IMAGE PARTITION REGULARITY OVER THE GAUSSIAN INTEGERS

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Abstract

A $u \times v$ matrix A with entries from $\mathbb{Q}$ is image partition regular provided that, whenever $\mathbb{N}$ is finitely colored, there is some $\vec{x} \in \mathbb{N}^v$ with all entries of $A\vec{x}$ lying in one color. Image partition regular matrices are natural tools for representing some classical theorems of Ramsey Theory, including theorems of Hilbert, Schur, and van der Waerden. Several characterizations and consequences of image partition regularity were investigated in the literature. Many natural analogues of known characterizations of image partition regularity of finite matrices with rational entries over the integers have been generalized for matrices with entries from *reals* over the ring $(\mathbb{R}, +)$. In both the cases of reals and integers, usual ordering played an important role. In the present work we shall prove that natural analogues of known characterizations of image partition regularity of finite matrices with rational entries over the integers are also valid for matrices with entries from *Gaussian rationals* $\mathbb{Q}[i]$ over the ring of Gaussian integers $\mathbb{Z}[i]$. The main hurdle for this generalization is the absence of ordering, and to overcome this hurdle we need some modifications of established techniques. We also prove that Milliken-Taylor Matrices with entries from $\mathbb{Z}[i]$ are also image partition regular over $\mathbb{Z}[i]$.

1. Introduction

In 1933, R. Rado published [10] his famous theorem characterizing those finite matrices A with rational entries that have the property that whenever $\mathbb{N}$ is finitely colored, there must be some $\overrightarrow{x}$ in the kernel of A all of whose entries have the same

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color (or are monochrome). This characterization was in terms of the columns condition which we shall describe below.

In 1943, R. Rado [11] published a paper among whose results was the fact that the same condition characterized those finite matrices with entries from any subring R of $\mathbb{C}$ that have the property that whenever R is finitely colored, there is some $\vec{x}$ in the kernel of A whose entries are monochrome.

Definition 1.1. Let $u, v \in \mathbb{N}$, and let R be any subring of $\mathbb{C}$ and F be the field generated by R. Let A be a $u \times v$ matrix with entries from F.
(a) The matrix A is *kernel partition regular* over R if and only if whenever $r \in \mathbb{N}$ and $R \setminus \{0\} = \bigcup_{i=1}^{r} E_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in (E_i)^v$ such that $A\vec{x} = \vec{0}$.
(b) The matrix A satisfies the *columns condition* over F if there exists some $m \in \{1, 2, \ldots, v\}$ and a partition $\langle I_t \rangle_{t=1}^{m}$ of $\{1, 2, \ldots, v\}$ such that

(1) $\sum_{i \in I_1} \vec{c_i} = 0$;

(2) for each $t \in \{2, 3, \ldots, m\}$ (if any remains), then $\sum_{i \in I_t} \vec{c_i}$ is a linear combination with coefficients from F (the field generated by R) of $\{\vec{c_i} : i \in \cup_{j=1}^{t-1} I_j\}$.

The results of Rado referred to above are that if R is any subring of the set $\mathbb{C}$ of complex numbers and the entries of A are from F, the field generated by R, then the system of linear equations is kernel partition regular over R if and only if the matrix A satisfies the columns condition over the field F.

Rado's Theorem, in any of its forms, is quite powerful. For example, it gives van der Waerden's Theorem [13] as a corollary, which says that whenever $\mathbb{N}$ is finitely colored, there must be arbitrarily long monochromatic arithmetic progressions.

The length four version of van der Waerden's Theorem states that whenever $\mathbb{N}$ is partitioned into finitely many cells then the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} a \\ d \end{pmatrix}$$

has monochromatic image. Many other theorems such as Schur's Theorem [12] are naturally represented in terms of images of matrices.

Definition 1.2. Let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is image partition regular over $\mathbb{N}$ if and only if whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} E_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in E_i^u$.

While there are several partial results, nothing near a characterization of either kernel or image partition regularity of infinite matrices has been obtained.

In this paper we are concerned with the extent to which several known results about image partition regularity over $\mathbb{N}$ can also be obtained over $\mathbb{Z}[i]$.

Definition 1.3. Let A be a $u \times v$ matrix with entries from $\mathbb{Q}[i]$. Then A is *image partition regular over* $\mathbb{Z}[i]$ if whenever $r \in \mathbb{N}$ and $\mathbb{Z}[i]\backslash\{0\} = \bigcup_{i=1}^{r} E_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in (\mathbb{Z}[i]\backslash\{0\})^v$ such that $A\vec{x} \in E_i^u$.

To establish some characterizations of image partition regular matrices we need the notion of *kernel partition regularity* as well. Here we shall deal with image partition regularity over $\mathbb{Z}[i]$ and we shall extend the definition of image partition regularity to allow entries from $\mathbb{Q}[i]$.

Definition 1.4. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $R = \mathbb{Z}[i]$ or $\mathbb{Q}[i]$. Then A is called a *first entries matrix* if:

(a) no row of A is 0,

(b) the first nonzero entries of any two rows are equal if they occur in the same column.

If A is a first entries matrix and d is the first nonzero entry of some row, then d is called a *first entry* of A.

For characterizations of image partition regular matrices with entries from $\mathbb{Q}[i]$ over $\mathbb{Z}[i]$ we need some basic facts about the algebra of the Stone-Čech compactification βS of a discrete semigroup S.

Some of the characterizations of image partition regularity that we shall give involve "central" sets. Central sets were introduced by Furstenberg [4] and defined in terms of notions of topological dynamics. These sets enjoy very strong combinatorial properties. They also have a nice characterization in terms of the algebraic structure of βS. We shall present this characterization below, after introducing the necessary background information.

We take the points of βS_d to be the ultrafilters on S, identifying the principal ultrafilters with the points of S and thus supposing that $S \subseteq \beta S_d$. Given $A \subseteq S$, we denote

$$c\ell A = \overline{A} = \{p \in \beta S_d : A \in p\}.$$

The set $\{\overline{A} : A \subset S\}$ is a basis for the closed sets of βS_d. The operation $\cdot$ on S can be extended to the Stone-Čech compactification βS_d of S so that $(\beta S_d, \cdot)$ is a compact right topological semigroup (meaning that for any $p \in \beta S_d$, the function $\rho_p : \beta S_d \to \beta S_d$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological center (meaning that for any $x \in S$, the function $\lambda_x : \beta S_d \to \beta S_d$ defined by $\lambda_x(q) = x \cdot q$ is continuous). A nonempty subset I of a semigroup T is called a *left ideal* of S if $TI \subset I$, a *right ideal* if $IT \subset I$, and a *two sided ideal* (or simply an *ideal*) if it is both a left and a right ideal. A *minimal left ideal* is a left ideal that does not contain any proper left ideal. Similarly, we can define *minimal right ideal* and *smallest ideal*.

Any compact Hausdorff right topological semigroup T has a smallest two-sided ideal

$$\begin{aligned} K(T) &= \bigcup\{L : L \text{ is a minimal left ideal of } T\} \\ &= \bigcup\{R : R \text{ is a minimal right ideal of } T\}, \end{aligned}$$

Given a minimal left ideal L and a minimal right ideal R, $L \cap R$ is a group, and in particular contains an idempotent. If p and q are idempotents in T we write $p \leq q$ if and only if $pq = qp = p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal $K(T)$ of T.

Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if the set $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. See [9] for an elementary introduction to the algebra of βS and for any unfamiliar details.

Definition 1.5. Let S be a discrete semigroup. A set $C \subset S$ is said to be a *central set* in S if there is an idempotent p in the smallest ideal $K(\beta S)$ of βS with $C \in p$.

The basic fact that we need about central sets is given by the Central Sets Theorem, which is due to Furstenberg [4, Proposition 8.21] for the case $S = \mathbb{Z}$.

Theorem 1.6 (Central Sets Theorem). *Let S be a semigroup. Let $\mathcal{T}$ be the set of sequences $\langle y_n \rangle_{n=1}^{\infty}$ in S. Let C be a subset of S which is central and let $F \in \mathcal{P}_f(\mathcal{T})$. Then there exist a sequence $\langle a_n \rangle_{n=1}^{\infty}$ in S and a sequence $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and for each $L \in \mathcal{P}_f(\mathbb{N})$ and each $f \in F$, $\sum_{n \in L}(a_n + \sum_{t \in H_n} f(t)) \in C$.*

However, the most general version of Central Sets Theorem is available in [1, Theorem 2.2]. Central sets are interesting combinatorial objects because of the fact that they contain images of any image partition regular matrix, and any finite partition of any infinite commutative semigroup $(S, +)$ is guaranteed to have one cell which is central.

Definition 1.7. Let S be a semigroup and let $A \subseteq S$. Then A is called a *central* set* in S if $A \cap C \neq \emptyset$ for every central set C in S.

The following obvious lemma will be useful for us.

Lemma 1.8. *Let S be a semigroup and let $A \subseteq S$. Then the following statements are equivalent.*

(*a*) *A is a central* set.*

(*b*) *A is a member of every minimal idempotent in βS.*

(*c*) *$A \cap C$ is a central set for every central set C of S.*

The paper is been organized as follows: In Section 2 we have discussed about some basic characterizations of image partition regular matrices over $\mathbb{Z}[i]$. In Section 3 we establish several characterizations of image partition regular matrices over $\mathbb{Z}[i]$. The results in this section are already established over $\mathbb{N}$ with entries of matrices from $\mathbb{Q}$ by N. Hindman, I. Leader, D. Strauss [7]. N. Hindman obtained similar results with entries from $\mathbb{R}$ over $(\mathbb{R},+)$ [5]. In both cases the usual ordering of $\mathbb{N}$ and $\mathbb{R}$ played important roles. But we can not adopt the same techniques and constructions as was used earlier in case of $\mathbb{Z}[i]$. The ideas are quite similar but we have to develop some stronger techniques and some new constructions. Finally in Section 4 we prove that Milliken Taylor matrices are the main sources of infinite image partition regular matrices.

