



ON ASYMPTOTIC APPROXIMATE GROUPS OF INTEGERS

Béla Bajnok

Department of Mathematics, Gettysburg College, Gettysburg, Pennsylvania
bbajnok@gettysburg.edu

Received: 10/11/16, Accepted: 11/19/17, Published: 12/1/17

Abstract

Let r be a positive integer, and let A be a nonempty finite set of at least two integers. We let $\tilde{C}_r(A)$ denote the *asymptotic r -covering number* of A , that is, the smallest integer value of l for which, for all sufficiently large positive integers h , the rh -fold sumset of A is contained in at most l translates of the h -fold sumset of A . Nathanson proved that $\tilde{C}_r(A)$ is always at most $r + 1$; here we extend this result to prove that $\tilde{C}_r(A)$ is always at least r , and determine all sets A for which $\tilde{C}_r(A) = r$.

1. Introduction

We start by recalling some standard terms and notations. Let A be a nonempty finite set of integers. If, for some integers a and b with $a \leq b$, A consists of all integers between a and b , inclusive, we say that A is an *interval* and write $A = [a, b]$; more generally, if for integers a , d , and m with $d > 0$ and $m > 0$, A consists of all integers of the form $a + id$ with $i = 0, 1, \dots, m - 1$, we say that A is an *arithmetic progression* of common difference d , first term a , and size m .

For integers x and c , we define the *translate of A by x* as

$$A + x = \{a + x \mid a \in A\}$$

and the *c -fold dilate of A* as

$$c \cdot A = \{c \cdot a \mid a \in A\}.$$

For example, an arithmetic progression of common difference d , first term a , and size m can be written as $d \cdot [0, m - 1] + a$.

We define the *sumset* of nonempty sets A_1, A_2, \dots, A_h of integers by

$$A_1 + A_2 + \dots + A_h = \{a_1 + a_2 + \dots + a_h \mid a_i \in A_i \text{ for } i = 1, 2, \dots, h\}.$$

If $A_i = A$ for all $i = 1, 2, \dots, h$, we denote this sumset by hA and call it the *h -fold sumset of A* . (Note that $h \cdot A \subseteq hA$, but this is usually a proper containment.)

Let r and l be positive integers. Nathanson [5] defines A to be an (r, l) -approximate group if the r -fold sumset of A is contained in the union of l translates of A ; that is, if there exists a set X consisting of l integers for which $rA \subseteq A + X$. Note that any (r, l) -approximate group is an (r, l') -approximate group for every integer $l' \geq l$. It is, therefore, of interest to find the smallest integer l for which A is an (r, l) -approximate group. We define this value as the r -covering number of A and denote it by $C_r(A)$. Since rA is a finite set, $C_r(A)$ is well-defined.

We make some trivial observations. If $r = 1$, then $rA = A$, so A is a $(1, l)$ -approximate group for any l ; in particular, $C_1(A) = 1$. Also, when $|A| = 1$, then $|rA| = 1$, so A is an (r, l) -approximate group for every r and l , and $C_r(A) = 1$. For these reasons, below we assume that $r \geq 2$ and $|A| \geq 2$.

Let us consider two examples.

Example 1 Let a_1 and a_2 be any two distinct integers, and set $A = \{a_1, a_2\}$. Then

$$\begin{aligned} rA &= \{(r-i)a_1 + ia_2 \mid i = 0, 1, \dots, r\} \\ &\subseteq A + \{(r-1-i)a_1 + ia_2 \mid i = 0, 2, 4, \dots, 2\lceil (r-1)/2 \rceil\}, \end{aligned}$$

and so $C_r(A) \leq \lceil (r+1)/2 \rceil$. (It is easy to see that equality holds here. In Section 2 we shall prove that $C_r(A) \geq \lceil (r+1)/2 \rceil$ for any set A .)

Example 2 Let a_1, a_2, \dots, a_m be positive integers so that $a_{i+1} > ra_i$ for $i = 1, 2, \dots, m-1$, and set $A = \{a_1, a_2, \dots, a_m\}$. Then

$$|rA| = \binom{m+r-1}{r},$$

so we see that A cannot be an (r, l) -approximate group for any $l < \binom{m+r-1}{r}/m$. In particular, we find that for any pair of positive integers r and l (with $r \geq 2$), there are finite sets of integers A with $C_r(A) > l$.

