INVARIANT MEASURES FOR CONTINUED FRACTIONS WITH VARIABLE NUMERATORS

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Abstract

Two one-parameter families of maps are described which give a generalization of the a/b-expansion which are considered in the paper Continued fraction expansions with variable numerators by K. Dajani, C. Kraaikamp, and N. D. S. Langeveld. These families are chosen in a way such that the invariant densities can be given explicitly.

1. Introduction

In [1], so-called a/b-expansions were considered. This expansion and some related continued fractions can be seen as a composition of the continued fraction map

$$Tx = \frac{1}{x} - a,\ \frac{1}{a+1} < x \leq \frac{1}{a},\ a \geq 1$$

and a second map $S : [0, 1] \to [0, 1]$. Take a fixed integer $N \geq 2$. The map

$$S_N x = Nx - j,\ \frac{j}{N} \leq x < \frac{j+1}{N},\ 0 \leq j < N$$

describes expansions to base N. Then the composition

$$(S_N \circ T)x = \frac{N}{x} - (Na + j) = \frac{N}{x} - \lfloor \frac{N}{x} \rfloor$$

gives the proper N-expansions. We note that for $S_N \circ T$ the invariant density

$$h(x) = \frac{1}{N+x}$$

is known.

If we replace N by $a = a(x) = \lfloor \frac{1}{x} \rfloor$ and consider

$$S_a x = ax - j,$$

the composition

$$(S_a \circ T)x = \frac{a}{x} - (a^2 + j) = \frac{a}{x} - \lfloor \frac{a}{x} \rfloor$$

is exactly the a/b-expansion of the paper [1].

It is almost obvious that the map $S_a \circ T$ is ergodic and admits an invariant density. However, an explicit form of the density is unknown. Therefore one may ask if a change of the map S gives a better result. A standard method is to try to find a *dual algorithm*. The dual algorithm has been used implicitly in Lévy's approach to the invariant measure for regular continued fractions (see [2]). Its definition is as follows.

Let (B, T) be a continued fraction with matrices $\{\alpha(k) : \ k \in I\}$, $I = \mathbf{N}$ or $I = \mathbf{N}_0$. These matrices describe the continued fraction as piecewise fractional linear maps. The continued fraction $(B^\#, T^\#)$ is called a *dual algorithm* if the following conditions hold:

(a) The block $(k_1, k_2, \ldots, k_n)$ is an admissible sequence of digits for (B, T) if and only if the block $(k_n, k_{n-1}, \ldots, k_1)$ is an admissible sequence of digits for $(B^\#, T^\#)$.

(b) There is a partition $B^\#(k)$, $k \in I$ of $B^\#$ such that the associated matrices $\alpha^\#(k)$ of $T^\#$ restricted $B^\#(k)$ are the transposed matrices of $\alpha(k)$.

Then we use the function

$$K(x, y) := \frac{1}{(1 + x_1 y_1 + x_2 y_2)^2}.$$

Let $B(k)$ be the set of all numbers $x \in B$ whose first digit in the given continued fraction expansion is k. If $V(k)$ denotes the inverse map of T restricted to the set $B(k)$ and $\omega(k; \cdot)$ its Jacobian then a straightforward calculation shows

$$K(V(k)x, y)\omega(k; x) = K(x, V^\#(k)y)\omega^\#(k; y).$$

The easiest way to obtain an invariant density is to show that the dual algorithm $T^\#$ is isomorphic to T by a fractional linear map

$$\psi(t) = \frac{B + Dt}{A + Bt}$$

which means

$$\psi \circ T = T^\# \circ \psi.$$

If we suppose that $TB(k) = B$ for all digits k then it is easy to show that

$$h(x) = \int_{B^\#} K(x,y)dy$$

is an invariant density for T (see [4]).
The Kuzmin equation

$$h(x) = \sum_{k\in I} h(V(k)x)\omega(k;x)$$

is verified as follows,

$$\sum_{k\in I} h(V(k)x)\omega(k;x) = \sum_{k\in I}\int_{B^\#} K(V(k)x),y)\omega(k;x)dy$$