2. Characterizations of Image Partition Regular Matrices

We now turn to some results about central subsets of $\mathbb{Z}[i]$.

Lemma 2.1. *Let p be an idempotent in $(\beta\mathbb{Z}[i],+)$. Then for every $a \in \mathbb{Z}[i] \setminus \{0\}$, $a\mathbb{Z}[i] \in p$.*

Proof. Let A be an IP set in $\mathbb{Z}[i]$. By division algorithm any element z can be expressed as $z = a \cdot w + r$, where w and r belong to $\mathbb{Z}[i]$ with $0 \leq\mid r \mid\leq\mid a \mid$, and there will be at most $\mid a \mid^2$ number of distinct remainders r for different z. Let us suppose that $a = \alpha + i\beta$, and denote all possible remainders when any z is divided by a as $r_1, r_2, \ldots, r_{\alpha^2+\beta^2}$. Since A is infinite there must be infinitely many elements congruent to $r_l (\text{mod } a)$ for some $l \in \{1, 2, \ldots, (\alpha^2+\beta^2)\}$. Choose $(\alpha^2+\beta^2)$ many of them as $z_{l_1}, z_{l_2}, \ldots, z_{l_{\alpha^2+\beta^2}}$ in A. Since A is an IP set the sum $\sum_{k=1}^{\alpha^2+\beta^2} z_{l_k}$ is also there, but this sum is congruent to $(\alpha^2+\beta^2) \cdot r_l (\text{mod } a) \equiv 0 (\text{mod } a)$. This trivially implies that $a\mathbb{Z}[i]$ is IP* as A is an arbitrary IP set. $\square$

The lemma above simply implies that $a\mathbb{Z}[i]$ is IP* for any $a \in \mathbb{Z}[i] \setminus \{0\}$, and so in particular central*. However, we need the following fact.

Lemma 2.2. *Let p be a minimal idempotent in $(\beta\mathbb{Z}[i],+)$ and let $\alpha \in \mathbb{Q}[i] \setminus \{0\}$. Then $\alpha \cdot p$ is also a minimal idempotent in $\beta\mathbb{Z}[i]$. Consequently, if C is central in $(\mathbb{Z}[i],+)$, then so is $(\alpha C) \cap \mathbb{Z}[i]$.*

Proof. The function $l_\alpha : \mathbb{Z}[i] \to \mathbb{Q}[i]$ defined by $l_\alpha(x) = \alpha \cdot x$ is a homomorphism, and hence so is its continuous extension $\tilde{l_\alpha} : \beta\mathbb{Z}[i] \to \beta\mathbb{Q}_{\mathrm{d}}[i]$ by [9, Corollary 4.22]. Furthermore, $\alpha \cdot p = \tilde{l_\alpha}(p)$. Thus $\alpha \cdot p$ is an idempotent and $\alpha \cdot p \in \tilde{l_\alpha}[K(\beta\mathbb{Z}[i])] = K(\overline{\alpha\mathbb{Z}[i]})$ (the latter equality holds by [9, Exercise 1.7.3]). Assume that $\alpha = \frac{a}{b}$ with $a, b \in \mathbb{Z}[i]$. Then $b\mathbb{Z}[i] \subseteq \alpha^{-1}a\mathbb{Z}[i]$ and thus $a\mathbb{Z}[i] \in \alpha \cdot p$ because $b\mathbb{Z}[i] \in p$ by Lemma 2.1 here. In particular, $\alpha \cdot p \in \beta\mathbb{Z}[i]$. Also $\alpha \cdot p \in K(\overline{\alpha\mathbb{Z}[i]}) \cap \overline{a\mathbb{Z}[i]}$ and $a\mathbb{Z}[i] \subseteq \alpha\mathbb{Z}[i]$

and consequently, $K(\overline{a\mathbb{Z}[i]}) = K(\overline{\alpha\mathbb{Z}[i]}) \cap \overline{a\mathbb{Z}[i]}$ by [9, Theorem 1.65]. Since every idempotent in $\beta\mathbb{Z}[i]$ is in $\overline{a\mathbb{Z}[i]}$ by Lemma 2.1, we have that $\overline{a\mathbb{Z}[i]} \cap K(\beta\mathbb{Z}[i]) \neq \emptyset$ and consequently, $K(\overline{a\mathbb{Z}[i]}) = \overline{a\mathbb{Z}[i]} \cap K(\beta\mathbb{Z}[i])$. Again by [9, Theorem 1.65]. Thus $(\alpha \cdot p) \in K(\beta\mathbb{Z}[i])$ as required. For the second assertion, let C be central in $(\mathbb{Z}[i], +)$ and pick a minimal idempotent p containing C. Then $\alpha C \cap \mathbb{Z}[i] \in \alpha \cdot p$. □

Theorem 2.3. *Let A be a $u \times v$ matrix with entries from $\mathbb{Z}[i]$, define $\varphi : \mathbb{Z}[i]^v \to \mathbb{Z}[i]^u$ by $\varphi(\vec{x}) = A\vec{x}$ and let $\tilde{\varphi} : \beta(\mathbb{Z}[i]^v) \to (\beta\mathbb{Z}[i])^u$ be its continuous extension. Let p be a minimal idempotent in $\beta\mathbb{Z}[i]$ with the property that for every $C \in p$ there exists $\vec{x} \in \mathbb{Z}[i]^v$ such that $A\vec{x} \in C^u$. Let $\vec{p} = (p, p, \ldots, p)^T$. Then there is a minimal idempotent $q \in \beta(\mathbb{Z}[i]^v)$ such that $\tilde{\varphi}(q) = \vec{p}$.*

Proof. Since $p \in K(\beta\mathbb{Z}[i])$ by [9, Theorem 2.23], $\vec{p} \in K(\beta(\mathbb{Z}[i])^u)$. Also by [9, Corollary 4.22], $\tilde{\varphi} : \beta(\mathbb{Z}[i]^v) \to \beta(\mathbb{Z}[i])^u$ is a homomorphism.

We claim that $\vec{p} \in \tilde{\varphi}[\beta(\mathbb{Z}[i]^v)]$. Suppose instead that $\vec{p} \notin \tilde{\varphi}[\beta(\mathbb{Z}[i]^v)]$. Since $\tilde{\varphi}[\beta(\mathbb{Z}[i]^v)]$ is closed, we can pick a neighborhood U of $\vec{p}$ such that $U \cap \tilde{\varphi}[\beta(\mathbb{Z}[i]^v)] = \emptyset$. Pick $D \in p$ such that $\overline{D}^u \subset U$ and pick $\vec{x} \in \mathbb{Z}[i]^v$ such that $A\vec{x} \in D^u$. Then $\tilde{\varphi}(\vec{x}) \in U \cap \tilde{\varphi}[\beta(\mathbb{Z}[i]^v)]$, a contradiction.

Let $M = \{q \in \beta(\mathbb{Z}[i]^v) : \tilde{\varphi}(q) = \vec{p}\}$. Then M is a compact subsemigroup of $\beta(\mathbb{Z}[i]^v)$. By [9, Theorem 2.5] pick an idempotent $w \in M$. By [9, Theorem 1.60], pick a minimal idempotent $q \in \beta(\mathbb{Z}[i]^v)$ with $q \leq w$. Since $\tilde{\varphi}$ is a homomorphism, $\tilde{\varphi}(q) \leq \tilde{\varphi}(w) = \vec{p}$. Since $\vec{p}$ is minimal in $(\beta\mathbb{Z}[i])^u$, we have that $\tilde{\varphi}(q) = \vec{p}$. □

In the following theorem we get a conclusion far stronger than the assertion that matrices satisfying the first entries condition are image partition regular. The stronger conclusion is of some interest in its won right. The technique of the proof is taken verbatim from [9, Theorem 15.5]. The first author of the present article also used this technique in several papers.

Theorem 2.4. *Let $u, v \in \mathbb{N}$, and let M be a $u \times v$ matrix with entries from $\mathbb{Z}[i]$ which satisfies the first entry condition. Let C be central set in $\mathbb{Z}[i]$. If for every first entry c of M, $c\mathbb{Z}[i]$ is a central*set, then there exist sequences $\langle x_{1,n}\rangle_{n=1}^{\infty}$, $\langle x_{2,n}\rangle_{n=1}^{\infty}, \ldots, \langle x_{v,n}\rangle_{n=1}^{\infty}$ in $\mathbb{Z}[i]$ such that for every $F \in \mathcal{P}_f(\mathbb{N})$ $A\vec{x_F} \in C^u$, where*

$$\vec{x_F} = \begin{pmatrix} \sum_{n\in F} x_{1,n} \\ \sum_{n\in F} x_{2,n} \\ \cdot \\ \cdot \\ \cdot \\ \sum_{n\in F} x_{v,n} \end{pmatrix}.$$

Proof. Let C be a central set in $\mathbb{Z}[i]$. We proceed by induction on v. Assume first that $v = 1$. We can assume that M has no repeated rows, so in this case we have

$M=(c)$ for some $c \in \mathbb{Z}[i]\setminus\{0\}$. Pick a sequence $\langle y_n\rangle_{n=1}^\infty$ with $FS(\langle y_n\rangle_{n=1}^\infty) \subseteq C\cap \mathbb{Z}[i]$. For each $n\in\mathbb{N}$, let $x_{1,n}=\frac{y_n}{c}$. The sequence$\langle x_{1,n}\rangle_{n=1}^\infty$ is as required.