As these two examples demonstrate, the r -covering number of sets may vary widely. We observe a much more subdued behavior if we consider sets ‘asymptotically’. Namely, Nathanson [5] defines A to be an *asymptotic* (r, l) -approximate group if there is an integer h_0 so that hA is an (r, l) -approximate group for every integer $h \geq h_0$. The main result of [5] is as follows.

Theorem 1 (Nathanson [5]). *Every nonempty finite set of integers is an asymptotic $(r, r+1)$ -approximate group.*

We extend this result by establishing the following.

Theorem 2. *Let $A = \{a_1, a_2, \dots, a_m\}$ be a set of (at least two) integers so that $a_1 < a_2 \cdots < a_m$, and set*

$$d = \gcd(a_2 - a_1, a_3 - a_1, \dots, a_m - a_1).$$

1. If A is an asymptotic (r, l) -approximate group, then $l \geq r$.
2. If A is an asymptotic (r, r) -approximate group, then $a_2 = a_1 + d$ and $a_m = a_{m-1} + d$.

Analogously to the r -covering number $C_r(A)$ of A introduced above, we define the *asymptotic r -covering number* of A as the smallest positive integer l for which A is an asymptotic (r, l) -approximate group, and we denote this quantity by $\tilde{C}_r(A)$. We may summarize Theorems 1 and 2 as follows.

Corollary 1. *Let A be any set of (at least two) integers. Keeping the notations of Theorem 2, we have*

$$\tilde{C}_r(A) = \begin{cases} r & \text{if } a_2 = a_1 + d \text{ and } a_m = a_{m-1} + d; \\ r+1 & \text{otherwise.} \end{cases}$$

Contrasting our two examples above with Corollary 1 we can point out that, for any fixed r , the r -covering number $C_r(A)$ may take up infinitely many different values (depending on A), while its asymptotic version, $\tilde{C}_r(A)$, can only equal r or $r + 1$.

Approximate groups were introduced by Tao [7]. In that notion the ambient group is not necessarily abelian, but the subset in question is symmetric. Approximate groups have been investigated extensively since; see, for example, papers by Breuillard [1]; Breuillard, Green, and Tao [2]; Green [3]; Sanders [6]; and their references.

2. Approximate Groups of Integers

We let A be a set of $m \geq 2$ integers. We will use the following version of a well-known result.

Lemma 1. *Let h be a positive integer. If A is an arithmetic progression, then $|hA| = hm - h + 1$, otherwise $|hA| \geq hm$.*

Proof. The first part of our claim is obvious. The second part is obvious as well for $h = 1$, and it is well-known for $h = 2$. In fact, more generally, we know that, when A_1 and A_2 are nonempty finite sets of size at least two and at least one of them is not an arithmetic progression, then $|A_1 + A_2| \geq |A_1| + |A_2|$ (see, for example, Lemma 1.3 and Theorem 1.3 in [4]). Suppose then that our claim holds for A and some positive integer k , and consider $(k + 1)A$. Since both A and kA have size at least two, we have

$$|(k + 1)A| = |kA + A| \geq |kA| + |A| \geq km + m = (k + 1)m,$$

and thus our claim holds for all h by induction. □

Proposition 1. *Let r and l be positive integers and A be an (r, l) -approximate group of size m . We then have $l \geq (rm - r + 1)/m$. In fact, if A is not an arithmetic progression, then $l \geq r$.*

Proof. Suppose that X is a set of l integers so that $rA \subseteq A + X$. Then

$$|rA| \leq |A + X| \leq |A| \cdot |X| = ml.$$

The result now follows from Lemma 1. □

Proposition 2. *Let r be a positive integer. If A is an arithmetic progression of size m , then A is an (r, l) -approximate group for every $l \geq (rm - r + 1)/m$; therefore,*

$$C_r(A) = \lceil (rm - r + 1)/m \rceil.$$

In particular, we have $C_r(A) \leq r$ for any arithmetic progression A .

