# AN IMPLICIT ZECKENDORF REPRESENTATION

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**Abstract**

We obtain here the Zeckendorf representation of a sum of Fibonacci numbers indexed by the Hofstadter G-sequence. The relative complexity of the situation means that this representation is given in an implicit form.

## 1. Introduction

Let $F_n$ denote the $n$th Fibonacci number. This may be defined by way of the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, where $F_0 = 0$ and $F_1 = 1$ [1, 4, 8]. It is well-known that any positive integer can be expressed as a sum of distinct positive Fibonacci numbers, called a *Fibonacci representation*. We note though that if there are no further restrictions placed on them, such representations are not necessarily unique. For example,

$$29 = F_4 + F_5 + F_8 = F_6 + F_8.$$

Zeckendorf's theorem, however, provides us with conditions under which Fibonacci representations are unique. It states that every $n \in \mathbb{N}$ can be represented in a unique way as the sum of one or more distinct positive Fibonacci numbers, excluding $F_1$, in such a way that the sum does not include any two consecutive Fibonacci numbers [1, 11, 12]. This is termed the *Zeckendorf representation* of $n$; from now on we term such a representation a *Z-rep*. Referring to the example above, we see that the Z-rep of 29 is given by $F_6 + F_8$.

In this paper we consider the Z-rep of a particular sum of Fibonacci numbers. In some cases the Z-reps of sums comprising Fibonacci numbers can be very straightforward to obtain. For example, the Z-rep of the sum on the left of (1) takes one of two simple forms, depending on whether $n$ is odd or even:

$$\sum_{k=1}^{n} F_k = \begin{cases} \sum_{k=1}^{\frac{n+1}{2}} F_{2k}, & \text{if } n \text{ is odd;} \\ \sum_{k=1}^{\frac{n}{2}} F_{2k+1}, & \text{if } n \text{ is even.} \end{cases} \tag{1}$$

In fact, even more straightforward are sums of the form

$$\sum_{k=1}^{n} F_{mk},$$

where $m \in \mathbb{N}$ is such that $m \geq 2$. Each of these sums is its own Z-rep.

The Fibonacci sum studied here is, with respect to obtaining its Z-reps, considerably more complicated than any of those illustrated above. Before defining this particular sum, we provide a result concerning one that is in some sense related to it. In [6] we obtained the Z-rep of

$$S_m = \sum_{k=1}^{m} F_{\lfloor k\phi \rfloor} \tag{2}$$

where $\phi$ denotes the golden ratio, given by

$$\phi = \frac{1+\sqrt{5}}{2}.$$

It was shown that, for $m \geq 1$,

$$S_m = F_{\lfloor m\phi \rfloor + 1} + \sum_{k=1}^{2m-\lfloor m\phi \rfloor - 1} F_{2\lfloor k\phi \rfloor + k - 1}, \tag{3}$$

where $\lfloor x \rfloor$ is the *floor function*, denoting the largest integer not exceeding $x$, and the sum is defined to be equal to 0 for $m = 1$ and $m = 2$. The sequence $(\lfloor j\phi \rfloor)_{j\geq 1}$, which indexes the Fibonacci numbers in (2), is a particular Beatty sequence [2, 3] known as the *lower Wythoff sequence* [10].

In this article we obtain the Z-rep of the following sum:

$$T_m = \sum_{k=1}^{m} F_{\lfloor \frac{k}{\phi} \rfloor}. \tag{4}$$

The sequence $(\lfloor j/\phi \rfloor)_{j\geq 1}$ is known as the *Hofstadter G-sequence* [5, 7, 9], noting that this is not in fact a Beatty sequence since $\frac{1}{\phi} < 1$. Incidentally, it is more conventional to denote this by way of the shifted sequence $(\lfloor (j+1)/\phi \rfloor)_{j\geq 0}$, but we adopt the former definition here for ease of notation.

The structure of the Z-reps for the sums $T_m$ is rather more intricate than was the case for $S_m$, and our approach in this article, therefore, is somewhat different. In a sense which will be made clear in due course, we obtain an implicit representation for (4) rather than an explicit representation of the type given by (3).

**2. Preliminary Definitions and Results**

We provide here a number of definitions and results that will be needed in order to state the Z-reps in Section 3 and then prove their validity in Section 4. We also expand somewhat on some of the introductory material mentioned in Section 1. As a first point, note that use will be made of the equality $\phi^2 = \phi + 1$ and its many rearrangements throughout this article. Next, we state Zeckendorf's theorem rather more formally, relatively straightforward proofs of which are given in [1, 11].

