A DIGIT REVERSAL PROPERTY FOR STERN POLYNOMIALS

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Abstract

We consider the following polynomial generalization of Stern’s diatomic series: let $s_1(x, y) = 1$, and for $n \geq 1$ set $s_{2n}(x, y) = s_n(x, y)$ and $s_{2n+1}(x, y) = x s_n(x, y) + y s_{n+1}(x, y)$. The coefficient $[x^y] s_n(x, y)$ is the number of hyperbinary expansions of $n-1$ with exactly $i$ occurrences of the digit 2 and $j$ occurrences of 0. We prove that the polynomials $s_n$ are invariant under digit reversal, that is, $s_n = s_n^R$, where $n^R$ is obtained from $n$ by reversing the binary expansion of $n$.

1. Introduction

The Stern sequence (also called Stern’s diatomic sequence or Stern–Brocot sequence) $s$ is defined by the recurrence $s_1 = 1$, $s_{2n} = s_n$ and $s_{2n+1} = s_n + s_{n+1}$. It was pointed out by Dijkstra [4], [3, pp.230–232] that this sequence satisfies a symmetry property with respect to reversal of the binary expansion of $n$. More precisely, for a positive integer $n$ having the proper base-2 expansion $n = (e_\nu e_{\nu-1} \ldots e_0)_2$ we define

$$n^R = \sum_{0 \leq i \leq \nu} e_{\nu-i} 2^i = (e_0 e_1 \ldots e_\nu)_2.$$

Theorem A (Dijkstra). $s_n = s_n^R$.

Stern’s diatomic sequence is closely related to continued fractions (see Stern [14], Lehmer [10], Lind [11], Graham, Knuth, Patashnik [8, Exercise 6.50]): if $n = (1^{k_0} 0^{k_1} \ldots 1^{k_{r-1}} 0^{k_r-1} 1^{k_r})_2$, then $s_n$ is the numerator of the continued fraction $[k_0; k_1, \ldots, k_r]$. Theorem A is therefore the same as the statement that $[k_0; k_1, \ldots, k_r]$ and $[k_r; k_{r-1}, \ldots, k_0]$ have the same numerator, which can be proved via continuants.

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In [12], Morgenbesser and the author proved a digit reversal property for the correlation
\[ \gamma_t(\alpha) = \lim_{N \to \infty} \frac{1}{N} \sum_{0 \leq n < N} e(\alpha \sigma_q(n + t) - \alpha \sigma_q(n)), \]
where \( e(x) = \exp(2\pi ix) \) and \( \sigma_q(n) \) is the sum of digits of \( n \) in base \( q \) \((q \geq 2 \text{ an integer}) \). That is, we proved that \( \gamma_t(\alpha) = \gamma_t(\nu)(\alpha) \), where the digit reversal is in base \( q \). We note that for the case \( q = 2 \) this statement is a special case of Theorem 1 below.

In this paper, we wish to give a refinement of Theorem A, concerning hyperbinary expansions. A hyperbinary expansion [5] of a nonnegative integer \( n \) is a sequence \((\varepsilon_{\nu-1}, \ldots, \varepsilon_0) \in \{0, 1, 2\}^\nu \) such that \( \sum_{0 \leq i < \nu} \varepsilon_i 2^i = n \). We call such an expansion proper if either \( \nu = 0 \) or \( \nu > 0 \) and \( \varepsilon_{\nu-1} \neq 0 \). Since we only work with proper expansions, we will usually omit the prefix “proper”. By induction, using the defining recurrence relation, it is not difficult to see that \( s_n \) is the number of hyperbinary expansions of \( n-1 \) (see also Proposition 1, which generalizes this property). This property is stated as Theorem 5.2 in [13], which seems to be the earliest (correct) appearance in the literature — we note that the statement on the bottom of page 275 of [2] is erroneous, which can be seen considering the case \( n = 5 \). The same statement can be found on page 57 of [11].