Proof. Let a and d be integers such that $A = a + d \cdot [0, m - 1]$. Set

$$X = (r - 1)a + md \cdot [0, l - 1].$$

If $l \geq (rm - r + 1)/m$, then

$$(m - 1) + (l - 1)m \geq r(m - 1),$$

and thus

$$\begin{aligned} rA &= ra + d \cdot [0, r(m - 1)] \\ &\subseteq ra + d \cdot [0, (m - 1) + (l - 1)m] \\ &= (a + d \cdot [0, m - 1]) + ((r - 1)a + md \cdot [0, l - 1]) \\ &= A + X, \end{aligned}$$

which implies the result. □

Theorem 3. *Let r be an integer so that $r \geq 2$, and let A be a finite set of integers of size $m \geq 2$. Then*

$$C_r(A) \geq \lceil (r + 1)/2 \rceil.$$

Furthermore, equality holds if and only if

- $m = 2$ (and thus A is an arithmetic progression), or
- $r = 2$ and A is an arithmetic progression (of any size), or
- $r = 4$ and A is an arithmetic progression of size three.

Proof. If A is an arithmetic progression, then $C_r(A)$ is given by Proposition 2. The verification of our claims is an easy exercise.

When A is not an arithmetic progression, then by Proposition 1 we have $C_r(A) \geq r \geq \lceil (r+1)/2 \rceil$. It remains to be shown that at least one of the inequalities is strict. If this were not the case, then we would have $r = 2$, so we only need to verify that if A is not an arithmetic progression, then it cannot be a $(2, 2)$ -approximate group. This was done as part of Theorem 3 in [4]. For the sake of completeness, we repeat the short proof here.

Suppose, indirectly, that $2A \subseteq A+X$ for some set X of size two. This implies that $|2A| \leq 2m$, so by Lemma 1, we must have $|2A| = 2m$ and thus $2A = A+X$. But then $\min 2A = \min(A+X)$, where $\min 2A = 2 \cdot \min A$ and $\min(A+X) = \min A + \min X$. Therefore, $\min A = \min X$; similarly, $\max A = \max X$. This implies that

$$\min A + \max X = \max A + \min X,$$

but then $|2A| = |A+X| \leq 2m - 1$, a contradiction. This completes our proof. \square

Regarding absolute upper bounds on r -covering numbers, we recall Example 2 to state that there are none.

Proposition 3. *Let r be an integer so that $r \geq 2$. Then for every positive integer l , there exists a nonempty finite set A of integers so that $C_r(A) \geq l$.*

3. Asymptotic Arithmetic Progressions of Integers

As we have seen, arithmetic progressions play a central role when studying approximate groups of integers. In this section, as preparation for the study of asymptotic approximate groups, we investigate the ‘asymptotic’ version of arithmetic sequences. We call a subset A of integers an *asymptotic arithmetic progression*, if hA is an arithmetic progression for all sufficiently large h .

Our main result about asymptotic arithmetic progressions is as follows.

Theorem 4. *Let $A = \{a_1, a_2, \dots, a_m\}$ be a set of (at least two) integers so that $a_1 < a_2 < \dots < a_m$, and set*

$$d = \gcd(a_2 - a_1, a_3 - a_1, \dots, a_m - a_1).$$

The following statements are equivalent.

1. *There is a positive integer h_0 for which h_0A is an arithmetic progression.*
2. *There is a positive integer h_0 for which hA is an arithmetic progression for every integer $h \geq h_0$.*
3. *$a_2 = a_1 + d$ and $a_m = a_{m-1} + d$.*

Before our proof, we recall that each nonempty finite set of integers can be ‘normalized’. The *normal form* of the set $A = \{a_1, a_2, \dots, a_m\}$ is the set $B = \{b_1 = 0, b_2, \dots, b_m\}$ defined by $b_i = (a_i - a_1)/d$ for $i = 1, 2, \dots, m$, where

$$d = \gcd(a_2 - a_1, a_3 - a_1, \dots, a_m - a_1).$$

Note that $A = d \cdot B + a_1$.

