly twice." This should be compared with Proposition 1 below. Moreover, $s_n(x, 1)$ appears as the special case $t = 1$ in [5] (see (1.2), (1.3) in that paper). The bivariate polynomial $s_n(x, y)$, however, appears to be a new object of study. We list the first few of these polynomials: we have

$s_1(x, y) = 1,$ $\qquad$ $s_2(x, y) = 1,$

$$
\begin{aligned}
s_3(x,y) &= x+y, & s_4(x,y) &= 1,\\
s_5(x,y) &= x+xy+y^2, & s_6(x,y) &= x+y,\\
s_7(x,y) &= x^2+xy+y, & s_8(x,y) &= 1,\\
s_9(x,y) &= x+xy+xy^2+y^3, & s_{10}(x,y) &= x+xy+y^2,\\
s_{11}(x,y) &= x^2+xy+y^2+x^2y+xy^2, & s_{12}(x,y) &= x+y,\\
s_{13}(x,y) &= x^2+xy+y^2+x^2y+xy^2, & s_{14}(x,y) &= x^2+xy+y,\\
s_{15}(x,y) &= y+xy+x^3+x^2y, & s_{16}(x,y) &= 1,\\
s_{17}(x,y) &= x+xy+xy^2+xy^3+y^4, & s_{18}(x,y) &= x+xy+xy^2+y^3,\\
s_{19}(x,y) &= x^2+xy+x^2y+xy^2+y^3+x^2y^2+xy^3, & s_{20}(x,y) &= x+xy+y^2,\\
s_{21}(x,y) &= x^2+2x^2y+2xy^2+y^3+x^2y^2+xy^3, & s_{22}(x,y) &= x^2+xy+y^2\\
& & &\quad +x^2y+xy^2,\\
s_{23}(x,y) &= xy+y^2+x^3+x^2y+xy^2+x^3y+x^2y^2, & s_{24}(x,y) &= x+y,\\
s_{25}(x,y) &= x^2+xy+x^2y+xy^2+y^3+x^2y^2+xy^3, & s_{26}(x,y) &= x^2+xy+y^2\\
& & &\quad +x^2y+xy^2
\end{aligned}
$$

and we see the notable identities $s_{11}(x,y) = s_{13}(x,y)$ and $s_{19}(x,y) = s_{25}(x,y)$, where $13 = 11^R$ and $25 = 19^R$. In fact, we have the following symmetry property generalizing Theorem A.

Theorem 1. *Let n be a positive integer. Then*

$$s_n(x,y) = s_{n^R}(x,y).$$

This theorem can be translated to a statement on hyperbinary expansions. For integers $i,j \geq 0$ and $t \geq 1$ let $h_{i,j}(t)$ be the number of proper hyperbinary expansions $(\varepsilon_{\nu-1},\ldots,\varepsilon_0)$ of $t-1$ such that $|\{\ell : 0 \leq \ell < \nu, \varepsilon_\ell = 2\}| = i$ and $|\{\ell : 0 \leq \ell < \nu, \varepsilon_\ell = 0\}| = j$. The following proposition connects the Stern polynomials $s_n(x,y)$ to hyperbinary expansions.

Proposition 1. *Assume that $n \geq 1$ is an integer. Then*

$$\sum_{i,j\geq 0} h_n(i,j)x^iy^j = s_n(x,y).$$

In other words, we have $h_n(i,j) = \left[x^iy^j\right]s_n(x,y)$, that is, the polynomial $s_n(x,y)$ encodes the number of hyperbinary expansions of $n-1$ having a given number of $\mathtt{2}$s and $\mathtt{0}$s. Let us illustrate this proposition by an example. The polynomial $s_{21}(x,y)$ can be obtained by listing the hyperbinary expansions of 20:

1212	x^2	1220	x^2y
2012	x^2y	2100	xy^2

10012	xy^2	10100	y^3
2020	x^2y^2	10020	xy^3.

Combining Theorem 1 and Proposition 1, we immediately get the following corollary.

Corollary 1. *Let $n \geq 1$ and $i, j \geq 0$. We have*

$$h_n(i,j) = h_{n^R}(i,j).$$

For example, we list the hyperbinary expansions of $18 = (\mathtt{10011})_2 - 1$ (first row) and $24 = (\mathtt{11001})_2 - 1$ (second row):

10010	2010	1210	10002	2002	1202	1122
11000	10120	10112	10200	2200	2120	2112

By the corollary there is a one-to-one correspondence between these expansions.