**Theorem 1.** *For any $n \in \mathbb{N}$ there exists an increasing sequence of positive integers of length $l \in \mathbb{N}$, $(c_1, c_2, \ldots, c_l)$ say, such that $c_1 \geq 2$, $c_k \geq c_{k-1}+2$ for $k = 2, 3, \ldots, l$, and*

$$n = \sum_{k=1}^{l} F_{c_k}.$$

We also make use of the fact that the aforementioned lower Wythoff sequence $(\lfloor j\phi \rfloor)_{j\geq 1}$ together with the *upper Wythoff sequence* $\left(\lfloor j\phi^2 \rfloor\right)_{j\geq 1}$ form a pair of complementary sequences [10]. Pairs of such sequences have no terms in common yet contain all the positive integers between them. In addition, we note here that the recursive relation $g(n) = n - g(g(n-1))$, where $g(0) = 0$, generates the Hofstadter G-sequence [5].

Finally, we will have cause to utilize the following lemmas. Proofs of Lemmas 1 and 3 appear in [6]. The notation $\{x\}$ will be used to denote $x - \lfloor x \rfloor$, the *fractional part* of $x$. It is the case that $0 \leq \{x\} < 1$ for any $x \in \mathbb{R}$, but, for each $n \in \mathbb{N}$, the irrationality of $\phi$ implies that $0 < \{n\phi\} < 1$.

**Lemma 1.** *Let $\alpha > 1$ be an irrational number. Then*

$$\left\{\frac{n}{\alpha}\right\} > 1 - \frac{1}{\alpha}$$

*if and only if $n = \lfloor j\alpha \rfloor$ for some $j \in \mathbb{N}$.*

**Lemma 2.** *We have*

$$n\phi - \frac{1}{\phi^2} > \lfloor n\phi \rfloor$$

*if and only if $n = \lfloor j\phi \rfloor$ for some $j \in \mathbb{N}$, and*

$$n\phi - \frac{1}{\phi^2} < \lfloor n\phi \rfloor$$

*if and only if $n = \lfloor j\phi^2 \rfloor$ for some $j \in \mathbb{N}$.*

*Proof.* First,

$$\left\{\frac{n}{\phi}\right\} > 1 - \frac{1}{\phi} \quad \text{if and only if} \quad \frac{n}{\phi} - \left\lfloor\frac{n}{\phi}\right\rfloor > \frac{1}{\phi^2}$$
$$\text{if and only if} \quad n(\phi-1) - \lfloor n(\phi-1)\rfloor > \frac{1}{\phi^2}$$
$$\text{if and only if} \quad n\phi - n - (\lfloor n\phi\rfloor - n) > \frac{1}{\phi^2}$$
$$\text{if and only if} \quad n\phi - \lfloor n\phi\rfloor > \frac{1}{\phi^2}.$$

It thus follows from Lemma 1 that

$$n\phi - \frac{1}{\phi^2} > \lfloor n\phi\rfloor$$

if and only if $n = \lfloor j\phi\rfloor$ for some $j \in \mathbb{N}$.

Next, note that the equality

$$\left\{\frac{n}{\phi^2}\right\} = 1 - \frac{1}{\phi^2}$$

can be rearranged to give

$$n = \phi^2\left(1 + \left\lfloor\frac{n}{\phi^2}\right\rfloor\right) - 1,$$

which contradicts the fact that $n$ is rational. By the complementarity property of the lower and upper Whythoff sequences, therefore, it follows that

$$n\phi - \frac{1}{\phi^2} < \lfloor n\phi\rfloor$$

if and only if $n = \lfloor j\phi^2\rfloor$ for some $j \in \mathbb{N}$ □

**Lemma 3.** *It is the case that $n = \lfloor j\phi\rfloor$ for some $j \in \mathbb{N}$ if and only if*

$$\lfloor (n+1)\phi\rfloor = \lfloor n\phi\rfloor + 2,$$

*and $n = \lfloor j\phi^2\rfloor$ if and only if*

$$\lfloor (n+1)\phi\rfloor = \lfloor n\phi\rfloor + 1.$$

## 3. The Zeckendorf Representation

In this section we describe the structure of the Z-reps of the terms in the sequence $(T_m)_{m\geq 8}$, and in the next we provide a proof of this result. As mentioned in Section

1, our representations will, because of their relatively complicated structure, each be expressed in an implicit form rather than in an explicit one. By this we mean that, instead of stating the Z-reps directly, we split them into blocks in such a way that, given any $m \in \mathbb{N}$, we may, via two very straightforward calculations, pinpoint both the block number and the row number in which the Z-rep of $T_m$ is situated, and hence obtain this Z-rep.