Now let $v\in\mathbb{N}$ and assume that the theorem is true for v. Let M be a $u\times(v+1)$ first entries matrix with entries from $\mathbb{Z}[i]$. By rearranging the rows of M and adding additional rows to M if needed, we may assume that we have some $r\in\{1,2,\cdots,u-1\}$ and some $d\in\mathbb{Z}[i]\setminus\{0\}$ such that

$$a_{i,1}=\begin{cases} 0 & \text{if } i\in\{1,2,\ldots,r\}, \\ d & \text{if } i\in\{r+1,r+2,\ldots,u\}. \end{cases}$$

Let B be the $r\times v$ matrix with entries $b_{i,j}=a_{i,j+1}$. Pick sequences $\langle z_{1,n}\rangle_{n=1}^\infty$, $\langle z_{2,n}\rangle_{n=1}^\infty$, $\ldots$,$\langle z_{v,n}\rangle_{n=1}^\infty$ in $\mathbb{Z}[i]\setminus\{0\}$ as guaranteed by the induction hypothesis for the matrix B. For each $i\in\{r+1,r+2,...,u\}$ and each $n\in\mathbb{N}$, let

$$y_{i,n}=\sum_{j=2}^{v+1}a_{i,j}z_{j-1,n}.$$

We take $y_{r,n}=0$ for all $n\in\mathbb{N}$.

Now C being a central set in $\mathbb{Z}[i]$, by Theorem 1.6 we can pick a sequence $\langle k_n\rangle_{n=1}^\infty$ in $\mathbb{Z}[i]$ and a sequence $\langle H_n\rangle_{n=1}^\infty$ of finite nonempty subsets of $\mathbb{N}$ such that $\max H_n<\min H_{n+1}$ for each n and for each $i\in\{r,r+1,...,u\}$, $FS(\langle k_n+\sum_{t\in H_n}y_{i,t}\rangle_{n=1}^\infty)\subseteq C$.

For each $n\in\mathbb{N}$, let $x_{1,n}=\frac{k_n}{c}$ and note that $k_n=k_n+\sum_{t\in H_n}y_{r,t}\in C\subseteq S$. For $j\in\{2,3,...,v+1\}$, let $x_{j,n}=\sum_{t\in H_n}z_{j-1,t}$. We claim that the sequences $\langle x_{j,n}\rangle_{n=1}^\infty$ are as required. To see this, let F be a finite nonempty subset of $\mathbb{N}$. We need to show that for each $i\in\{1,2,\cdots,u\}$, $\sum_{j=1}^{v+1}a_{i,j}\sum_{n\in F}x_{j,n}\in C$. So let $i\in\{1,2,\cdots,u\}$ be given.

Case 1. $i\le r$. Then

$$\begin{aligned}\textstyle\sum_{j=1}^{v+1}a_{i,j}.\sum_{n\in F}x_{j,n} &= \textstyle\sum_{j=2}^{v+1}a_{i,j}.\sum_{n\in F}\sum_{t\in H_n}z_{j-1,t}\\ &= \textstyle\sum_{j=1}^{v}b_{i,j}.\sum_{t\in G}z_{j,t}\in C.\end{aligned}$$

where $G=\cup_{n\in F}H_n$

Case 2. $i>r$. Then

$$\begin{aligned}\textstyle\sum_{j=1}^{v+1}a_{i,j}.\sum_{n\in F}x_{j,n} &= \textstyle c.\sum_{n\in F}x_{1,n}+\sum_{j=2}^{v+1}a_{i,j}.\sum_{n\in F}x_{j,n}\\ &= \textstyle\sum_{n\in F}cx_{1,n}+\sum_{n\in F}\sum_{t\in H_n}\sum_{j=2}^{v+1}a_{i,j}z_{j-1,t}\\ &= \textstyle\sum_{n\in F}(k_n+\sum_{t\in H_n}y_{i,t})\in C.\end{aligned}$$

□

Corollary 2.5. *Any finite matrix with entries from $\mathbb{Z}[i]$ which satisfies the first entries condition is image partition regular over $\mathbb{Z}[i]$.*

3. Image Partition Regularity Over $\mathbb{Z}[i]$

In this section we provide several characterizations of image partition regularity over $\mathbb{Z}[i]$.

Theorem 3.1. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}[i]$. The following statements are equivalent.*

(a) A is image partition regular over $\mathbb{Z}[i]$.

(b) Let $\vec{c_1}, \vec{c_2}, \ldots, \vec{c_v}$ be the columns of A. There exist $s_1, s_2, \ldots, s_v \in \mathbb{Q}[i]\backslash\{0\}$ such that the matrix

$$M = \begin{pmatrix} & & & & & & -1 & 0 & . & . & . & 0 \\ & & & & & & 0 & -1 & . & . & . & 0 \\ & & & & & & . & . & . & . & . & . \\ s_1 \cdot \vec{c_1} & s_2 \cdot \vec{c_2} & . & . & . & s_v \cdot \vec{c_v} & . & . & . & . & . & . \\ & & & & & & . & . & . & . & . & . \\ & & & & & & 0 & 0 & . & . & . & -1 \end{pmatrix}$$

is kernel partition regular over $\mathbb{Z}[i]$.

(c) There exist $m \in \mathbb{N}$, and a $v \times m$ matrix G with entries from $\mathbb{Q}[i]$ and no row equal to $\vec{0}$, and a $u \times m$ first entries matrix B, with all its first entries equal to 1, such that $AG = B$.

(d) There exist $m \in \mathbb{N}$, and a $v \times m$ matrix H with entries from $\mathbb{Z}[i]$ and no row equal to $\vec{0}$, and a $u \times m$ first entries matrix C with entries from $\mathbb{Z}[i]$, and $c \in \mathbb{Z}[i]\backslash\{0\}$ as the only the first entries of C, such that $AH = C$.

(e) There exist $m \in \mathbb{N}$, and a $u \times m$ first entries matrix B with entries from $\mathbb{Q}[i]$ and the first entries all are equal Gaussian integers such that given any $\vec{y} \in (\mathbb{Z}[i]\backslash\{0\})^m$ there is some $\vec{x} \in (\mathbb{Z}[i]\backslash\{0\})^v$, with $A\vec{x} = B\vec{y}$.

(f) There exist $m \in \mathbb{N}$, and a $u \times m$ first entries matrix C with entries from $\mathbb{Z}[i]$ and the first entries all are equal Gaussian integers such that given any $\vec{y} \in (\mathbb{Z}[i]\backslash\{0\})^m$ there is some $\vec{x} \in (\mathbb{Z}[i]\backslash\{0\})^v$ with $A\vec{x} = C\vec{y}$.

(g) There exist $m \in \mathbb{N}$, and a $u \times m$ matrix B with entries from $\mathbb{Z}[i]$ which satisfies first entries condition such that given any $\vec{y} \in (\mathbb{Z}[i]\backslash\{0\})^m$ there is some $\vec{x} \in (\mathbb{Z}[i]\backslash\{0\})^v$ with $A\vec{x} = B\vec{y}$.

(h) For every central set C in $\mathbb{Z}[i]$, there exists $\vec{x} \in \big(\mathbb{Z}[i]\backslash\{0\}\big)^v$ such that $A\vec{x} \in C^u$.

(i) For every central set C in $\mathbb{Z}[i]$, the set $\big\{\vec{x} \in \big(\mathbb{Z}[i]\backslash\{0\}\big)^v : A\vec{x} \in C^u\big\}$ is central in $\big(\mathbb{Z}[i]\big)^v$.

(j) There exist $s_1, s_2, \ldots, s_v \in \mathbb{Q}[i]\backslash\{0\}$ such that the matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & .0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \\ s_1a_{1,1} & s_2a_{1,2} & s_3a_{1,3} & . & . & . & s_va_{1,v} \\ s_1a_{2,1} & s_2a_{2,2} & s_3a_{2,3} & . & . & . & s_va_{2,v} \\ s_1a_{3,1} & s_2a_{3,2} & s_3a_{3,3} & . & . & . & s_va_{3,v} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ s_1a_{u,1} & s_2a_{u,2} & s_3a_{u,3} & . & . & . & s_va_{u,v} \end{pmatrix}$$

is image partition regular.

(k) *There exist* $b_1, b_2, \ldots, b_v \in \mathbb{Q}[i]\backslash\{0\}$ *such that the matrix*

$$Q = \begin{pmatrix} b_1 & 0 & 0 & . & . & . & 0 \\ 0 & b_2 & 0 & . & . & . & 0 \\ 0 & 0 & b_3 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & b_v \\ & & & A & & & \end{pmatrix}$$

is image partition regular.

(l) *For each* $\vec{r} \in (\mathbb{Q}[i])^v\backslash\{\vec{0}\}$ *there exists* $b \in \mathbb{Q}[i]\backslash\{0\}$ *such that* $\begin{pmatrix} b\vec{r} \\ A \end{pmatrix}$ *is image partition regular.*

(m) *Whenever* $m \in \mathbb{N}$, $\phi_1, \phi_2, \ldots, \phi_m$ *are nonzero linear mappings from* $(\mathbb{Q}[i])^v$ *to* $\mathbb{Q}[i]$, *there exists* $\vec{b} \in (\mathbb{Q}[i])^m$ *such that, whenever* C *is central in* $\mathbb{Z}[i]$, *there exists* $\vec{x} \in (\mathbb{Z}[i]\backslash\{0\})^v$ *for which* $A\vec{x} \in C^u$, *and for each* $i \in \{1, 2, \ldots, m\}$, $b_i\phi_i(\vec{x}) \in C$, *in particular* $\phi_i(\vec{x}) \neq 0$.

(n) *For every central set* C *in* $\mathbb{Z}[i]$, *there exists* $\vec{x} \in (\mathbb{Z}[i]\backslash\{0\})^v$ *such that* $\vec{y} = A\vec{x} \in C^u$, *all entries of* $\vec{x}$ *are distinct, and for all* $i, j \in \{1, 2, \ldots, u\}$, *if rows* i *and* j *of* A *are unequal, then* $y_i \neq y_j$.