Lemma 2. *Let A be a nonempty finite set of integers, and let B be its normal form. Then for any positive integer h , hA is an arithmetic progression if, and only if, hB is an arithmetic progression.*

Proof. Note that hA is an arithmetic progression if, and only if, $hA - ha_1$ is an arithmetic progression, and hB is an arithmetic progression if, and only if, $d \cdot hB$ is an arithmetic progression. Since

$$\begin{aligned} hA - ha_1 &= \{\sum_{i=1}^m h_i a_i - ha_1 \mid \sum h_i = h\} \\ &= \{\sum_{i=1}^m (h_i a_i - h_i a_1) \mid \sum h_i = h\} \\ &= \{\sum_{i=1}^m d h_i b_i \mid \sum h_i = h\} \\ &= d \cdot \{\sum_{i=1}^m h_i b_i \mid \sum h_i = h\} \\ &= d \cdot hB, \end{aligned}$$

the result follows. □

Proof of Theorem 4. We follow the scheme $1 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$. Since $2 \Rightarrow 1$ is trivially true, we only need to establish the first two implications.

Suppose that h_0A is an arithmetic progression for some positive integer h_0 . By Lemma 2, we may assume that A is in normal form, so we need to prove that $a_2 = 1$ and $a_m = a_{m-1} + 1$.

Note that $a_i \in h_0A$ for every $i = 1, 2, \dots, m$. In particular, the smallest element of h_0A is $a_1 = 0$, and its smallest positive element is a_2 . Therefore, if h_0A is an arithmetic progression, then each of its elements must be divisible by a_2 . In particular, a_i is divisible by a_2 for every $i = 1, 2, \dots, m$. Since A is in normal form, this implies that $a_2 = 1$. This means that h_0A is an arithmetic progression with a common difference of 1.

Observe also that the largest element of h_0A is h_0a_m , and the second-largest element is $(h_0 - 1)a_m + a_{m-1}$. Since h_0A is an arithmetic progression with a common difference of 1, we have $h_0a_m = (h_0 - 1)a_m + a_{m-1} + 1$ and thus $a_m = a_{m-1} + 1$.

Suppose now that we have a set A with $a_2 = a_1 + d$ and $a_m = a_{m-1} + d$. Then the normal form of A is the set B of the form

$$B = \{0, 1, b_3, \dots, b_{m-2}, b - 1, b\}$$

for some integers $1 < b_3 < \dots < b_{m-2} < b - 1$. We shall prove that

$$hB = \{0, 1, 2, \dots, hb\}$$

for every $h \geq b - 2$. By Lemma 2, this implies that hA is an arithmetic progression for every integer $h \geq b - 2$. Since $\min hB = 0$ and $\max hB = hb$, it suffices to prove that $[0, hb] \subseteq hB$.

Let $h \geq b - 2$, let $0 \leq k \leq h$, and let $0 \leq j \leq k$. Note that the integer

$$(k - j) \cdot (b - 1) + j \cdot b$$

is an element of kB , so the interval $[k(b - 1), kb]$ is contained in kB . But $0 \in B$, so $kB \subseteq hB$, and thus $[k(b - 1), kb] \subseteq hB$.

Now let $0 \leq t \leq h - k$. Note that the integer

$$(h - k - t) \cdot 0 + t \cdot 1 + k \cdot b$$

is an element of hB , and thus the interval $[kb, kb + h - k]$ is contained in hB . Combining this with the previous paragraph, we conclude that, for every integer $0 \leq k \leq h$, hB contains the interval $[k(b - 1), kb + h - k]$.

But, since $h \geq b - 2$,

$$kb + h - k \geq (k + 1)(b - 1) - 1,$$

we have

$$[0, hb] = [0(b - 1), hb + h - h] = \cup_{k=0}^h [k(b - 1), kb + h - k] \subseteq hB,$$

as claimed. □

4. Asymptotic Approximate Groups of Integers

We start by proving that, in contrast to the r -covering number $C_r(A)$ of a set A , which may be less than r , the asymptotic r -covering number $\tilde{C}_r(A)$ of A (of size at least two) is always at least r .

Theorem 5. *Let A be any set of integers of finite size at least two, and let r and l be positive integers. If A is an asymptotic (r, l) -approximate group, then $l \geq r$.*

Proof. Suppose, indirectly, that $l < r$. Let $m = |A| \geq 2$, h be a positive integer, and suppose that hA is an (r, l) -approximate group. Let X be a set of l integers for which $rhA \subseteq hA + X$.