**Definition 1.** When $n = \lfloor j\phi \rfloor$ for some $j \in \mathbb{N}$, we use $\mathcal{B}_n$ to denote the set of rows appearing in Table 1, and when $n = \lfloor j\phi^2 \rfloor$ for some $j \in \mathbb{N}$, $\mathcal{B}_n$ denotes the set of rows appearing in Table 2. In other words, for the former case, $\mathcal{B}_n$ is a block comprising the Z-reps for terms $T_{3+2n+3\lfloor n\phi \rfloor}$ to $T_{10+2n+3\lfloor n\phi \rfloor}$ inclusive, while for the latter case, $\mathcal{B}_n$ is a block comprising the Z-reps for terms $T_{3+2n+3\lfloor n\phi \rfloor}$ to $T_{7+2n+3\lfloor n\phi \rfloor}$ inclusive.

On considering Definition 1, we see that when $n$ is of the form $\lfloor j\phi \rfloor$ then $\mathcal{B}_n$ comprises 8 rows while if $n$ is of the form then $\lfloor j\phi^2 \rfloor$ then $\mathcal{B}_n$ comprises 5 rows; these two types of blocks are thus termed 8-blocks and 5-blocks, respectively. The values of $n$ corresponding to the 8-blocks are precisely those in the lower Wythoff sequence $(\lfloor j\phi \rfloor)_{j\geq 1}$, while those corresponding to the 5-blocks are precisely those in the upper Wythoff sequence $\left(\lfloor j\phi^2 \rfloor\right)_{j\geq 1}$. It is thus the case that $\mathcal{B}_n$ is an 8-block when $n = 1, 3, 4, 6, 8, 9, 11, \ldots$, while $\mathcal{B}_n$ is a 5-block when $n = 2, 5, 7, 10, \ldots$. Since the lower and upper Wythoff sequences are a pair of complementary sequences, $\mathcal{B}_n$ is well-defined for each $n \in \mathbb{N}$. Note also that the set of additional summands is defined to be empty when $n = 1$. Once the block structure is known, it is, given any particular $m \geq 8$, a straightforward matter to extract the representation for $T_m$, as is demonstrated in Section 5.

| Term number | First summand | Extra summand | Additional summands |
|---|---|---|---|
| $3+2n+3\lfloor n\phi \rfloor$ | $F_{4+n+2\lfloor n\phi \rfloor}$ | | $\sum_{k=2}^{n} F_{2\lfloor k\phi \rfloor+k-1}$ |
| $4+2n+3\lfloor n\phi \rfloor$ | $F_{4+n+2\lfloor n\phi \rfloor}$ | $F_{2+n+2\lfloor n\phi \rfloor}$ | $\sum_{k=2}^{n} F_{2\lfloor k\phi \rfloor+k-1}$ |
| $5+2n+3\lfloor n\phi \rfloor$ | $F_{5+n+2\lfloor n\phi \rfloor}$ | $F_{2+n+2\lfloor n\phi \rfloor}$ | $\sum_{k=2}^{n} F_{2\lfloor k\phi \rfloor+k-1}$ |
| $6+2n+3\lfloor n\phi \rfloor$ | $F_{6+n+2\lfloor n\phi \rfloor}$ | | $\sum_{k=2}^{n} F_{2\lfloor k\phi \rfloor+k-1}$ |
| $7+2n+3\lfloor n\phi \rfloor$ | $F_{6+n+2\lfloor n\phi \rfloor}$ | $F_{4+n+2\lfloor n\phi \rfloor}$ | $\sum_{k=2}^{n} F_{2\lfloor k\phi \rfloor+k-1}$ |
| $8+2n+3\lfloor n\phi \rfloor$ | $F_{7+n+2\lfloor n\phi \rfloor}$ | $F_{4+n+2\lfloor n\phi \rfloor}$ | $\sum_{k=2}^{n} F_{2\lfloor k\phi \rfloor+k-1}$ |
| $9+2n+3\lfloor n\phi \rfloor$ | $F_{8+n+2\lfloor n\phi \rfloor}$ | | $\sum_{k=2}^{n} F_{2\lfloor k\phi \rfloor+k-1}$ |
| $10+2n+3\lfloor n\phi \rfloor$ | $F_{8+n+2\lfloor n\phi \rfloor}$ | $F_{6+n+2\lfloor n\phi \rfloor}$ | $\sum_{k=2}^{n} F_{2\lfloor k\phi \rfloor+k-1}$ |

Table 1: The $n$th block, $\mathcal{B}_n$, of Z-reps of the terms in $(T_m)_{m\geq 8}$ when $n = \lfloor j\phi \rfloor$ for some $j \in \mathbb{N}$.