Proof. (a) implies (b). Given any $p \in \mathbb{N}\backslash\{1\}$, let the *start base* p coloring of $\mathbb{Z}[i]$ be the function

$$\sigma_p : \mathbb{Z}[i]\backslash\{0\} \to X \times Y \times \{0, 1\}$$

where $Y = \{a+ib : a, b \in \{-(p-1), -(p-2), \ldots, 0, 1, \ldots, p-1\}\}$ and $X = Y\backslash\{0\}$, defined as follows: given any $y \in \mathbb{Z}[i]\backslash\{0\}$, write $y = \sum_{t=0}^{n} a_t p^t$, where each $a_t \in Y$ for $t = 0, 1, \ldots, n-1$ and $a_n \neq 0$ (idea to construct a_t directly comes from the *start* *base* p coloring of $\mathbb{N}$ componentwise without changing the sign, i.e., take $y = y_1 + iy_2$ and write $\mid y_1 \mid = \sum_{t=0}^{n_1} a_{1,t}p^t$ & $\mid y_2 \mid = \sum_{t=0}^{n_2} a_{2,t}p^t$, where for $i \in \{1, 2\}$ each $a_{i,t} \in \{0, 1, \ldots, p-1\}$ for $0 \leq t < n_i$ and $a_{i,n_i} \neq 0$; now take $n = \max\{n_1, n_2\}$, $n' = \min\{n_1, n_2\}$ and then define $a_{i,t} = 0$ for $n_i < t \leq n$ where $i \in \{1, 2\}$ with $n_i = n'$. Now we can write $y_i = \sum_{t=0}^{n} sgn(y_i)a_{i,t}p^t$ for $i = 1, 2$ and so clearly our $a_t = sgn(y_1)a_{1,t} + i\, sgn(y_2)a_{2,t}$ for $t = 0, 1, \ldots, n$.); if $n > 0$, $\sigma_p(y) = (a_n, a_{n-1}, i)$ where $i \equiv n (\mathrm{mod}\, 2)$; if $n = 0$, $\sigma_p(y) = (a_0, 0, 0)$ and $\sigma_p(0) = (0, 0, 0)$.

Let $\vec{c_1}, \vec{c_2}, \ldots, \vec{c_v}$ be the columns of A and let $\vec{d_1}, \vec{d_2}, \ldots, \vec{d_u}$ denote the columns of the $u \times u$ identity matrix. Let B be the matrix

$$\begin{pmatrix} s_1 \cdot \vec{c_1} & s_2 \cdot \vec{c_2} & . & . & . & s_v \cdot \vec{c_v} & -\vec{d_1} & -\vec{d_2} & . & . & . & -\vec{d_u} \end{pmatrix},$$

where $s_1, s_2, \ldots, s_v$ are yet unspecified nonzero *Gaussian rationals*. Denote the columns of B by $\vec{b_1}, \vec{b_2}, \ldots, \vec{b_{u+v}}$. Then

$$\vec{b_t} = \begin{cases} s_t \cdot \vec{c_t} & \text{if } t \leq v \\ -\vec{d_{t-v}} & \text{if } t > v. \end{cases}$$

Given any $p \in \mathbb{N}\backslash\{1\}$ and any $x \in \mathbb{Z}[i]\backslash\{0\}$, let $\gamma(p, x) = \max\big\{n \in \omega : p^n \leq \max\{\mid Re(x) \mid, \mid Im(x) \mid\}\big\}$. Now temporarily fix some $p \in \mathbb{N}\backslash\{1\}$. We obtain $m = m(p)$ and an ordered partition $\big(D_1(p), D_2(p), \ldots, D_m(p)\big)$ of $\{1, 2, \ldots, u\}$ as follows. Pick $\vec{x} \in \big(\mathbb{Z}[i]\backslash\{0\}\big)^v$ such that $A\vec{x}$ is monochrome with respect to the start base p coloring and let $\vec{y} = A\vec{x}$. Now divide up $\{1, 2, \ldots, u\}$ according to which of the y_i's start furthest to the left in their base p representation. That is, we get $D_1(p), D_2(p), \ldots, D_m(p)$ so that

(1) if $k \in \{1, 2, \ldots, m\}$ and $i, j \in D_k(p)$, then $\gamma(p, y_i) = \gamma(p, y_j)$, and

(2) if $k \in \{2, 3, \ldots, m\}$, $i \in D_k(p)$, and $j \in D_{k-1}(p)$, then $\gamma(p, y_j) > \gamma(p, y_i)$.

We also observe that as $\sigma_p(y_i) = \sigma_p(y_j)$, we have that $\gamma(p, y_i) \equiv \gamma(p, y_j) (\mathrm{mod}\ 2)$ and hence $\gamma(p, y_j) \geq \gamma(p, y_i) + 2$.

There are only finitely many ordered partitions of $\{1, 2, \ldots, u\}$. Therefore we may pick an infinite subset P of $\mathbb{N}\backslash\{1\}$, $m \in \mathbb{N}$, and an odered partition $(D_1, D_2, \ldots, D_m)$ of $\{1, 2, \ldots, u\}$ so that for all $p \in P$, $m(p) = m$ and

$$\big(D_1(p), D_2(p), \ldots, D_m(p)\big) = (D_1, D_2, \ldots, D_m).$$

We shall utilize $(D_1, D_2, \ldots, D_m)$ to find $s_1, s_2, \ldots, s_v$ and to get a partition of $\{1, 2, \ldots, u + v\}$ as required for the columns condition.

We proceed by induction. First we shall find $E_1 \subseteq \{1, 2, \ldots, v\}$, specify $s_i \in \mathbb{Q}[i]\backslash\{0\}$ for each $i \in E_1$, let $I_1 = E_1 \cup (v + D_1)$, and show that $\sum_{i \in I_1} \vec{b_i} = 0$. That is, we shall show that $\sum_{i \in E_1} s_i \cdot \vec{c_i} = \sum_{i \in D_1} \vec{d_i}$. In order to do this, we show that $\sum_{i \in D_1} \vec{d_i}$ is in the span of $(\vec{c_1}, \vec{c_2}, \ldots, \vec{c_v})$. For then one has $\sum_{i \in D_1} \vec{d_i} =$

$\sum_{i=1}^{v} \alpha_i \cdot \vec{c_i}$, where $\alpha_i \in \mathbb{Q}[i]$ (which is true because one is solving linear equations with rational coefficients) and not all α_i's be zero. Let $E_1 = \{i \in \{1, 2, \ldots, v\} : \alpha_i \neq 0\}$ and for $i \in E_1$, let $s_i = \alpha_i$.

Let S be the linear span of $(\vec{c_1}, \vec{c_2}, \ldots, \vec{c_v})$. In order to show that $\sum_{i \in D_1} \vec{d_i}$ is in S, it suffices to show that $\sum_{i \in D_1} \vec{d_i}$ is in $cl\, S$. To this end, let $\epsilon > 0$ be given and pick $p \in P$ with $p > \sqrt{2}u/\epsilon$. Pick $\vec{x} \in \big(\mathbb{Z}[i]\backslash\{0\}\big)^v$ and $\vec{y} \in \big(\mathbb{Z}[i]\big)^u$ that we used to get $\big(D_1(p), D_2(p), \ldots, D_m(p)\big)$. That is, $A\vec{x} = \vec{y}$, $\vec{y}$ is monochrome with respect to the the start base p coloring, and $(D_1, D_2, \ldots, D_m) = \big(D_1(p), D_2(p), \ldots, D_m(p)\big)$ is the ordered partition of $\{1, 2, \ldots, u\}$ induced by the starting positions of y_i's. Pick γ so that for all $i \in D_1$, $\gamma(p, y_i) = \gamma$. Pick $(a, b, c) \in X \times Y \times \{0, 1\}$ such that $\sigma_p(y_i) = (a, b, c)$ for all $i \in \{1, 2, \ldots, u\}$. Let $\ell = a + b/p$ and observe that $1 \leq |\,\ell\,| \leq \sqrt{2}p$. For $i \in D_1$, $y_i = a \cdot p^{\gamma} + b \cdot p^{\gamma\text{-}1} + z_i \cdot p^{\gamma\text{-}2}$ where $0 \leq |z_i| < \sqrt{2}p$, and hence $y_i/p^{\gamma} = \ell + z_i/p^2$; let $\lambda_i = z_i/p^2$ and note that $0 \leq |\lambda_i| < \sqrt{2}/p$. For $i \in \cup_{j=2}^{m} D_j$, we have $\gamma(p, y_i) \leq \gamma - 2$; let $\lambda_i = y_i/p^{\gamma}$ and note that $0 <|\,\lambda_i\,|< \sqrt{2}/p$.

Now $A\vec{x} = \vec{y}$ and so

$$\sum_{i=1}^{v} x_i \cdot \vec{c_i} = \vec{y} = \sum_{i=1}^{v} y_i \cdot \vec{d_i} = \sum_{i \in D_1} y_i \cdot \vec{d_i} + \sum_{j=2}^{m} \sum_{i \in D_j} y_i \cdot \vec{d_i}.$$

Thus $\sum_{i=1}^{v} (x_i/p^{\gamma}) \cdot \vec{c_i} = \sum_{i \in D_1} \ell \cdot \vec{d_i} + \sum_{i=1}^{u} \lambda_i \cdot \vec{d_i}$ and consequently $\| \sum_{i \in D_1} \vec{d_i} - \sum_{i=1}^{v} \big(x_i/(\ell p^{\gamma})\big) \cdot \vec{c_i} \| = \| \sum_{i=1}^{u} (\lambda_i/\ell) \cdot \vec{d_i} \| \leq \sum_{i=1}^{u} |\,\lambda_i/\ell\,| < \sqrt{2}u/p < \epsilon$. Since $\sum_{i=1}^{v} \big(x_i/(\ell p^{\gamma})\big) \cdot \vec{c_i}$ is in S, $\sum_{i \in D_1} \vec{d_i} \in c\ell\, S$ as required.