3. Proofs

3.1. Proof of Theorem 1

The main argument, which represents the induction step in the proof of the theorem, is the following lemma.

Lemma 1. *Let*

$$A = \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.$$

If ${}^tA$ denotes the transpose of the matrix A, the following identities for 1×2-matrices hold.

$$\begin{aligned}
\begin{pmatrix} x & y \end{pmatrix} AA &= (-y)\begin{pmatrix} x & y \end{pmatrix} + (y+1)\begin{pmatrix} x & y \end{pmatrix} A,\\
\begin{pmatrix} 1 & 1 \end{pmatrix} {}^tA{}^tA &= (-y)\begin{pmatrix} 1 & 1 \end{pmatrix} + (y+1)\begin{pmatrix} 1 & 1 \end{pmatrix} {}^tA,\\[1ex]
\begin{pmatrix} x & y \end{pmatrix} AB &= x\begin{pmatrix} x & y \end{pmatrix} + y\begin{pmatrix} x & y \end{pmatrix} B,\\
\begin{pmatrix} 1 & 1 \end{pmatrix} {}^tA{}^tB &= x\begin{pmatrix} 1 & 1 \end{pmatrix} + y\begin{pmatrix} 1 & 1 \end{pmatrix} {}^tB,\\[1ex]
\begin{pmatrix} x & y \end{pmatrix} BA &= y\begin{pmatrix} x & y \end{pmatrix} + x\begin{pmatrix} x & y \end{pmatrix} A,\\
\begin{pmatrix} 1 & 1 \end{pmatrix} {}^tB{}^tA &= y\begin{pmatrix} 1 & 1 \end{pmatrix} + x\begin{pmatrix} 1 & 1 \end{pmatrix} {}^tA,\\[1ex]
\begin{pmatrix} x & y \end{pmatrix} BB &= (-x)\begin{pmatrix} x & y \end{pmatrix} + (x+1)\begin{pmatrix} x & y \end{pmatrix} B,\\
\begin{pmatrix} 1 & 1 \end{pmatrix} {}^tB{}^tB &= (-x)\begin{pmatrix} 1 & 1 \end{pmatrix} + (x+1)\begin{pmatrix} 1 & 1 \end{pmatrix} {}^tB.
\end{aligned}$$

The proof is by simple calculation and is left to the reader. □

To prove Theorem 1, we let $n \geq 1$ be an odd integer. This is no loss of generality since we can deal with the even case by repeatedly using the relation $s_{2n}(x,y) = s_n(x,y)$. Let $n = \sum_{i \leq \nu} \varepsilon_i 2^i$ be the binary representation of n and $\varepsilon_\nu \neq 0$. We prove the theorem by induction on ν. The case $\nu \leq 1$ is trivial, since in this case we have $n^R = n$. We write $A(\mathtt{0}) = \left(\begin{smallmatrix} 1 & 0 \\ x & y \end{smallmatrix}\right)$ and $A(\mathtt{1}) = \left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right)$. By a simple application of the recurrence relation we have $\left(\begin{smallmatrix} s_{2n} \\ s_{2n+1} \end{smallmatrix}\right) = A(\mathtt{0}) \left(\begin{smallmatrix} s_n \\ s_{n+1} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} s_{2n+1} \\ s_{2n+2} \end{smallmatrix}\right) = A(\mathtt{1}) \left(\begin{smallmatrix} s_n \\ s_{n+1} \end{smallmatrix}\right)$ for all $n \geq 1$. Since n is odd and $s_1(x,y) = s_2(x,y) = 1$, it follows from these identities that

$$s_n(x,y) = \begin{pmatrix} x & y \end{pmatrix} A(\varepsilon_1) \cdots A(\varepsilon_{\nu-1}) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{1}$$