$$= \sum_{k\in I}\int_{B^\#} K(x,V^\#(k)y)\omega^\#(k;y)$$

$$= \sum_{k\in I}\int_{B^\#(k)} K(x,y)dy = \int_{B^\#} K(x,y)dy = h(x).$$

In our case we apply this device to $S\circ T$. If the map $S\circ T$ is given piecewise by matrices

$$M = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$$

then this amounts to the equation

$$(\mathbf{C}) \qquad \begin{pmatrix} A & B \\ B & D \end{pmatrix}\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} = \begin{pmatrix} \alpha_i & \gamma_i \\ \beta_i & \delta_i \end{pmatrix}\begin{pmatrix} A & B \\ B & D \end{pmatrix}.$$

In this case in [3] the dual algorithm was called a *natural dual*. More information on the dual algorithm can be found in [2] and [4]. If $S\circ T[0,1] = [\gamma,\delta]$ then the invariant density is given as

$$h(x) = \frac{\delta}{1+\delta x} - \frac{\gamma}{1+\gamma x}.$$

In this note we restrict to piecewise linear maps S with time-1-partition $[\frac{j}{k}, \frac{j+1}{k}[$, $0\le j<k$, and show that there are only two types of maps S such that $S\circ T$ has a natural dual.

2. The Main Result

As a kind of easy exercise we first consider the map

$$S_{k,\varepsilon}x = \frac{-jk + j\varepsilon + (k^2 - k\varepsilon)x}{k + j\varepsilon - k\varepsilon x}$$

with a fixed parameter $0 \le \varepsilon < k$. The equation (C) is satisfied with

$$M = \begin{pmatrix} k-\varepsilon-1 & \varepsilon \\ \varepsilon & \frac{\varepsilon^2}{k-\varepsilon} \end{pmatrix}.$$

For the composed map

$$(S_{k,\varepsilon} \circ T)x = \frac{-k^2 + k\varepsilon + (ak^2 - ak\varepsilon + jk - j\varepsilon)x}{k\varepsilon - (ak\varepsilon + k + j\varepsilon)x},$$

equation (C) is also solvable with

$$M = \begin{pmatrix} k-\varepsilon & \varepsilon \\ \varepsilon & 1+\frac{\varepsilon^2}{k-\varepsilon} \end{pmatrix}.$$

For $\varepsilon = 0$ (and $k = N$) we obtain the known density for N-expansions. However, if we replace k by $a = a(x) = \lfloor \frac{1}{x} \rfloor$ and form

$$(S_{a,\varepsilon} \circ T)x = \frac{-a^2 + a\varepsilon + (a^3 - a^2\varepsilon + ja - j\varepsilon)x}{a\varepsilon - (a^2\varepsilon + a + j\varepsilon)x},$$

we see that equation (C) has no solution.

Equation (C) leads to

$$-A(a^2\varepsilon + a + j\varepsilon) + B(a^3 - a^2\varepsilon + ja - j\varepsilon) = Ba\varepsilon + D(-a^2 + a\varepsilon).$$

We compare the terms containing j and we get

$$-A\varepsilon + B(a - \varepsilon) = 0.$$

Since $a - \varepsilon \neq 0$ we get $A = a - \varepsilon$ and $B = \varepsilon$, and hence $A + B = a$, a contradiction.

The main theorem of this note is as follows.

Theorem 1. *(a) There is a family of maps which depend on a parameter $\rho > -1$*

$$R_\rho = \frac{j + \eta x}{a + \varepsilon x}, \quad \frac{j}{a} \le x < \frac{j+1}{a}, \quad 0 \le j < a$$

$$\varepsilon = \frac{\rho a^2 - a}{a^2 + j + 1}, \quad \eta = \frac{a^2 + a\rho j + a\rho}{a^2 + j + 1}, \quad a\eta = a + \varepsilon + \varepsilon j$$

such that $R_\rho \circ T$ has a natural dual.
(b) There is a family of maps which depend on a parameter $\rho \ge -1$:

$$Q_\rho = \frac{j + 1 + \lambda x}{a + \kappa x}, \; \frac{j}{a} \le x < \frac{j+1}{a}, \; 0 \le j < a$$

$$\kappa = \frac{\rho a^2 + a}{a^2 + j}, \; \lambda = \frac{-a^2 + a\rho j}{a^2 + j}$$

such that $Q_\rho \circ T$ has a natural dual.