Now let $k \in \{2, 3, \ldots, m\}$ and assume that we have chosen $E_1, E_2, \ldots, E_{k-1} \subseteq \{1, 2, \ldots, v\}$, $s_i \in \mathbb{Q}[i]\backslash\{0\}$ for $i \in \cup_{j=1}^{k-1} E_j$, and $I_j = E_j \cup (v + D_j)$ as required for the columns condition. Let $L_k = \cup_{j=1}^{k-1} E_j$ and let $M_k = \cup_{j=1}^{k-1} D_j$ and enumerate M_k in order as $q(1), q(2), \ldots, q(r)$. We claim that it suffices to show that $\sum_{i \in D_k} \vec{d_i}$ is in the span of $\big(\vec{c_1}, \vec{c_2}, \ldots, \vec{c_v}, \vec{d_{q(1)}}, \vec{d_{q(2)}}, \ldots, \vec{d_{q(r)}}\big)$, which we will again denote by S. Indeed, assume that we have done this and can pick $\alpha_1, \alpha_2, \ldots, \alpha_v \in \mathbb{Q}[i]$, not all zero and $\delta_{q(1)}, \delta_{q(2)}, \ldots, \delta_{q(r)}$ in $\mathbb{Q}[i]$ such that $\sum_{i \in D_k} \vec{d_i} = \sum_{i=1}^{v} \alpha_i \cdot \vec{c_i} + \sum_{i=1}^{r} \delta_{q(i)} \cdot \vec{d_{q(i)}}$. Let $E_k = \{i \in \{1, 2, \ldots, v\}\backslash L_k : \alpha_i \neq 0\}$ and for $i \in E_k$, let $s_i = \alpha_i$. Let $I_k = E_k \cup (v + D_k)$.

Then

$$\sum_{i \in I_k} \vec{b_i} = \sum_{i \in E_k} \alpha_i \cdot \vec{c_i} + \sum_{i \in D_k} -\vec{d_i} = \sum_{i \in E_k} \alpha_i \cdot \vec{c_i} + \sum_{i=1}^{v} -\alpha_i \cdot \vec{c_i} + \sum_{i=1}^{r} -\delta_{q(i)} \cdot \vec{d_{q(i)}}.$$

Now, if $i \in \{1, 2, \ldots, v\}$ and $\alpha_i \neq 0$, then $i \in L_k \cup E_k$ so

$$\sum_{i\in E_k} \alpha_i \cdot \vec{c_i} + \sum_{i=1}^{v} -\alpha_i \cdot \vec{c_i} + \sum_{i=1}^{r} -\delta_{q(i)} \cdot \vec{d_{q(i)}} = \sum_{i\in L_k} -\alpha_i \cdot \vec{c_i} + \sum_{i=1}^{r} -\delta_{q(i)} \cdot \vec{d_{q(i)}}$$

and

$$\begin{aligned}\sum_{i\in L_k} -\alpha_i \cdot \vec{c_i} + \sum_{i=1}^{r} -\delta_{q(i)} \cdot \vec{d_{q(i)}} &= \sum_{i\in L_k} -\alpha_i \cdot \vec{c_i} + \sum_{i\in M_k} -\delta_i \cdot \vec{d_i} \\ &= \sum_{i\in L_k} (-\alpha_i/s_i) \cdot \vec{b_i} + \sum_{i\in M_k} \delta_i \cdot \vec{b_{v+i}}.\end{aligned}$$

Let $\beta_i = -\alpha_i/s_i$ if $i \in L_k$ and let $\beta_{v+i} = \delta_i$ if $i \in M_k$. Then we have

$$\sum_{i\in \cup_{j=1}^{k-1} I_j} \beta_i \cdot \vec{b_i} = \sum_{i\in L_k} (-\alpha_i/s_i) \cdot \vec{b_i} + \sum_{i\in M_k} \delta_i \cdot \vec{b_{v+i}} = \sum_{i\in I_k} \vec{b_i}$$

as required for the columns condition.

In order to show that $\sum_{i\in D_k} \vec{d_i}$ is in S, it suffices to show that $\sum_{i\in D_k} \vec{d_i}$ is in $c\ell S$ as S is closed in $\mathbb{C}^u$. To this end, let $\epsilon > 0$ be given and pick $p \in P$ with $p > \sqrt{2}u/\epsilon$. Pick $\vec{x} \in \big(\mathbb{Z}[i]\backslash\{0\}\big)^v$ and $\vec{y} \in \big(\mathbb{Z}[i]\big)^u$ that we used to get $\big(D_1(p), D_2(p), \ldots, D_m(p)\big)$. Pick γ so that for all $i \in D_k$, $\gamma(p, y_i) = \gamma$. Pick $(a, b, c) \in X \times Y \times \{0, 1\}$ such that $\sigma_p(y_i) = (a, b, c)$ for all $i \in \{1, 2, \ldots, u\}$. Let $\ell = a + b/p$ and observe that $1 \leq |\ \ell\ | \leq \sqrt{2}p$. For $i \in D_k$, $y_i = a \cdot p^\gamma + b \cdot p^{\gamma-1} + z_i \cdot p^{\gamma-2}$ where $0 \leq |\ z_i\ | < \sqrt{2}p$, and hence $y_i/p^\gamma = \ell + z_i/p^2$; let $\lambda_i = z_i/p^2$ and note that $0 \leq |\ \lambda_i\ | < \sqrt{2}/p$. For $i \in \cup_{j=k+1}^m D_j$, we have $\lambda(p, y_i) \leq \gamma - 2$; let $\lambda_i = y_i/p^\gamma$ and note that $0 < |\ \lambda_i\ | < \sqrt{2}/p$. (Of course we have no control on the size of y_i/p^γ for $i \in M_k$.)

Now $A\vec{x} = \vec{y}$ and so

$$\sum_{i=1}^{v} x_i \cdot \vec{c_i} = \sum_{i=1}^{u} y_i \cdot \vec{d_i} = \sum_{i\in M_k} y_i \cdot \vec{d_i} + \sum_{i\in D_k} y_i \cdot \vec{d_i} + \sum_{j=k+1}^{m} \sum_{i\in D_j} y_i \cdot \vec{d_i},$$

where $\sum_{j=k+1}^m \sum_{i\in D_j} y_i \cdot \vec{d_i} = 0$ if $k = m$.

Thus $\sum_{i=1}^v (x_i/p^\gamma) \cdot \vec{c_i} = \sum_{i\in M_k} (y_i/p^\gamma) \cdot \vec{d_i} + \sum_{i\in D_k} \ell \cdot \vec{d_i} + \sum_{j=k}^m \sum_{i\in D_j} \lambda_i \cdot \vec{d_i}$. Consequently,

$$\begin{aligned}&\| \sum_{i\in D_k} \vec{d_i} - \left(\sum_{i=1}^{v} \big(x_i/(\ell p^\gamma) \cdot \vec{c_i}\big) + \sum_{i\in M_k} \big(-y_i/(\ell p^\gamma)\big) \cdot \vec{d_i} \right) \| \\ &= \| \sum_{j=k}^{m} \sum_{i\in D_j} (\lambda_i/\ell) \cdot \vec{d_i} \| \leq \sum_{j=k}^{m} \sum_{i\in D_j} |\ \lambda_i/\ell\ | < u\sqrt{2}/p < \epsilon,\end{aligned}$$

which suffices, since $x_i/(\ell p^\gamma) \neq 0$ for $i \in \{1, 2, \ldots, v\}$.

Having chosen $I_1, I_2, \ldots, I_m$, if $\{1, 2, \ldots, u+v\} = \bigcup_{j=1}^m I_j$, we are done. So assume $\{1, 2, \ldots, u+v\} \neq \bigcup_{j=1}^m I_j$. Let $I_{m+1} = \{1, 2, \ldots, u+v\} \setminus \bigcup_{j=1}^m I_j$. Now $\{1, 2, \ldots, u\} = \bigcup_{j=1}^m D_j$, so $\{-\vec{d_1}, -\vec{d_2}, \ldots, -\vec{d_u}\} \subseteq \{\vec{b_i} : i \in \cup_{j=1}^m I_j\}$ and hence we can write $\sum_{i \in I_{m+1}} \vec{b_i}$ as a linear combination of $\{\vec{b_i} : i \in \cup_{j=1}^m I_j\}$.