We then have

$$|rhA| \leq |hA| \cdot l;$$

since, by Lemma 1,

$$|rhA| = |r(hA)| \geq r \cdot |hA| - r + 1,$$

we get

$$|hA| \leq \frac{r-1}{r-l}.$$

But, again by Lemma 1, $|hA| \geq hm - h + 1$, so

$$hm - h + 1 \leq \frac{r-1}{r-l},$$

or

$$h \leq \frac{l-1}{(r-l)(m-1)}.$$

This implies that there can only be finitely many values of h for which hA is an (r, l) -approximate group, which is a contradiction. \square

In the rest of this section, we characterize sets with asymptotic r -covering number equal to r . We first reduce the problem to sets in normal form via the following lemma.

Lemma 3. *Let A be a nonempty finite set of integers, and let B be its normal form. Then, for any pair of positive integers r and l , A is an (r, l) -approximate group if, and only if, B is an (r, l) -approximate group.*

Proof. Suppose first that B is an (r, l) -approximate group, and let X be a set of l integers for which $rB \subseteq B + X$. Recall that, with $a_1 = \min A$ and

$$d = \gcd(a_2 - a_1, \dots, a_m - a_1),$$

we have $A = d \cdot B + a_1$, and thus

$$\begin{aligned} rA &= r(d \cdot B + a_1) = d \cdot rB + ra_1 \subseteq d \cdot (B + X) + ra_1 = d \cdot B + d \cdot X + ra_1 \\ &= A + d \cdot X + (r-1)a_1. \end{aligned}$$

Here $d \cdot X + (r-1)a_1$ is a set of l integers, which implies that A is an (r, l) -approximate group.

Conversely, assume that A is an (r, l) -approximate group. Let $A = \{a_1, a_2, \dots, a_m\}$ with a_1 and d defined as above. The covering number $C_r(A)$ of A is at most l . Let X be a set of exactly $C_r(A)$ integers for which $rA \subseteq A + X$.

First, we prove that, for every $x \in X$, the integer $x - (r-1)a_1$ is divisible by d . There must be an element $c \in rA$ for which $c \in A + x$, since otherwise we would have

$$rA \subseteq A + (X \setminus \{x\}),$$

contradicting $|X| = C_r(A)$. Suppose then that nonnegative integers $1 \leq j \leq m$ and h_1, h_2, \dots, h_m are such that $\sum_{i=1}^m h_i = r$ and

$$c = \sum_{i=1}^m h_i a_i = a_j + x.$$

Then

$$x - (r - 1)a_1 = \sum_{i=1}^m h_i a_i - ra_1 - (a_j - a_1) = \sum_{i=1}^m h_i (a_i - a_1) - (a_j - a_1),$$

which is then divisible by d since $a_i - a_1$ is divisible by d for every $i = 1, 2, \dots, m$.

Let

$$Y = \{(x - (r - 1)a_1)/d \mid x \in X\}.$$

As we just proved, Y is a set of $C_r(A)$ integers. Furthermore,

$$d \cdot rB = r(d \cdot B + a_1) - ra_1 = rA - ra_1 \subseteq A + X - ra_1 = d \cdot B + X - (r - 1)a_1 = d \cdot B + d \cdot Y,$$

so $rB \subseteq B + Y$. Since $|Y| = C_r(A) \leq l$, we can conclude that B is an (r, l) -approximate group, as claimed. \square

For the proof of Theorem 7 below, we employ the following result, sometimes referred to as the Fundamental Theorem of Additive Number Theory.

Theorem 6 (Nathanson; cf. Theorem 1.1 in [4]). *Suppose that A is a nonempty finite set of integers in normal form and $\max A = a_m$. Then there are nonnegative integers h_0, c , and d , so that hA contains the interval $[c, ha_m - d]$ for every integer $h \geq h_0$.*

For easier reference, we state our characterization of sets with $\tilde{C}_r(A) = r$ in the following form.

Theorem 7. *Let $A = \{a_1, a_2, \dots, a_m\}$ be a set of (at least two) integers so that $a_1 < a_2 \cdots < a_m$, and set*

$$d = \gcd(a_2 - a_1, a_3 - a_1, \dots, a_m - a_1).$$

Let r be any positive integer. The following statements are equivalent.