For example, $\mathcal{B}_1$ is an 8-block, the term numbers of which range from 8 to 15

| Term number | First summand | Extra summand | Additional Summands |
|---|---|---|---|
| $3+2n+3\lfloor n\phi\rfloor$ | $F_{4+n+2\lfloor n\phi\rfloor}$ | | $\sum_{k=2}^{n} F_{2\lfloor k\phi\rfloor+k-1}$ |
| $4+2n+3\lfloor n\phi\rfloor$ | $F_{4+n+2\lfloor n\phi\rfloor}$ | $F_{2+n+2\lfloor n\phi\rfloor}$ | $\sum_{k=2}^{n} F_{2\lfloor k\phi\rfloor+k-1}$ |
| $5+2n+3\lfloor n\phi\rfloor$ | $F_{5+n+2\lfloor n\phi\rfloor}$ | $F_{2+n+2\lfloor n\phi\rfloor}$ | $\sum_{k=2}^{n} F_{2\lfloor k\phi\rfloor+k-1}$ |
| $6+2n+3\lfloor n\phi\rfloor$ | $F_{6+n+2\lfloor n\phi\rfloor}$ | | $\sum_{k=2}^{n} F_{2\lfloor k\phi\rfloor+k-1}$ |
| $7+2n+3\lfloor n\phi\rfloor$ | $F_{6+n+2\lfloor n\phi\rfloor}$ | $F_{4+n+2\lfloor n\phi\rfloor}$ | $\sum_{k=2}^{n} F_{2\lfloor k\phi\rfloor+k-1}$ |

Table 2: The $n$th block, $\mathcal{B}_n$, of Z-reps of the terms in $(T_m)_{m\geq 8}$ when $n = \lfloor j\phi^2\rfloor$ for some $j \in \mathbb{N}$.

inclusive. The first row of $\mathcal{B}_1$, therefore, gives the Z-rep for

$$\sum_{k=1}^{8} F_{\lfloor \frac{k}{\phi}\rfloor}.$$

Noting that there is no extra summand in this case, and that the additional sum is empty when $n = 1$, we see that the Z-rep is $F_{4+1+2\lfloor\phi\rfloor} = F_7$. The second row of $\mathcal{B}_1$ gives the representation for

$$\sum_{k=1}^{9} F_{\lfloor \frac{k}{\phi}\rfloor},$$

which is $F_{4+1+2\lfloor\phi\rfloor} + F_{2+1+2\lfloor\phi\rfloor} = F_7 + F_5$, and so on.

Next, $\mathcal{B}_2$ is a 5-block, the term numbers of which range from 16 to 20 inclusive. Notice that for $n \geq 2$, the additional sum is non-empty. The first row of $\mathcal{B}_2$ provides the Z-rep for

$$\sum_{k=1}^{16} F_{\lfloor \frac{k}{\phi}\rfloor},$$

which can be seen to be $F_{4+2+2\lfloor 2\phi\rfloor} + F_{-1+2+2\lfloor 2\phi\rfloor} = F_{12} + F_7$, and similarly the second row gives

$$\begin{aligned}\sum_{k=1}^{17} F_{\lfloor \frac{k}{\phi}\rfloor} &= F_{4+2+2\lfloor 2\phi\rfloor} + F_{2+2+2\lfloor 2\phi\rfloor} + F_{-1+2+2\lfloor 2\phi\rfloor}\\ &= F_{12} + F_{10} + F_7.\end{aligned}$$

As a final example, $\mathcal{B}_3$ is another 8-block. The term numbers range from 21 to 28 inclusive. This time the additional sum contributes two summands to each row. We have

$$\begin{aligned}\sum_{k=1}^{21} F_{\lfloor \frac{k}{\phi}\rfloor} &= F_{4+3+2\lfloor 3\phi\rfloor} + F_{-1+3+2\lfloor 3\phi\rfloor} + F_{-1+2+2\lfloor 2\phi\rfloor}\\ &= F_{15} + F_{10} + F_7,\end{aligned}$$

$$\sum_{k=1}^{22} F_{\lfloor \frac{k}{\phi} \rfloor} = F_{4+3+2\lfloor 3\phi \rfloor} + F_{2+3+2\lfloor 3\phi \rfloor} + F_{-1+3+2\lfloor 3\phi \rfloor} + F_{-1+2+2\lfloor 2\phi \rfloor}$$
$$= F_{15} + F_{13} + F_{10} + F_{7},$$

and so on.