2. Main Results

Our main theorem generalizes Theorem A. We define bivariate polynomials \( s_n(x, y) \) by
\[
\begin{align*}
s_1(x, y) & = 1, \\
s_{2n}(x, y) & = s_n(x, y), \\
s_{2n+1}(x, y) & = x s_n(x, y) + y s_{n+1}(x, y).
\end{align*}
\]
Note that this definition of Stern polynomials differs from the one given in [6] and also from the definition in [9]. However, the univariate polynomials \( s_n(x, 1) \) appear, up to a shift of the index \( n \), in the article [1] by Bates and Mansour: the authors write “[…]the \( n \)th term \( f(n; q) \) of the \( q \)-analogue of the Calkin–Wilf sequence is the generating function for the number of hyperbinary expansions of \( n \) according to the number of powers that are used exactly twice.” This should be compared with Proposition 1 below. Moreover, \( s_n(x, 1) \) appears as the special case \( t = 1 \) in [5] (see (1.2), (1.3) in that paper). The bivariate polynomial \( s_n(x, y) \), however, appears to be a new object of study. We list the first few of these polynomials: we have
\[
\begin{align*}
s_1(x, y) & = 1, \\
s_2(x, y) & = 1.
\end{align*}
\]
\[ s_3(x, y) = x + y, \]
\[ s_5(x, y) = x + xy + y^2, \]
\[ s_7(x, y) = x^2 + xy + y, \]
\[ s_9(x, y) = x + xy + x^2 + y^2, \]
\[ s_{11}(x, y) = x^2 + xy + y^2 + x^2y + xy^2, \]
\[ s_{13}(x, y) = x^2 + xy + y^2 + x^2y + xy^2, \]
\[ s_{15}(x, y) = y + xy + x^3 + x^2y, \]
\[ s_{17}(x, y) = x + xy + x^2 + x^2y + x^3y + y^3 + x^2y^2 + x^3y^2 + xy^3, \]
\[ s_{19}(x, y) = x^2 + xy + x^2y + xy^2 + y^3 + x^2y^2 + x^3y^2 + xy^3, \]
\[ s_{21}(x, y) = x^2 + 2xy + y^2 + x^2y^2 + y^3 + x^2y^2 + x^3y^2 + xy^3, \]
\[ s_{23}(x, y) = xy + y^2 + x^3 + x^2y + xy^2 + x^3y + x^2y^2, \]
\[ s_{25}(x, y) = x^2 + xy + x^2y + xy^2 + y^3 + x^2y^2 + xy^3, \]
\[ s_{27}(x, y) = x^2 + xy + y^2 + x^2y + x^3y + y^3 + x^2y^2 + x^3y^2 + xy^3. \]

and we see the notable identities \( s_{11}(x, y) = s_{13}(x, y) \) and \( s_{19}(x, y) = s_{25}(x, y) \), where \( 13 = 11^2 \) and \( 25 = 19^2 \). In fact, we have the following symmetry property generalizing Theorem A.

**Theorem 1.** Let \( n \) be a positive integer. Then

\[ s_n(x, y) = s_{n^2}(x, y). \]

This theorem can be translated to a statement on hyperbinary expansions. For integers \( i, j \geq 0 \) and \( t \geq 1 \) let \( h_{i,j}(t) \) be the number of proper hyperbinary expansions \((e_{t-1}, \ldots, e_0)\) of \( t - 1 \) such that \( \{ \ell : 0 \leq \ell < \nu, e_\ell = 2 \} \) = \( i \) and \( \{ \ell : 0 \leq \ell < \nu, e_\ell = 0 \} \) = \( j \). The following proposition connects the Stern polynomials \( s_n(x, y) \) to hyperbinary expansions.