1. *A is an asymptotic (r, r) -approximate group.*
2. *A is an asymptotic arithmetic progression.*
3. *$a_2 = a_1 + d$ and $a_m = a_{m-1} + d$.*

Proof. We follow the scheme $1 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$. The implication $3 \Rightarrow 2$ was already established in Theorem 4. Furthermore, if A is an asymptotic arithmetic progression, that is, if hA is an arithmetic progression for all sufficiently large integers h , then for these h values hA is an (r, r) -approximate group by Proposition 2, proving the implication $2 \Rightarrow 1$. Therefore, we only need to prove that $1 \Rightarrow 3$.

Suppose then that A is an asymptotic (r, r) -approximate group. By Lemma 3 we may assume that A is in normal form. Our goal is to prove that $a_1 = 1$ and $a_m = a_{m-1} + 1$. We shall first prove that $a_2 = 1$.

Suppose, indirectly, that $a_2 \geq 2$. Then, by Theorem 4, there is no positive integer h for which hA is an arithmetic progression, so, by Lemma 1, $|rhA| \geq r \cdot |hA|$ holds for every h .

Now let h_0 , c , and d be nonnegative integers for which, by Theorem 6, rhA contains the interval $[c, rha_m - d]$ for every integer $h \geq h_0/r$. Then, with

$$h_1 = \max\{h_0/r, (c - 1)/a_m, (d + 1)/a_m\},$$

we have

$$[ha_m + 1, (r - 1)ha_m + 1] \subseteq rhA$$

for every $h \geq h_1$.

Since A is an asymptotic (r, r) -approximate group, we have an integer h_2 for which hA is an (r, r) -approximate group for every integer $h \geq h_2$. We set

$$h_3 = \max\{h_1, h_2, r/a_m\},$$

and choose h to be any integer with $h \geq h_3$.

Let X be a set of r integers so that $rhA \subseteq hA + X$. We then have $|rhA| \leq |hA| \cdot r$. Since above we have already established the reverse inequality, we must have $|rhA| = |hA| \cdot r$, and thus $rhA = hA + X$. In particular,

$$0 = \min rhA = \min(hA + X) = \min hA + \min X = 0 + \min X,$$

so $\min X = 0$; similarly, $\max X = (r - 1)ha_m$. Then we can write

$$X = \{x_1, x_2, x_3, \dots, x_{r-1}, x_r\},$$

where

$$0 = x_1 < x_2 < x_3 < \dots < x_{r-1} < x_r = (r - 1)ha_m.$$

Claim 1. There is at least one value of $i \in \{1, 2, \dots, r - 1\}$ for which $x_{i+1} - x_i \geq ha_m$.

Proof of Claim 1. This follows from the fact that

$$\sum_{i=1}^{r-1} (x_{i+1} - x_i) = x_r - x_1 = (r - 1)ha_m.$$

◇

Claim 2. For all values of $i \in \{1, 2, \dots, r - 1\}$, we have $x_{i+1} - x_i \leq ha_m + 1$.

Proof of Claim 2. Let us assume, to the contrary, that $x_{i+1} - x_i \geq ha_m + 2$ for some i . In this case the interval

$$I = [x_i + ha_m + 1, x_{i+1} - 1]$$

is nonempty. Note that $hA + X$ and I are disjoint. But each element of I is at least $x_1 + ha_m + 1 = ha_m + 1$ and at most $x_r - 1 = (r - 1)ha_m - 1$, so

$$I \subseteq [ha_m + 1, (r - 1)ha_m + 1] \subseteq rhA,$$

which contradicts our assumption that $rhA = hA + X$. ◇

Claim 3. There is no value of $i \in \{1, 2, \dots, r - 1\}$ for which $x_{i+1} - x_i = ha_m$.

Proof of Claim 3. If, to the contrary, we were to have $x_{i+1} - x_i = ha_m$ for some i , then $x_{i+1} + 1 = x_i + ha_m + 1$ would not be generated. Since this integer is at least $ha_m + 1$ and at most $(r - 1)ha_m + 1$, we get a contradiction as in the proof of Claim 2. ◇

Claim 4. There is no value of $i \in \{1, 2, \dots, r - 1\}$ for which $x_{i+1} - x_i = 1$.