There are some observations to be made here. First, the formulas appearing in Tables 1 and 2 only provide the Z-rep of $T_m$ for $m \geq 8$. However, it is a trivial matter to obtain the representations for $m = 1, 2, \ldots, 7$. Next, we have allowed the structure of the blocks to be dictated to by the structure of the additional summands. Indeed, two rows appear in the same block if and only if one possesses precisely the same set of additional summands as the other; we deemed this approach to be the most judicious one. Finally, it may be seen that the structure of each 5-block is identical to that of the first five rows of each 8-block.

## 4. A Proof

The following theorem concerns the structure of the Z-reps of the terms in $(T_m)_{m\geq 8}$, as was outlined in Section 3. In Section 5 it is shown how these Z-reps may be obtained in a straightforward manner

**Theorem 2.** *For each $T_m$, where $m \geq 8$, there is a unique ordered pair $(k, n)$ such that the Zeckendorf representation of $T_m$ is given by the $k$th row of $\mathcal{B}_n$.*

*Proof.* We proceed by induction on $n$, the block number. Since the set of additional summands is defined to be empty for the special case $n = 1$, we will use $n = 2$ as the base case. It is straightforward to check that the 8-block $\mathcal{B}_1$ gives, in order, the correct Z-reps for $T_8$ to $T_{15}$, and the 5-block $\mathcal{B}_2$ gives, in order, the correct Z-reps for $T_{16}$ to $T_{20}$. So let us now assume that when, for some $n \geq 3$, $n-1$ is of the form $\lfloor j\phi \rfloor$, $\mathcal{B}_{n-1}$ gives, in order, the Z-reps of $T_{3+2(n-1)+3\lfloor (n-1)\phi \rfloor}$ to $T_{10+2(n-1)+3\lfloor (n-1)\phi \rfloor}$ inclusive, while if, for some $n \geq 3$, $n-1$ is of the form $\lfloor j\phi^2 \rfloor$, $\mathcal{B}_{n-1}$ gives, in order, the Z-reps of $T_{3+2(n-1)+3\lfloor (n-1)\phi \rfloor}$ to $T_{7+2(n-1)+3\lfloor (n-1)\phi \rfloor}$. There are just three possibilities that can arise:

(i) $\mathcal{B}_{n-1}$ is a 5-block and $\mathcal{B}_n$ is an 8-block;

(ii) $\mathcal{B}_{n-1}$ is an 8-block and $\mathcal{B}_n$ is also an 8-block;

(iii) $\mathcal{B}_{n-1}$ is an 8-block and $\mathcal{B}_n$ is a 5-block.

Let us consider the situation given by (i) above. From the definition of $\mathcal{B}_{n-1}$ and the complementarity property of the lower and upper Wythoff sequences, $\mathcal{B}_{n-1}$ is a 5-block if and only if $n-1=\lfloor j\phi^2\rfloor$ for some $j\in\mathbb{N}$. Thus, on using Lemma 3, we have

$$\begin{aligned}7+2(n-1)+3\lfloor(n-1)\phi\rfloor+1&=7+2n-2+3\left(\lfloor n\phi\rfloor-1\right)+1\\&=3+2n+3\lfloor n\phi\rfloor,\end{aligned}$$

which shows that the term number corresponding to the final row of $\mathcal{B}_{n-1}$ is indeed 1 less than the term number corresponding to the first row of $\mathcal{B}_n$.

Next,

$$F_{\left\lfloor\frac{3+2n+3\lfloor n\phi\rfloor}{\phi}\right\rfloor}$$

is the numerical expression that is added to $T_{7+2(n-1)+3\lfloor(n-1)\phi\rfloor}$ in order to obtain the term $T_{3+2n+3\lfloor n\phi\rfloor}$. By the inductive hypothesis, $F_{6+(n-1)+2\lfloor(n-1)\phi\rfloor}$ and $F_{4+(n-1)+2\lfloor(n-1)\phi\rfloor}$ are the first and extra summands in row 5 of $\mathcal{B}_{n-1}$, respectively, while the additional summands for this row are given by $\sum_{k=2}^{n-1}F_{2\lfloor k\phi\rfloor+k-1}$.