The statement of the theorem is equivalent to the assertion that

$$\begin{pmatrix} x & y \end{pmatrix} A(\varepsilon_1) \cdots A(\varepsilon_{\nu-1}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} {}^t\!A(\varepsilon_1) \cdots {}^t\!A(\varepsilon_{\nu-1}) \begin{pmatrix} x \\ y \end{pmatrix} \tag{2}$$

for all $\nu \geq 1$ and all finite sequences $(\varepsilon_1, \ldots, \varepsilon_{\nu-1})$ in $\{\mathtt{0}, \mathtt{1}\}$. We prove the identity (2) by induction on ν, using Lemma 1. This identity is obvious for $\nu \leq 2$. For $\nu > 2$ we have four cases, corresponding to the four possible values of $(\varepsilon_1, \varepsilon_2)$. By Lemma 1, in each of the four cases there exist coefficients α and β such that

$$\begin{pmatrix} x & y \end{pmatrix} A(\varepsilon_1) A(\varepsilon_2) = \alpha \begin{pmatrix} x & y \end{pmatrix} + \beta \begin{pmatrix} x & y \end{pmatrix} A(\varepsilon_2)$$

and

$$\begin{pmatrix} 1 & 1 \end{pmatrix} {}^t\!A(\varepsilon_1) {}^t\!A(\varepsilon_2) = \alpha \begin{pmatrix} 1 & 1 \end{pmatrix} + \beta \begin{pmatrix} 1 & 1 \end{pmatrix} {}^t\!A(\varepsilon_2).$$

By applying the induction hypothesis (2) to the sequences $(\varepsilon_2, \ldots, \varepsilon_{\nu-1})$ and $(\varepsilon_3, \ldots, \varepsilon_{\nu-1})$ we obtain the statement of the theorem. □

3.2. Proof of Proposition 1

In order to prove Proposition 1, we use the following recurrence for $h_n(i,j)$ (see [7]).

Proposition 2. *Let $n \geq 1$. Then*

$$\begin{array}{lcll} h_1(0,0) & = & 1, & \\ h_1(i,j) & = & 0 & \textit{for } (i,j) \neq (0,0), \\ h_{2n}(i,j) & = & h_n(i,j) & \textit{for } i,j \geq 0, \\ h_{2n+1}(i,0) & = & h_n(i-1,0) & \textit{for } i \geq 1, \\ h_{2n+1}(0,j) & = & h_{n+1}(0,j-1) & \textit{for } j \geq 1, \\ h_{2n+1}(i,j) & = & h_n(i-1,j) + h_{n+1}(i,j-1) & \textit{for } i,j \geq 1. \end{array} \tag{3}$$

Moreover, $h_t(0,0) = 0$ if t is not a power of 2.

Proof. The first two lines follow from the fact that the empty tuple () is the only hyperbinary expansion of 0.

The hyperbinary expansions of $2n-1$ are in bijection with the hyperbinary expansions of $n-1$ by deleting the lowest digit, which is a `1`. This explains the third line of (3).

A hyperbinary expansion of $2n$ without `0` necessarily ends with the digit `2`, and the bijection is by deleting this digit. This gives line number four.

There is exactly one hyperbinary expansion of $2n$ without `2`, and it ends with `0`. The same argument applies here.

The sixth line is a combination of arguments as above.

Finally, integers having the binary expansion $(\mathtt{1}\cdots\mathtt{1})$ are the only ones without `0` and `2`, which proves the last line. $\square$

In order to prove Proposition 1, we proceed by induction. The identity is trivial for $n=1$. Assume that $n=2u$ for some $u \geq 1$. We have $h_n(i,j) = h_u(i,j)$ and $s_n(x,y) = s_u(x,y)$. If $n = 2u+1$, we have

$$\begin{aligned}
\sum_{\substack{i\geq 0\\ j\geq 0}} h_n(i,j)x^iy^j &= h_n(0,0) + \sum_{j\geq 1} h_n(0,j)y^j + \sum_{i\geq 1} h_n(i,0)x^i + \sum_{i,j\geq 1} h_n(i,j)x^iy^j \\
&= \sum_{j\geq 1} h_{u+1}(0,j-1)y^j + \sum_{i\geq 1} h_u(i-1,0)x^i \\
&+ \sum_{i,j\geq 1} \big(h_u(i-1,j) + h_{u+1}(i,j-1)\big)x^iy^j \\
&+ \sum_{\substack{i\geq 1\\ j\geq 0}} h_u(i-1,j)x^iy^j + \sum_{\substack{i\geq 0\\ j\geq 1}} h_u(i,j-1)x^iy^j \\
&= xs_u(x,y) + ys_{u+1}(x,y) = s_n(x,y),
\end{aligned}$$

which proves the desired assertion. $\square$

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