**Proposition 1.** Assume that \( n \geq 1 \) is an integer. Then

\[ \sum_{i,j \geq 0} h_n(i, j)x^i y^j = s_n(x, y). \]

In other words, we have \( h_n(i, j) = [x^i y^j] s_n(x, y) \), that is, the polynomial \( s_n(x, y) \) encodes the number of hyperbinary expansions of \( n - 1 \) having a given number of \( 2s \) and \( 0s \). Let us illustrate this proposition by an example. The polynomial \( s_{21}(x, y) \) can be obtained by listing the hyperbinary expansions of 20:

\[
\begin{align*}
1212 & \quad x^2 \\
2012 & \quad x^2 y
\end{align*}
\]

\[
\begin{align*}
1220 & \quad x^2 y \\
2100 & \quad xy^2
\end{align*}
\]
Combining Theorem 1 and Proposition 1, we immediately get the following corollary.

**Corollary 1.** Let \( n \geq 1 \) and \( i,j \geq 0 \). We have

\[
  h_n(i,j) = h_n^n(i,j).
\]

For example, we list the hyperbinary expansions of \( 18 = (10011)_2 - 1 \) (first row) and \( 24 = (11001)_2 - 1 \) (second row):

\[
\begin{align*}
  10010 & 2010 & 1210 & 10002 & 2002 & 1202 & 1122 \\
  11000 & 10120 & 10112 & 10200 & 2200 & 2120 & 2112
\end{align*}
\]

By the corollary there is a one-to-one correspondence between these expansions.

3. Proofs

3.1. Proof of Theorem 1

The main argument, which represents the induction step in the proof of the theorem, is the following lemma.

**Lemma 1.** Let

\[
  A = \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.
\]

If \( A^t \) denotes the transpose of the matrix \( A \), the following identities for \( 1 \times 2 \)-matrices hold.

\[
\begin{align*}
  (x \ y) AA &= (-y) (x \ y) + (y + 1) (x \ y) A, \\
  (1 \ 1)^t A^t A &= (-y) (1 \ 1) + (y + 1) (1 \ 1)^t A, \\
  (x \ y) AB &= x (x \ y) + y (x \ y) B, \\
  (1 \ 1)^t A^t B &= x (1 \ 1) + y (1 \ 1)^t B, \\
  (x \ y) BA &= y (x \ y) + x (x \ y) A, \\
  (1 \ 1)^t B^t A &= y (1 \ 1) + x (1 \ 1)^t A, \\
  (x \ y) BB &= (-x) (x \ y) + (x + 1) (x \ y) B, \\
  (1 \ 1)^t B^t B &= (-x) (1 \ 1) + (x + 1) (1 \ 1)^t B.
\end{align*}
\]
The proof is by simple calculation and is left to the reader. 

To prove Theorem 1, we let $n \geq 1$ be an odd integer. This is no loss of generality since we can deal with the even case by repeatedly using the relation $s_{2n}(x, y) = s_n(x, y)$. Let $n = \sum_{i \leq \nu} \varepsilon_i 2^i$ be the binary representation of $n$ and $\varepsilon_\nu \neq 0$. We prove the theorem by induction on $\nu$. The case $\nu \leq 1$ is trivial, since in this case we have $n^R = n$. We write $A(0) = \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}$ and $A(1) = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$. By a simple application of the recurrence relation we have $(s_{2n+1}) = A(0)(s_{n+1})$ and $(s_{2n+2}) = A(1)(s_{n+1})$ for all $n \geq 1$. Since $n$ is odd and $s_1(x, y) = s_2(x, y) = 1$, it follows from these identities that

$$s_n(x, y) = (x \ y) A(\varepsilon_1) \cdots A(\varepsilon_{\nu-1}) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (1)$$