Thus, bearing in mind the structure of row 1 of $\mathcal{B}_n$, we would like first to be able to show that the following is true:

$$\begin{aligned}&F_{\left\lfloor\frac{3+2n+3\lfloor n\phi\rfloor}{\phi}\right\rfloor}\\&\quad=F_{4+n+2\lfloor n\phi\rfloor}-F_{6+(n-1)+2\lfloor(n-1)\phi\rfloor}-F_{4+(n-1)+2\lfloor(n-1)\phi\rfloor}+F_{-1+n+2\lfloor n\phi\rfloor}.\end{aligned}\tag{5}$$

The definition of $\mathcal{B}_n$, in conjunction with Lemma 3, implies that

$$\lfloor n\phi\rfloor=\lfloor(n-1)\phi\rfloor+1$$

when $\mathcal{B}_{n-1}$ is a 5-block. The right-hand side of (5) therefore simplifies to

$$\begin{aligned}&F_{4+n+2\lfloor n\phi\rfloor}-F_{3+n+2\lfloor n\phi\rfloor}-F_{1+n+2\lfloor n\phi\rfloor}+F_{-1+n+2\lfloor n\phi\rfloor}\\&\qquad=F_{2+n+2\lfloor n\phi\rfloor}-F_{1+n+2\lfloor n\phi\rfloor}+F_{-1+n+2\lfloor n\phi\rfloor}\\&\qquad=F_{n+2\lfloor n\phi\rfloor}+F_{-1+n+2\lfloor n\phi\rfloor}\\&\qquad=F_{1+n+2\lfloor n\phi\rfloor},\end{aligned}$$

so we are now left with the task of proving that

$$\left\lfloor\frac{3+2n+3\lfloor n\phi\rfloor}{\phi}\right\rfloor=1+n+2\lfloor n\phi\rfloor.\tag{6}$$

This will be carried out by showing that

$$1+n+2\lfloor n\phi\rfloor<\frac{3+2n+3\lfloor n\phi\rfloor}{\phi}<2+n+2\lfloor n\phi\rfloor.\tag{7}$$

To this end, we have

$$\begin{aligned}
\frac{3+2n+3\lfloor n\phi\rfloor}{\phi} > 1+n+2\lfloor n\phi\rfloor \quad &\text{if and only if} \quad 3-\phi+n(2-\phi) > \lfloor n\phi\rfloor(2\phi-3)\\
&\text{if and only if} \quad \frac{3-\phi}{2-\phi}+n > \lfloor n\phi\rfloor\left(\frac{2\phi-3}{2-\phi}\right)\\
&\text{if and only if} \quad \frac{3-\phi}{2-\phi}+n > \frac{\lfloor n\phi\rfloor}{\phi}\\
&\text{if and only if} \quad 3\phi+1+n\phi > \lfloor n\phi\rfloor \qquad (8)
\end{aligned}$$

and

$$\begin{aligned}
\frac{3+2n+3\lfloor n\phi\rfloor}{\phi} < 2+n+2\lfloor n\phi\rfloor \quad &\text{if and only if} \quad 3-2\phi+n(2-\phi) < \lfloor n\phi\rfloor(2\phi-3)\\
&\text{if and only if} \quad \frac{3-2\phi}{2-\phi}+n < \lfloor n\phi\rfloor\left(\frac{2\phi-3}{2-\phi}\right)\\
&\text{if and only if} \quad \frac{3-2\phi}{2-\phi}+n < \frac{\lfloor n\phi\rfloor}{\phi}\\
&\text{if and only if} \quad -1+n\phi < \lfloor n\phi\rfloor. \qquad (9)
\end{aligned}$$

Note that since the inequalities given by (8) and (9) are both true, we have shown that (7), and hence (6), is true.

Next, consider the second row of Table 1. By comparing this to the first row, it may be seen that our aim in this case is to demonstrate the truth of

$$\left\lfloor\frac{4+2n+3\lfloor n\phi\rfloor}{\phi}\right\rfloor = 2+n+2\lfloor n\phi\rfloor. \qquad (10)$$

Adopting an identical procedure to that used in the previous paragraph, we obtain

$$\frac{4+2n+3\lfloor n\phi\rfloor}{\phi} > 2+n+2\lfloor n\phi\rfloor \quad \text{if and only if} \quad 2\phi+n\phi > \lfloor n\phi\rfloor$$

and

$$\frac{4+2n+3\lfloor n\phi\rfloor}{\phi} < 3+n+2\lfloor n\phi\rfloor \quad \text{if and only if} \quad -\phi-2+n\phi < \lfloor n\phi\rfloor,$$

from which it follows that (10) is true. Rows 3, 4, and 5 of Table 1 may be dealt with in a similar manner.