Proof. (a) The map $R_\rho \circ T$ is given piecewise as

$$(R_\rho \circ T)x = \frac{-a + (a^2 + j)x}{\varepsilon - (a\varepsilon + \eta)x}.$$

Here $a = a(x) = \lfloor \frac{1}{x} \rfloor$. We try to satisfy condition (C). This leads to

$$-A(a\varepsilon + \eta) + B(a^2 + j) = \varepsilon B - aD.$$

When we first put $B = 0$ we see $A(a\varepsilon + \eta) = aD$. Then we put $A = 1$ and $D = \rho$ which explains the origin of the parameter ρ. Then $a\epsilon + \eta = \rho a$. Then an easy calculation gives our values for ε and η. The required matrix is given as

$$M = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}.$$

The (non-normalized) invariant measure has density

$$h(x) = \frac{1}{1 + \rho x}.$$

Note that for $\rho = 0$ the map $R_0 \circ T$ is piecewise linear. When we put $B = 1$ then we get $-A(a\varepsilon + \eta) + a^2 + j = \varepsilon - aD$. We multiply this equation by a and substitute $a\eta = a + \varepsilon + \varepsilon j$. This leads to

$$Aa^2\varepsilon + A(a + \varepsilon + \varepsilon j) + a\varepsilon = a^3 + aj + a^2 D.$$

Then $A\varepsilon = a$, hence $\varepsilon = \frac{a}{A}$. But the equation

$$Aa + a + \frac{a}{A} = a^2 D$$

has no solution.
(b) The proof runs along the same lines. One finds $a\kappa + \lambda = a\rho$. In this case $\rho = -1$ leads to the density $h(x) = \frac{1}{1-x}$. This result corresponds to the fixed point $\xi = 1$ for $(Q_{-1} \circ T)x = 2 - \frac{1}{x}$ near $x = 1$. $\square$

Remarks.
(1) Only for $\rho = 1$ is there a natural dual for the maps R_ρ and Q_ρ. The equation

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \eta & -\varepsilon \\ -j & a \end{pmatrix} = \begin{pmatrix} \eta & -j \\ -\varepsilon & a \end{pmatrix} \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$$

shows that R_ρ has, for a fixed value of $a \geq 2$, the invariant density

$$h(x) = \frac{1}{a + x} - \frac{1}{a + 1 + x}.$$

A similar calculation holds for Q_1.
(2) If one fixes the parameter $a = k$ then the composed maps have no natural dual.
(3) The following example shows that it is not obvious that a suitable choice of parameters will lead to a natural dual:

$$Px = \frac{2x}{\mu + (2-2\mu)x},\ 0 \le x \le \frac{1}{2}$$

$$Px = \frac{2-2x}{-\lambda + (2+2\lambda)x},\ \frac{1}{2} \le x \le 1.$$

No natural dual can be found for the allowed range of parameters. However, one can find *exceptional duals* (see [3]). An example is given by the piecewise linear map $\mu = 1$ and $\lambda = -1$. Then

$$(P \circ T)x = \frac{2}{2a+x},\ \frac{2}{2a+1} < x < \frac{1}{a}$$

$$(P \circ T)x = \frac{2}{2a+2-x},\ \frac{1}{a+1} < x < \frac{2}{2a+1}.$$

The dual is defined on $[-1+\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$ which gives the invariant density as

$$\int_{-1+\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \frac{dy}{(1+xy)^2}.$$

One could produce a lot of new examples in this way, but the aim of this note is to raise the hope on more research about the connection between a given map and its invariant density.

References

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