The statement of the theorem is equivalent to the assertion that

$$(x \ y) A(\varepsilon_1) \cdots A(\varepsilon_{\nu-1}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1 \ 1)^t A(\varepsilon_1) \cdots A(\varepsilon_{\nu-1}) \begin{pmatrix} x \\ y \end{pmatrix} \quad (2)$$

for all $\nu \geq 1$ and all finite sequences $(\varepsilon_1, \ldots, \varepsilon_{\nu-1})$ in \{0, 1\}. We prove the identity (2) by induction on $\nu$, using Lemma 1. This identity is obvious for $\nu \leq 2$. For $\nu > 2$ we have four cases, corresponding to the four possible values of $(\varepsilon_1, \varepsilon_2)$. By Lemma 1, in each of the four cases there exist coefficients $\alpha$ and $\beta$ such that

$$(x \ y) A(\varepsilon_1) A(\varepsilon_2) = \alpha (x \ y) + \beta (x \ y) A(\varepsilon_2)$$

and

$$(1 \ 1)^t A(\varepsilon_1)^t A(\varepsilon_2) = \alpha (1 \ 1) + \beta (1 \ 1)^t A(\varepsilon_2).$$

By applying the induction hypothesis (2) to the sequences $(\varepsilon_2, \ldots, \varepsilon_{\nu-1})$ and $(\varepsilon_3, \ldots, \varepsilon_{\nu-1})$ we obtain the statement of the theorem. 

3.2. Proof of Proposition 1

In order to prove Proposition 1, we use the following recurrence for $h_n(i, j)$ (see [7]).

**Proposition 2.** Let $n \geq 1$. Then

$$h_1(0, 0) = 1,$$
$$h_1(i, j) = 0 \quad \text{for } (i, j) \neq (0, 0),$$
$$h_{2n}(i, j) = h_n(i, j) \quad \text{for } i, j \geq 0,$$
$$h_{2n+1}(i, 0) = h_n(i - 1, 0) \quad \text{for } i \geq 1,$$
$$h_{2n+1}(0, j) = h_{n+1}(0, j - 1) \quad \text{for } j \geq 1,$$
$$h_{2n+1}(i, j) = h_n(i - 1, j) + h_{n+1}(i, j - 1) \quad \text{for } i, j \geq 1. \quad (3)$$

Moreover, $h_t(0, 0) = 0$ if $t$ is not a power of 2.
Proof. The first two lines follow from the fact that the empty tuple () is the only hyperbinary expansion of 0.

The hyperbinary expansions of $2n - 1$ are in bijection with the hyperbinary expansions of $n - 1$ by deleting the lowest digit, which is a 1. This explains the third line of (3).

A hyperbinary expansion of $2n$ without 0 necessarily ends with the digit 2, and the bijection is by deleting this digit. This gives line number four.

There is exactly one hyperbinary expansion of $2n$ without 2, and it ends with 0. The same argument applies here.

The sixth line is a combination of arguments as above.

Finally, integers having the binary expansion $1 \cdots 1$ are the only ones without 0 and 2, which proves the last line. \[\square\]

In order to prove Proposition 1, we proceed by induction. The identity is trivial for $n = 1$. Assume that $n = 2u$ for some $u \geq 1$. We have $h_n(i, j) = h_u(i, j)$ and $s_n(x, y) = s_u(x, y)$. If $n = 2u + 1$, we have

$$\sum_{i \geq 0, j \geq 0} h_n(i, j)x^iy^j = h_n(0, 0) + \sum_{j \geq 1} h_n(0, j)y^j + \sum_{i \geq 1} h_n(i, 0)x^i + \sum_{i, j \geq 1} h_n(i, j)x^iy^j$$

$$= \sum_{j \geq 1} h_{u+1}(0, j-1)y^j + \sum_{i \geq 1} h_u(i-1, 0)x^i$$

$$+ \sum_{i, j \geq 1} (h_u(i-1, j) + h_{u+1}(i, j-1))x^iy^j$$

$$+ \sum_{i \geq 1, j \geq 0} h_u(i-1, j)x^iy^j + \sum_{i \geq 1, j \geq 0} h_u(i, j-1)x^iy^j$$

$$= xs_u(x, y) + s_{u+1}(x, y) = s_n(x, y),$$

which proves the desired assertion. \[\square\]

References