A little more care is needed when considering row 6. Our task in this case is to show that

$$\left\lfloor\frac{8+2n+3\lfloor n\phi\rfloor}{\phi}\right\rfloor = 5+n+2\lfloor n\phi\rfloor. \qquad (11)$$

This time we have

$$\frac{8+2n+3\lfloor n\phi\rfloor}{\phi} > 5+n+2\lfloor n\phi\rfloor \quad \text{if and only if} \quad -\frac{1}{\phi^2}+n\phi > \lfloor n\phi\rfloor \qquad (12)$$

and

$$\frac{8+2n+3\lfloor n\phi\rfloor}{\phi} < 6+n+2\lfloor n\phi\rfloor \quad \text{if and only if} \quad -2\phi-4+n\phi<\lfloor n\phi\rfloor. \qquad (13)$$

Whilst it is clear that the inequality on the right-hand side of (13) is always true, the same cannot be said for the right-hand side of (12). However, from Lemma 2 we know that the right-hand side of (12) is true if and only if $n=\lfloor j\phi\rfloor$ for some $j\in\mathbb{N}$. Thus, since $n$ will always be of this form in the situation given by (i), it follows that (11) is true in this case. Rows 7 and 8 of Table 1 may subsequently be dealt with in a manner similar to that of the earlier rows.

Let us now consider the situation given by (ii). From the definition of $\mathcal{B}_{n-1}$ and the complementarity property of the lower and upper Wythoff sequences once more, $\mathcal{B}_{n-1}$ is an 8-block if and only if $n-1=\lfloor j\phi\rfloor$ for some $j\in\mathbb{N}$. Thus, on using Lemma 3, we have

$$\begin{aligned} 10+2(n-1)+3\lfloor(n-1)\phi\rfloor+1 &= 10+2n-2+3\left(\lfloor n\phi\rfloor-2\right)+1 \\ &= 3+2n+3\lfloor n\phi\rfloor. \end{aligned}$$

This shows that the term number corresponding to the final row of $\mathcal{B}_{n-1}$ is 1 less than the term number corresponding to the first row of $\mathcal{B}_n$.

In this case

$$F_{\left\lfloor\frac{3+2n+3\lfloor n\phi\rfloor}{\phi}\right\rfloor}$$

is added to $T_{10+2(n-1)+3\lfloor(n-1)\phi\rfloor}$ to obtain $T_{3+2n+3\lfloor n\phi\rfloor}$. By the inductive hypothesis, the first and extra summands in row 8 of $\mathcal{B}_{n-1}$ are given by $F_{8+(n-1)+2\lfloor(n-1)\phi\rfloor}$ and $F_{6+(n-1)n+2\lfloor(n-1)\phi\rfloor}$, respectively, while the additional summands for this row are given by $\sum_{k=2}^{n-1}F_{2\lfloor k\phi\rfloor+k-1}$.

We would like to be able to show, therefore, that

$$\begin{aligned} &F_{\left\lfloor\frac{3+2n+3\lfloor n\phi\rfloor}{\phi}\right\rfloor} \\ &\quad = F_{4+n+2\lfloor n\phi\rfloor}-F_{8+(n-1)+2\lfloor(n-1)\phi\rfloor}-F_{6+(n-1)+2\lfloor(n-1)\phi\rfloor}+F_{-1+n+2\lfloor n\phi\rfloor} \qquad (14)\end{aligned}$$

is true. As mentioned above, it follows, from Lemma 3 and the definition of $\mathcal{B}_n$, that $\lfloor n\phi\rfloor=\lfloor(n-1)\phi\rfloor+2$ in the present case. Thus, on simplifying the right-hand side of (14), we obtain

$$F_{4+n+2\lfloor n\phi\rfloor}-F_{3+n+2\lfloor n\phi\rfloor}-F_{1+n+2\lfloor n\phi\rfloor}+F_{-1+n+2\lfloor n\phi\rfloor}=F_{1+n+2\lfloor n\phi\rfloor}.$$

This leads to (6) once more, which has already been shown to be true. It follows from this that the series of calculations concerning the structure of rows 2 to 8 in this case is identical to the series of calculations carried out for situation (i). Note that, as in situation (i), it is the case that $n=\lfloor j\phi\rfloor$ for some $j\in\mathbb{N}$ here. Thus, by way of Lemma 2 once more, in the calculation associated with row 6 the right-hand side of (12) is true. This implies that (11) is true in this case.