Proof of Claim 4. If, to the contrary, we were to have $x_{i+1} - x_i = 1$ for some value of i , then, by Claim 2, we would have

$$(r - 1)ha_m = \sum_{i=1}^{r-1} (x_{i+1} - x_i) \leq 1 + (r - 2)(ha_m + 1),$$

contradicting the fact that $h \geq h_3 \geq r/a_m$. ◇

Claim 5. $x_r - x_{r-1} \neq ha_m + 1$.

Proof of Claim 5. Suppose, indirectly, that $x_r - x_{r-1} = ha_m + 1$ and thus $x_{r-1} = (r - 2)ha_m - 1$. Since

$$(r - 1)ha_m + 1 \in rhA = hA + X,$$

we have

$$a := (r - 1)ha_m + 1 - x_i \in hA$$

for some $i = 1, 2, \dots, r$. We cannot have $i = r$, since then $a = 1$, and $1 \notin hA$ as the smallest positive element of hA is $a_2 \geq 2$. Also, we cannot have $i \leq r - 1$, since then

$$a \geq (r - 1)ha_m + 1 - x_{r-1} = ha_m + 2,$$

but the largest element of hA is ha_m . This proves our claim. ◇

According to our five claims, we then must have a value of $i \in \{1, 2, \dots, r - 2\}$ for which $x_{i+1} - x_i = ha_m + 1$ and $x_{i+2} - x_{i+1} \geq 2$. Consider the integer $a = x_i + ha_m + 2$. We have

$$ha_m + 2 \leq a = x_{i+1} + 1 \leq x_r = (r - 1)ha_m,$$

so $a \in rhA = hA + X$.

However, we show that $a - X$ and hA are disjoint. Indeed, for all integers $0 \leq j \leq i$, we have $a - x_j \geq ha_m + 2$; for all $i + 2 \leq j \leq r$, we have

$$a - x_j \leq x_i + ha_m + 2 - x_{i+2} \leq x_i + ha_m + 2 - (x_{i+1} + 2) = -1;$$

and $a - x_{i+1} = 1$. But

$$hA \subseteq \{0\} \cup [2, ha_m],$$

so $a - X$ and hA are disjoint, which is a contradiction. This proves that $a_2 = 1$.

To prove that $a_m = a_{m-1} + 1$, set $C = -A + a_m$. Then C is in normal form, and it is an easy exercise to verify that it is also an asymptotic (r, r) -approximate group; indeed, for any integer $h \geq h_3$, we have

$$\begin{aligned} rhC &= rh(-A + a_m) \\ &= -rhA + rha_m \\ &\subseteq -(hA + X) + rha_m \\ &= h(-A + a_m) + (-X + (r-1)ha_m) \\ &= hC + (-X + (r-1)ha_m), \end{aligned}$$

where $-X + (r-1)ha_m$ is a set of r integers. But then, by our argument above, the smallest positive element of C equals 1, that is, $a_m - a_{m-1} = 1$. This completes our proof. \square

References

- [1] E. Breuillard, A brief introduction to approximate groups. Thin groups and superstrong approximation. *Math. Sci. Res. Inst. Publ.*, **61**, Cambridge Univ. Press, Cambridge, 2014, 23–50.
- [2] E. Breuillard, B. Green, and T. Tao, The structure of approximate groups. *Publ. Math. Inst. Hautes Études Sci.* **116** (2012), 115–221.
- [3] B. Green, What is . . . an approximate group? *Notices Amer. Math. Soc.* **59**:5 (2012), 655–656.
- [4] M. B. Nathanson, Additive Number Theory: Inverse Problems and the Geometry of Sumsets. *Graduate Texts in Mathematics* **165**, Springer–Verlag (1996).
- [5] M. B. Nathanson, Every finite set of integers is an asymptotic approximate group. *Mosc. J. Comb. Number Theory* **6** (2016), no. 4, 33–42.
- [6] T. Sanders, Approximate groups and doubling metrics. *Math. Proc. Cambridge Philos. Soc.* **152**:3 (2012), 385–404.
- [7] T. Tao, Product set estimates for non-commutative groups. *Combinatorica* **28**:5 (2008), 547–594.