$(b) \implies (c)$. By Rado's extended result, M satisfies the columns condition over $\mathbb{Q}[i]$. Pick $m \in \mathbb{N}$, $\langle I_t \rangle_{t=1}^m$, $\langle J_t \rangle_{t=2}^m$, and $\left\langle \langle \delta_{t,i} \rangle_{i \in J_t} \right\rangle_{t=2}^m$ as guaranteed by the columns condition for M. Let B' be the $(u+v) \times m$ matrix whose entry in i'th row and t'th column is given by

$$b'_{i,t} = \begin{cases} -\delta_{t,i} & \text{if } i \in J_t \\ 1 & \text{if } i \in I_t \\ 0 & \text{if } i \notin \cup_{j=1}^t I_j \end{cases}$$

We observe that B' satisfies the first entries condition, with the first non-zero entry in each row being 1. We also observe that $MB' = \mathbf{0}$, the $u \times m$ matrix whose entries are all zero (Indeed, let $j \in \{1, 2, \ldots, u\}$ and $t \in \{1, 2, \ldots, m\}$. If $t = 1$, then $\sum_{i=1}^{u+v} c_{j,i} \cdot b'_{i,t} = \sum_{i \in I_1} c_{j,i} = 0$ and if $t > 1$, then $\sum_{i=1}^{u+v} c_{j,i} \cdot b'_{i,t} = \sum_{i \in J_t} -\delta_{t,i} \cdot c_{j,i} + \sum_{i \in I_t} c_{j,i} = 0$). Let S denote the $v \times v$ diagonal matrix whose diagonal entries are $s_1, s_2, \ldots, s_v$. Then M can be written in the block form as $\begin{pmatrix} AS & -I_u \end{pmatrix}$ and B' can be written in the block form as $\begin{pmatrix} C \\ B \end{pmatrix}$, where I_u denotes the $u \times u$ identity matrix and C and B denote the $v \times m$ and $u \times m$ matrices respectively. We observe that C and B both are first entries matrices with the first non-zero entry in each row is 1, and that $ASC = B$, because $MB' = \mathbf{0}$. So take $G = SC$, which is a $v \times m$ matrix with coefficients from $\mathbb{Q}[i]$ and having no row equals to $\vec{0}$. Hence we have $AG = B$, as required.

$(c) \implies (d)$. We can choose $c \in \mathbb{Z}[i] \setminus \{0\}$ in such a way, so that all entries of cG and cAG are in $\mathbb{Z}[i]$. In fact it can be achieved by multiplying suitable positive integers, which is trivially a common multiple of the denominators of each entry of the corresponding matrices . Let $C = cB$, then C is again a first entry matrix with all the first non-zero entries in each row equal to c (clearly, the entries of C are from $\mathbb{Z}[i]$). Take $H = cG$, and we have $AH = C$, as required.

$(c) \implies (e)$. Assume that (c) holds. We can choose a $d \in \mathbb{Z}[i] \setminus \{0\}$ such that all the entries of dG are in $\mathbb{Z}[i]$. Since B is a given $u \times m$ first entries matrix, then $B' = dB$ is also a first entries matrix with all the first entries equal to d. Let $\vec{y} \in \left(\mathbb{Z}[i] \setminus \{0\}\right)^m$ be given and let us set $\vec{x} = dG\vec{y}$. Then $A\vec{x} = B'\vec{y}$. Replacing B' by B, we have $A\vec{x} = B\vec{y}$.

$(e) \implies (f)$. Let B be a given first entries matrix as in (e). We choose $c \in \mathbb{Z}[i] \setminus \{0\}$ to be a common multiple of the denominators in the entries of B and set $C = cB$. Given $\overrightarrow{y}$, let $\overrightarrow{z} = c\overrightarrow{y}$ and pick $\overrightarrow{x}$ such that $A\overrightarrow{x} = B\overrightarrow{z} = C\overrightarrow{y}$. Clearly the first entries in each row of C are equal Gaussian integers.

$(d) \implies (f)$. Given $\overrightarrow{y}$, set $\overrightarrow{x} = H\overrightarrow{y}$, and the result follows immediately.

$(f) \implies (g)$. Trivial.

$(g) \implies (f)$. Let B be given as in (g), and for each $j \in \{1, 2, \ldots, m\}$, pick $w_j \in \mathbb{Z}[i]\backslash\{0\}$ such that for any $i \in \{1, 2, \ldots, v\}$, if $j = \min\{t \in \{1, 2, \ldots, m\} : b_{i,t} \neq 0\}$ then $b_{i,j} = w_j$. (That is, w_j is the first entry associated with the column j.) Let c be the common multiple of $\{w_1, w_2, \ldots, w_m\}$. Define the $u \times m$ matrix C as follows: for $(i,j) \in \{1, 2, \ldots, u\} \times \{1, 2, \ldots, m\}$, $c_{i,j} = (c/w_j)b_{i,j}$. Clearly C is a first entries matrix with entries from $\mathbb{Z}[i]$ and all the first entries are equal to c, a fixed Gaussian integer. Now, given $\overrightarrow{y} \in \left(\mathbb{Z}[i]\backslash\{0\}\right)^m$, we define $\overrightarrow{z} \in \left(\mathbb{Z}[i]\backslash\{0\}\right)^m$ by the rule: for $j \in \{1, 2, \ldots, m\}$, $z_j = (c/w_j)y_j$. Then $B\overrightarrow{z} = C\overrightarrow{y}$. Now again applying (g) there exists $\overrightarrow{x} \in \left(\mathbb{Z}[i]\backslash\{0\}\right)^v$ such that $B\overrightarrow{z} = A\overrightarrow{x}$. Therefore we have $A\overrightarrow{x} = C\overrightarrow{y}$.

$(f) \implies (a)$. By Lemma 2.1 for each $a \in \mathbb{Z}[i]\backslash\{0\}$, $a\mathbb{Z}[i]$ is an IP^*-set, and therefore a *central** set. Thus by Corollary 2.5, C being a first entries matrix is image partition regular over $\mathbb{Z}[i]$. To see that A is image partition regular over $\mathbb{Z}[i]$, let $r \in \mathbb{N}$ and let $\mathbb{Z}[i] = \bigcup_{i=1}^{r} E_i$. Pick $i \in \{1, 2, \ldots, r\}$ and $\overrightarrow{y} \in \left(\mathbb{Z}[i]\backslash\{0\}\right)^m$ such that $C\overrightarrow{y} \in E_i^u$. Then pick $\overrightarrow{x} \in \left(\mathbb{Z}[i]\backslash\{0\}\right)^v$ such that $A\overrightarrow{x} = C\overrightarrow{y}$.

$(g) \implies (h)$. Let B be as guaranteed by (g) and let C be a central set in $\mathbb{Z}[i]$. Pick by Theorem 2.4, some $\overrightarrow{y} \in \left(\mathbb{Z}[i]\backslash\{0\}\right)^m$ such that $B\overrightarrow{y} \in C^u$, and pick $\overrightarrow{x}$ such that $A\overrightarrow{x} = B\overrightarrow{y}$. Therefore $A\overrightarrow{x} \in C^u$.

$(h) \implies (a)$. This is obvious because for any finite partition of $\mathbb{Z}[i]$, at least one of the cells must be central.

$(h) \implies (i)$. Pick $d \in \mathbb{Z}[i]\backslash\{0\}$ such that all entries of dA are in $\mathbb{Z}[i]$. We claim that for every central set C in $\mathbb{Z}[i]$, there exists $\overrightarrow{x} \in (\mathbb{Z}[i]\backslash\{0\})^v$ such that $dA\overrightarrow{x} \in C^u$. By Lemma 2.2, $\left(\frac{1}{d}C \cap \mathbb{Z}[i]\right)$ is central, so pick $\overrightarrow{x} \in (\mathbb{Z}[i]\backslash\{0\})^v$ such that $A\overrightarrow{x} \in \left(\frac{1}{d}C \cap \mathbb{Z}[i]\right)^u$. Then $dA\overrightarrow{x} \in C^u$.

Let C be a central subset of $\mathbb{Z}[i]$ and pick a minimal idempotent $p \in \beta\mathbb{Z}[i]$ such that $C \in p$. Define $\varphi : (\mathbb{Z}[i])^v \to (\mathbb{Z}[i])^u$ by $\varphi\left(\overrightarrow{x}\right) = dA\overrightarrow{x}$ and let $\tilde{\varphi} : \beta\left(\mathbb{Z}[i]^v\right) \to (\beta\mathbb{Z}[i])^u$ be its continuous extension. Now dp is a minimal idempotent by Lemma 2.2. Define $\overline{dp} = (dp, dp, \ldots, dp)^T$ and pick by Lemma 2.3, a minimal idempotent $q \in \beta\left(\mathbb{Z}[i]^v\right)$ such that $\tilde{\varphi}(q) = \overline{dp}$. Now $\times_{i=1}^{u}\overline{dC}$ is a neighborhood of $\overline{dp}$ and pick $B \in q$ such that $\tilde{\varphi}[\overline{B}] \subseteq \times_{i=1}^{u}\overline{dC}$. Then $B \subseteq \left\{\overrightarrow{x} \in (\mathbb{Z}[i]\backslash\{0\})^v : A\overrightarrow{x} \in C^u\right\}$, so $\left\{\overrightarrow{x} \in (\mathbb{Z}[i]\backslash\{0\})^v : A\overrightarrow{x} \in C^u\right\}$ is central in $(\mathbb{Z}[i])^v$.

$(i) \implies (a)$. This is immediate because for any finite partition of $\mathbb{Z}[i]$, at least one of the cells must be central.

$(b) \implies (j)$. Let

$$B = \begin{pmatrix} s_1a_{1,1} & s_2a_{1,2} & s_3a_{1,3} & . & . & . & s_va_{1,v} \\ s_1a_{2,1} & s_2a_{2,2} & s_3a_{2,3} & . & . & . & s_va_{2,v} \\ s_1a_{3,1} & s_2a_{3,2} & s_3a_{3,3} & . & . & . & s_va_{3,v} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ s_1a_{u,1} & s_2a_{u,2} & s_3a_{u,3} & . & . & . & s_va_{u,v} \end{pmatrix}$$

and let I_u and I_v be the identity matrices of order u and v respectively. Then $P = \begin{pmatrix} I_v \\ B \end{pmatrix}$ and $M = \begin{pmatrix} B & -I_u \end{pmatrix}$. To see that P is image partition regular, let $\mathbb{Z}[i] \setminus \{0\}$ be finitely colored and pick $\vec{z} \in \big(\mathbb{Z}[i]\backslash\{0\}\big)^{u+v}$ such that $M\vec{z} = 0$ and $\vec{z}$ is monochrome. Let $\vec{x} \in \big(\mathbb{Z}[i]\backslash\{0\}\big)^{v}$ and $\vec{y} \in \big(\mathbb{Z}[i]\backslash\{0\}\big)^{u}$ such that $\vec{z} = \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}$. Then $\vec{0} = M\vec{z} = B\vec{x} - \vec{y}$ and so $P\vec{x} = \vec{z}$.