For situation (iii), the calculations associated with rows 1 to 5 of $\mathcal{B}_n$ are identical to those performed for situation (ii). This time, however, $n = \lfloor j\phi^2 \rfloor$ for some $j \in \mathbb{N}$. Thus, from Lemma 2, it follows that the right-hand side of (12) is actually false. From this we see that (11) is false in this case.

Finally, on considering the rows in Tables 1 and 2, it is clear that the given representations are indeed all Z-reps. This completes the proof of the theorem. □

## 5. Obtaining the Representations

In order to utilize Tables 1 and 2 to obtain the Z-rep of $T_m$ for some $m \geq 8$, we need simply to identify both the block $\mathcal{B}_n$ in which the term number $m$ lies and the row that this term number occupies within $\mathcal{B}_n$.

To this end, it is clear that for any given $m \geq 8$ there exists some $n \in \mathbb{N}$, not necessarily unique, such that

$$3 + 2n + 3\lfloor n\phi \rfloor \leq m \leq 10 + 2n + 3\lfloor n\phi \rfloor, \tag{15}$$

noting that the inequality on the right-hand side allows for $\mathcal{B}_n$ to be either an 8-block or a 5-block. From (15) we obtain

$$3 + 2n + (3n\phi - 3) < m < 10 + 2n + 3n\phi,$$

which can be rearranged to give

$$\frac{m-10}{2+3\phi} < n < \frac{m}{2+3\phi},$$

and hence

$$\left\lceil \frac{m-10}{2+3\phi} \right\rceil \leq n \leq \left\lfloor \frac{m}{2+3\phi} \right\rfloor,$$

where $\lceil x \rceil$ is the *ceiling function*, denoting the smallest integer at least as large as $x$.

Then, since

$$0 \leq \left\lfloor \frac{m}{2+3\phi} \right\rfloor - \left\lceil \frac{m-10}{2+3\phi} \right\rceil \leq 1,$$

it may be seen that either $n = a(m)$ or $n = a(m) - 1$, where

$$a(m) = \left\lfloor \frac{m}{2+3\phi} \right\rfloor.$$

Recalling once more that $3+2n+3\lfloor n\phi\rfloor$ gives the term number of the first row of $\mathcal{B}_n$, it follows that if

$$m \geq 3+2a(m)+3\lfloor a(m)\phi\rfloor$$

then $a(m)$ gives the block number, otherwise the block number will be given by $a(m)-1$.

For the case in which $n=a(m)$, the term number $m$ corresponds to row

$$m-(3+2a(m)+3\lfloor a(m)\phi\rfloor)+1 = m-2-2a(m)-3\lfloor a(m)\phi\rfloor$$

of $\mathcal{B}_{a(m)}$. Otherwise, $m$ corresponds to row

$$m-(3+2(a(m)-1)+3\lfloor (a(m)-1)\phi\rfloor)+1 = m-2a(m)-3\lfloor (a(m)-1)\phi\rfloor)$$

of $\mathcal{B}_{a(m)-1}$.

To demonstrate this with $m=44$, for example, we have

$$3+2a(m)+3\lfloor a(m)\phi\rfloor = 3+2a(44)+3\lfloor a(44)\phi\rfloor = 42 \leq 44,$$

so that the block number is given by $a(m)=a(44)=6$ and the row number within this block is

$$m-2-2a(m)-3\lfloor a(m)\phi\rfloor = 44-2-2a(44)-3\lfloor a(44)\phi\rfloor = 3.$$

Therefore,

$$T_{44} = F_{29}+F_{26}+F_{23}+F_{20}+F_{15}+F_{10}+F_7.$$

On the other hand, if $m=48$ then

$$3+2a(m)+3\lfloor a(m)\phi\rfloor = 3+2a(48)+3\lfloor a(48)\phi\rfloor = 50 > 44.$$

In this case the block number is given by $a(m)-1=a(48)-1=7-1=6$ and the row number within this block is

$$m-2a(m)-3\lfloor (a(m)-1)\phi\rfloor = 48-2a(48)-3\lfloor (a(48)-1)\phi\rfloor = 7.$$

We then have

$$T_{48} = F_{32}+F_{23}+F_{20}+F_{15}+F_{10}+F_7.$$

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