$(j) \implies (k)$. For each $i \in \{1, 2, \ldots, v\}$, let $b_i = \frac{1}{s_i}$ and let

$$S = \begin{pmatrix} s_1 & 0 & . & . & . & 0 \\ 0 & s_2 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & s_v \end{pmatrix}.$$

Then $P = QS$. Pick $d \in \mathbb{Z}[i]\backslash\{0\}$ such that $\{ds_1, ds_2, \ldots, ds_v\} \subseteq \mathbb{Z}[i]\backslash\{0\}$. We show that statement (h) holds for the matrix Q. Let C be central in $\mathbb{Z}[i]$ and pick a minimal idempotent $p \in \beta\mathbb{Z}[i]$ such that $C \in p$. By Lemma 2.1, $d\mathbb{Z}[i] \in p$ and so $C \cap d\mathbb{Z}[i]$ is central. We have already shown that statement (a) implies statement (h), so the statement (h) holds for the matrix P. Pick $\vec{x} \in \big(\mathbb{Z}[i]\backslash\{0\}\big)^{v}$ such that $P\vec{x} \in \big(C \cap d\mathbb{Z}[i]\big)^{u+v}$. Then the entries of $\vec{x}$ are the first v entries of $P\vec{x}$, hence are multiples of d. Therefore $\vec{y} = S\vec{x} \in \big(\mathbb{Z}[i]\backslash\{0\}\big)^{v}$ and $Q\vec{y} = P\vec{x} \in C^{u+v}$.

$(k) \implies (a)$. Follows directly.

$(c) \implies (l)$. If $\vec{r}G \neq \vec{0}$, we can choose b so that the first entry of $b\vec{r}G$ is 1. If $\vec{r}G = \vec{0}$, we can choose $\vec{c} \in \big(\mathbb{Z}[i]\backslash\{0\}\big)^{u}$ such that $\vec{r} \cdot \vec{c} \neq 0$ and add $\vec{c}$ to G as a new final column. In this case we choose b so that $b\vec{r} \cdot \vec{c} = 1$. In either case, $\begin{pmatrix} b\vec{r} \\ A \end{pmatrix} G$ is a first entries matrix with all its first entries equal to 1 and so the statement (c) holds for $\begin{pmatrix} b\vec{r} \\ A \end{pmatrix}$.

$(l) \implies (m)$. For each $i \in \{1, 2, \ldots, m\}$, there exists $\vec{r_i} \in \mathbb{Q}[i]^v\backslash\{0\}$ such that $\phi_i(\vec{x}) = \vec{r_i} \cdot \vec{x}$ for all $\vec{x} \in \mathbb{Q}[i]^v$. By applying statement (l) m times in succession (using the fact that at each stage the new matrix satisfies (l) because (a) implies (l) as (c) implies (l)), we can choose $b_1, b_2, \ldots, b_m \in \mathbb{Q}[i]\backslash\{0\}$ for which the matrix

$$T = \begin{pmatrix} b_1\overrightarrow{r_1} \\ b_2\overrightarrow{r_2} \\ . \\ . \\ . \\ b_m\overrightarrow{r_m} \\ A \end{pmatrix}$$

is image partition regular. Since we know from (h) that every image partition regular matrix has an image in any central set, for any central set C in $\mathbb{Z}[i]$, there exists $\overrightarrow{x} \in \big(\mathbb{Z}[i]\backslash\{0\}\big)^v$ for which $T\overrightarrow{x} \in C^{m+u}$. This implies $A\overrightarrow{x} \in C^u$. Also, for $i \in \{1, 2, \ldots, m\}$, $b_i\overrightarrow{r_i} \cdot \overrightarrow{x} \in C$ and so $b_i\phi_i(\overrightarrow{x}) \in C$. In particular we have $\phi_i(\overrightarrow{x}) \neq 0$.

$(m) \implies (n)$. We may presume that A has no repeated rows so that the conclusion regarding $\overrightarrow{y}$ becomes the statement that all entries of $\overrightarrow{y}$ are distinct. For $i \neq j$ in $\{1, 2, \ldots, v\}$, let $\overrightarrow{\phi_{i,j}}$ be the linear mapping from $\mathbb{Q}[i]^v$ to $\mathbb{Q}[i]$ taking $\overrightarrow{x}$ to $x_i - x_j$. For $i \neq j$ in $\{1, 2, \ldots, u\}$, let $\overrightarrow{\psi_{i,j}}$ be the linear mapping from $\mathbb{Q}[i]^v$ to $\mathbb{Q}[i]$ taking $\overrightarrow{x}$ to $\sum_{t=1}^{v}(a_{i,t} - a_{j,t}) \cdot x_t$. Applying statement (m) to the set $\{\phi_{i,j} : i, j \in \{1, 2, \ldots, v\}, i \neq j\} \cup \{\psi_{i,j} : i, j \in \{1, 2, \ldots, u\}, i \neq j\}$, we reach the desired conclusion.

$(n) \implies (h)$. This follows immediately. □

4. Infinite Matrices

We now prove that certain infinite matrices with entries from Gaussian integers are also image partition regular over $\mathbb{Z}[i] \setminus \{0\}$. K. Milliken and A.Taylor independently proved a theorem from which it can be derived that certain infinite matrices, called Milliken-Taylor Matrices, are image partition regular over $\mathbb{N}$. Some generalizations of this celebrated theorem are also available in [8, Corollary 3.6], [**?**, Theorem 5.7], [2, Theorem 2.6].

Definition 4.1. Let $m \in \omega$, and $\vec{a} = \langle a_i \rangle_{i=0}^{m}$ be a sequence in $\mathbb{Z}[i] \setminus \{0\}$, and let $\vec{z} = \langle z_n \rangle_{n=0}^{\infty}$ be a sequence in $\mathbb{Z}[i]$. The *Milliken-Taylor system* determined by $\vec{a}$ and $\vec{z}$ is $MT(\vec{a}, \vec{z}) = \{\sum_{i=0}^{m} a_i \cdot \sum_{t \in F_i} z_t$: each $F_i \in \mathcal{P}_f(\omega)$ and if $i < m$, then $\max F_i < \min F_{i+1}\}$.

If $\vec{c}$ is obtained from $\vec{a}$ by deleting repetitions, then for any infinite sequence $\vec{z}$, one has $MT(\vec{a}, \vec{z}) \subseteq MT(\vec{c}, \vec{z})$, so it suffices to consider sequences $\vec{c}$ without adjacent repeated entries.

Definition 4.2. Let $\vec{a}$ be a finite or infinite sequence in $\mathbb{Z}[i]$ with only finitely many nonzero entries. Then $c(\vec{a})$ is the sequence obtained from $\vec{a}$ by deleting all zeros and then deleting all adjacent repeated entries. The sequence $c(\vec{a})$ is the *compressed form* of $\vec{a}$. If $\vec{a} = c(\vec{a})$, then $\vec{a}$ is called a *compressed sequence*.

Definition 4.3. Let $\vec{a}$ be a compressed sequence in $\mathbb{Z}[i] \setminus \{0\}$. A *Milliken-Taylor matrix* determined by $\vec{a}$ is an $\omega \times \omega$ matrix M such that the rows of M are all possible rows with finitely many nonzero entries and compressed form equal to $\vec{a}$.

Theorem 4.4. *Let $m \in \omega$ and $\vec{a} = \langle a_i \rangle_{i=0}^m$ be a compressed sequence in $\mathbb{Z}[i] \setminus \{0\}$ with $a_0 \neq 0$, and let M be a Milliken-Taylor matrix determined by $\vec{a}$. Then M is image partition regular over $\mathbb{Z}[i]$. In fact, given any sequence $\langle w_n \rangle_{n=0}^\infty$ in $\mathbb{Z}[i] \setminus \{0\}$ such that whenever $r \in \mathbb{N}$, $\mathbb{Z}[i] \setminus \{0\} = \bigcup_{i=1}^r C_i$, there exist $i \in \{1, 2, \ldots, r\}$ and a sum subsystem $\langle x_n \rangle_{n=0}^\infty$ of $\langle w_n \rangle_{n=0}^\infty$ such that $MT(\vec{a}, \vec{x}) \subseteq C_i$.*

Proof. By [9, Lemma 5.11], choose an idempotent $p \in \bigcap_{k=0}^\infty c\ell_{\beta\mathbb{Z}[i]} FS(\langle w_n \rangle_{n=k}^\infty)$. Let $q = a_0 \cdot p + a_1 \cdot p + \ldots + a_m \cdot p$. So it suffices to show that whenever $Q \in q$, there is a sum subsystem $\langle x_n \rangle_{n=0}^\infty$ of $\langle w_n \rangle_{n=0}^\infty$ such that $MT(\vec{a}, \vec{x}) \subseteq Q$. Let $Q \in q$ be given. Assume first that $m = 0$. Then $(a_0)^{-1}Q \in p$, where $(a_0)^{-1}Q = \{z \in \mathbb{Z}[i] : a_0 \cdot z \in Q\}$, so by [9, Theorem 5.14] there is a sum subsystem$\langle x_n \rangle_{n=0}^\infty$ of $\langle w_n \rangle_{n=0}^\infty$ such that $FS(\langle x_n \rangle_{n=0}^\infty) \subseteq (a_0)^{-1}Q$. Then $MT(\vec{a}, \vec{x}) \subseteq Q$. Now assume that $m > 0$. Define

$$P(\emptyset) = \{z \in \mathbb{Z}[i] : -(a_0 \cdot z) + Q \in a_1 \cdot p + a_2 \cdot p + \ldots + a_m \cdot p\}.$$

Since $Q \in q$ we have

$$a_0^{-1} \cdot \{z \in \mathbb{Z}[i] : -z + Q \in a_1 \cdot p + a_2 \cdot p + \ldots + a_m \cdot p\} \in p$$

which shows that

$$P(\emptyset) = \{z \in \mathbb{Z}[i] : -(a_0 \cdot z) + Q \in a_1 \cdot p + a_2 \cdot p + \ldots + a_m \cdot p\} \in p.$$

Given z_0 define

$$P(z_0) = \{u \in \mathbb{Z}[i] : -(a_0 \cdot z_0 + a_1 \cdot u) + Q \in a_2 \cdot p + a_3 \cdot p + \ldots + a_m \cdot p\}.$$

If $z_0 \in P(\emptyset)$, then

$$-(a_0 \cdot z_0) + Q \in a_1 \cdot p + a_2 \cdot p + \ldots + a_m \cdot p$$

and so

$$\{u \in \mathbb{Z}[i] : -(a_1 \cdot u) + (-(a_0 \cdot z_0) + Q) \in a_2 \cdot p + a_3 \cdot p + \ldots + a_m \cdot p\} \in p$$

and thus $P(z_0) \in p$.

Given $n \in \{1, 2, \ldots, m-1\}$ and $z_0, z_1, \ldots, z_{n-1}$, let $P(z_0, z_1, \ldots, z_{n-1}) =$

$$\{u \in \mathbb{Z}[i] : -(a_0 \cdot z_0 + \ldots + a_{n-1} z_{n-1} + a_n \cdot u + Q \in a_{n+1} \cdot p + \ldots + a_m \cdot p\}.$$

If $z_0 \in P(\emptyset)$ and for each $i \in \{1, 2, \ldots, n-1\}$, $z_i \in P(z_0, z_1, \ldots, z_{i-1})$, then $P(z_0, z_1, \ldots, z_{n-1}) \in p$.

Now given $z_0, z_1, \ldots, z_{m-1}$, let

$$P(z_0, z_1, \ldots, z_{m-1}) = \{u \in \mathbb{Z}[i] : a_0 \cdot z_0 + a_1 \cdot z_1 + \ldots + a_{m-1} \cdot z_{m-1} + a_m \cdot u \in Q\}.$$

If $z_0 \in P(\emptyset)$ and for each $i \in \{1, 2, \ldots, m-1\}$, $z_i \in P(z_0, z_1, \ldots, z_{i-1})$, then $P(z_0, z_1, \ldots, z_{m-1}) \in p$. Given any $B \in p$, let $B^\star = \{z \in B : -z + B \in p\}$. Then $B^\star \in p$ and by [9, Lemma 4.14], for each $z \in B^\star$, $-z + B^\star \in p$.

Choose $z_0 \in P(\emptyset)^\star \cap FS(\langle w_n \rangle_{n=0}^\infty)$ and choose $H_0 \in \mathcal{P}_f(\mathbb{N})$ such that $z_0 = \sum_{t \in H_0} w_t$. Let $n \in \omega$. We further assume that we have chosen $z_0, z_1, \ldots, z_n$ and $H_0, H_1, \ldots, H_n$ such that

1. if $k \in \{0, 1, \ldots, n\}$, then $H_k \in \mathcal{P}_f(\omega)$ and $z_k = \sum_{t \in H_k} w_t$,
2. if $k \in \{0, 1, \ldots, n-1\}$, then $\max H_k < \min H_{k+1}$,
3. if $\emptyset \neq F \subseteq \{0, 1, \ldots, n\}$, then $\sum_{t \in F} z_t \in P(\emptyset)^\star$, and
4. if $k \in \{1, 2, \ldots, \min\{m, n\}\}$, $F_0, F_1, \ldots, F_k \in \mathcal{P}_f(\{0, 1, \ldots, n\})$, and for each $j \in \{0, 1, \ldots, k-1\}$, $\max F_j < \min F_{j+1}$, then

$$\sum_{t \in F_k} z_t \in P(\sum_{t \in F_0} z_t, \sum_{t \in F_1} z_t \ldots, \sum_{t \in F_{k-1}} z_t)^\star.$$

All hypotheses hold at $n = 0$, with (2) and (4) holding vacuously. Let $v = \max H_n$. For $r \in \{0, 1, \ldots, n\}$, let

$$E_r = \left\{ \sum_{t \in F} z_t : \emptyset \neq F \subseteq \{r, r+1, \ldots, n\} \right\}.$$

For $k \in \{0, 1, \ldots, m-1\}$ and $r \in \{0, 1, \ldots, n\}$, let

$$W_{k,r} = \{ \begin{array}{l} (\sum_{t \in F_0} z_t, \ldots, \sum_{t \in F_k} z_t) : F_0, F_1, \ldots, F_k \in \mathcal{P}_f(\{0, 1, \ldots, r\}) \\ \text{and for each } i \in \{0, 1, \ldots, k-1\}, \max F_i < \min F_{i+1} \} \end{array}$$

Note that $W_{k,r} \neq \emptyset$ if and only if $k \leq r$. If $u \in E_0$, then $u \in P(\emptyset)^\star$, so $-u + P(\emptyset)^\star \in p$ and $P(u) \in p$. If $k \in \{1, 2, \ldots, m-1\}$ and $(u_0, u_1, \ldots, u_k) \in W_{k,m}$, then $u_k \in P(u_0, u_1, \ldots, u_{k-1})$, and so $P(u_0, u_1, \ldots, u_k) \in p$ and thus $P(u_0, u_1, \ldots, u_k)^\star \in p$. If $r \in \{0, 1, \ldots, n-1\}$, $k \in \{0, 1, \ldots, \min\{m-1, r\}\}$, $(u_0, u_1, \ldots, u_k) \in W_{k,r}$, and $z \in E_{r+1}$, then $z \in P(u_0, u_1, \ldots, u_k)^\star$ and so $-z + P(u_0, u_1, \ldots, u_k)^\star \in p$. If $n = 0$, let

$$z_1 \in FS(\langle w_t \rangle_{t=v+1}^\infty) \cap P(\emptyset)^\star \cap \{-z_0 + P(\emptyset)^\star\} \cap P(z_0)^\star$$

and pick $H_1 \in \mathcal{P}_f(\mathbb{N})$ such that $\min H_1 > v$ and $z_1 = \sum_{t\in H_1} w_t$. The hypotheses are satisfied. Now assume that $n \geq 1$ and pick

$$\begin{aligned} z_{n+1} \quad \in \quad & FS(\langle w_t\rangle_{t=v+1}^{\infty}) \cap P(\emptyset)^\star \cap \textstyle\bigcap_{u\in E_0}\{-u+P(\emptyset)^\star\} \\ & \cap \textstyle\bigcap_{k=0}^{\min\{m-1,n\}} \bigcap_{(u_0,u_1,\ldots,u_k)\in W_{k,m}} P(u_0,u_1,\ldots,u_k)^\star \\ & \cap \textstyle\bigcap_{r=0}^{n-1} \bigcap_{k=0}^{\min\{m-1,r\}} \bigcap_{(u_0,u_1,\ldots,u_k)\in W_{k,r}} \\ & \textstyle\bigcap_{z\in E_{r+1}} (-z+P(u_0,u_1,\ldots,u_k)^\star)\,. \end{aligned}$$

Pick $H_{n+1} \in \mathcal{P}_f(\mathbb{N})$ such that $\min H_{n+1} > v$ and $z_{n+1} = \sum_{t\in H_{n+1}} w_t$. Hypotheses (1) and (2) hold directly. For hypothesis (3) assume that $\emptyset \neq F \subseteq \{0,1,\ldots,n+1\}$ and $n+1 \in F$. If $F = \{n+1\}$ we have directly that $z_{n+1} \in P(\emptyset)^\star$, so assume that $\{n+1\} \subsetneq F$ and let $G = F \setminus \{n+1\}$. Let $u = \sum_{t\in G} z_t$. Then $u \in E_0$ and so $z_{n+1} \in -u + P(\emptyset)^\star$ and thus $\sum_{t\in F} z_t \in P(\emptyset)^\star$.

To verify hypothesis (4), let $k \in \{1,2,\ldots,\min\{m,n+1\}\}$ and assume that $F_0, F_1, \ldots, F_k \in \mathcal{P}_f(\{0,1,\ldots,n+1\})$ and for each $j \in \{0,1,\ldots,k-1\}$, $\max F_j < \min F_{j+1}$. We can assume that $n+1 \in F_k$. For $l \in \{0,1,\ldots,k-1\}$ let $u_l = \sum_{t\in F_l} z_t$. Then $k-1 \leq \min\{m-1,n\}$ and $(u_0,u_1,\ldots,u_{k-1}) \in W_{k-1,m}$. If $F_k = \{n+1\}$, then $\sum_{t\in F_k} z_t = z_{n+1} \in P(u_0,u_1,\ldots,u_{k-1})^\star$. So assume that $\{n+1\} \subsetneq F_k$ and let $F'_k = F_k \setminus \{n+1\}$. Let $r = \max F_{k-1}$. Then $r < \min F'_k$ and so $r \leq n-1$, $k-1 \leq \min\{m-1,r\}$, and $(y_0,y_1,\ldots,y_{k-1}) \in W_{k-1,r}$. Let $z = \sum_{t\in F'_k} z_t$. Then $z \in E_{r+1}$ and so

$$z_{n+1} \in -z + P(u_0,u_1,\ldots,u_{k-1})^\star.$$

Hence we have

$$\sum_{t\in F_k} z_t \in P(\sum_{t\in F_0} z_t, \sum_{t\in F_1} z_t, \ldots, \sum_{t\in F_{k-1}} z_t)^\star.$$

□

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