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FINITE REPRESENTATIONS OF THE SUMMATORY DIVISOR FUNCTION AND RAMANUJAN'S τ-FUNCTION

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Abstract

Both the partition and the divisor function are known to admit finite closed-form representations in terms of a sparse upper Hessenberg matrix. It is shown that this matrix determines other arithmetical functions as well; in particular, finite representations for the summatory divisor function, Ramanujan's τ-function, and several related functions are established.

1. Introduction

1.1. Preliminaries and Notation

An explicit representation of a prescribed arithmetical function sheds light on the nature of this function; however, it may be a difficult task to establish a *finite* representation such as a finite sum or product. As an illustration, consider the problem of finding a finite representation of the partition function which assigns the number of partitions $p(n)$ to each nonnegative integer n. (As usual, a partition of n is an additive decomposition of n into positive integers where the order of the summands does not matter.) This problem has recently been solved by Bruinier and Ono [2] who established the formula

$$p(n) = \frac{1}{24n-1} \cdot \sum_{Q \in \mathcal{Q}_n} P(\alpha_Q)$$

comprising a certain nonholomorphic weak Maass form P, a collection $\mathcal{Q}_n$ of prescribed positive definite quadratic forms, and complex CM points α_Q.

A different approach to this problem may be based on pentagonal numbers in conjunction with a specific upper Hessenberg matrix. The set of generalized pentagonal numbers (without zero) is defined by the union $P_1 \cup P_2$ where

$$P_1 = \{r \in \mathbb{N} : r = \frac{1}{2}\,(3k^2 \pm k) \text{ for an odd integer } k \in \mathbb{N}\},$$

$$P_2 = \{r \in \mathbb{N} : r = \frac{1}{2}(3k^2 \pm k) \text{ for an even integer } k \in \mathbb{N}\},$$

and $\mathbb{N}$ denotes the set of positive integers.

In the sequel the following notation will be used: A_{jk} denotes the entry in the jth row and kth column of a given matrix A, and $\mathrm{tr}(A)$ stands for the trace of A; moreover, for each pentagonal number $M \in P_1 \cup P_2$ its successor M' is the pentagonal number $M' = \min\{r \in P_1 \cup P_2 : r > M\}$. I_M always stands for the unity matrix with M rows and columns. Setting

$$\mathtt{1}_{12}(r) = \begin{cases} +1 & \text{if } r \in P_1 \\ -1 & \text{if } r \in P_2 \\ 0 & \text{otherwise,} \end{cases}$$

the companion matrix associated with the partition function is the infinite matrix H with entries

$$H_{jk} = \begin{cases} \mathtt{1}_{12}(k) & \text{if } j = 1 \\ +1 & \text{if } j \geq 2 \text{ and } k = j-1 \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

and for each positive integer $m \geq 1$ the truncated matrix H_m is the $m \times m$ matrix being obtained from H by deleting the jth row and kth column for all $j, k > m$. The companion matrix and its truncated versions are sparse upper Hessenberg matrices, and it turns out that H completely determines the partition function.

By Theorem 2.3 in Weba [7], one obtains the finite matrix representations

$$p(n) = (H^n)_{11} \quad \text{for all integers } n \geq 1, \tag{2}$$

i.e., $p(n)$ coincides with the entry in the first row and first column of the nth power H^n. Likewise, if M is an arbitrary pentagonal number with successor M' then the nth power H_M^n of the truncated matrix H_M satisfies

$$p(n) = (H_M^n)_{11} \quad \text{for all } 1 \leq n \leq M' - 1. \tag{3}$$

If, in addition, H_M has M pairwise distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_M$ then $p(n)$ also admits the spectral representation

$$p(n) = \sum_{j=1}^{M} \alpha_j \cdot \lambda_j^n \quad \text{for all integers } 1 \leq n \leq M' - 1, \tag{4}$$

where coefficients

$$\alpha_j = \left(\sum_{r=1}^{M} \mathtt{1}_{12}(r) \cdot r \cdot \lambda_j^{-r} \right)^{-1}, \quad 1 \leq j \leq M, \tag{5}$$

have the sum $\sum_{j=1}^{M} \alpha_j = 1$. For a proof, see again Theorem 2.3 in [7]. Hence the partition function is determined by powers or eigenvalues of the Hessenberg matrix H_M. Moreover, this matrix also determines the divisor function

$$\sigma(n) = \sum_{d|n} d, \quad n \in \mathbb{N},$$

via its trace or its eigenvectors. More precisely, let M be an arbitrary pentagonal number with successor M'; then the divisor function can be expressed by

$$\sigma(n) = \operatorname{tr}(H_M^n) \quad \text{for all} \quad 1 \leq n \leq M' - 1. \tag{6}$$

In particular, the divisor function admits the spectral representation

$$\sigma(n) = \sum_{j=1}^{M} \lambda_j^n \quad \text{for all} \quad 1 \leq n \leq M' - 1 \tag{7}$$

where $\lambda_1, \ldots, \lambda_M$ are the (not necessarily distinct) eigenvalues of H_M; see Theorem 3.2 in [7]. Therefore, the partition function and the divisor function are closely related, and a comparison between their spectral representations shows that both arithmetical functions can be viewed as superpositions of sinusoids with different amplitudes and phase angles but with identical frequencies. This is an explanation for the oscillating behaviour of these functions.

1.2. Main Results

The Hessenberg matrix H and its truncated versions $H_M, M \in P_1 \cup P_2$, determine not only $\sigma(n)$ and $p(n)$ but also their respective summatory functions and other arithmetical functions such as Ramanujan's τ-function and divisor functions

$$\sigma_k(n) = \sum_{d|n} d^k, \quad n \in \mathbb{N},$$

for certain odd integers $k \geq 1$. In either case, it will be possible to derive matrix representations as well as equivalent spectral representations in terms of eigenvalues.

2. Finite Representations of Summatory Functions

The summatory function $\sum_{n=1}^{N} \sigma(n)$ associated with the divisor function $\sigma(n)$ satisfies the well-known relation

$$\sum_{n=1}^{N} \sigma(n) = \frac{\pi^2}{12} \cdot N^2 + \mathcal{O}(N \cdot \log N)$$

which yields the simple but imprecise asymptotic approximation

$$\sum_{n=1}^{N} \sigma(n) \approx \frac{\pi^2}{12} \cdot N^2.$$

Theorem 2.1 establishes exact representations of this summatory function.

Theorem 2.1 *Let $M \in P_1 \cup P_2$ be a pentagonal number and N a prescribed integer with $1 \le N \le M' - 1$. Then the following assertions hold:*
(i) The summatory function of the divisor function $\sigma(n)$ satisfies

$$\sum_{n=1}^{N} \sigma(n) = \sum_{n=1}^{N} \operatorname{tr}(H_M^n). \tag{8}$$

(ii) If M has the form $M = (3m^2+m)/2$ for some $m \in \mathbb{N}$, then $+1$ is not an eigenvalue of H_M, and the summatory function admits both the matrix representation

$$\sum_{n=1}^{N} \sigma(n) = \operatorname{tr}\left((H_M - I_M)^{-1} \cdot (H_M^N - I_M) \cdot H_M \right) \tag{9}$$

and the spectral representation

$$\sum_{n=1}^{N} \sigma(n) = -M + \sum_{j=1}^{M} \frac{\lambda_j^{N+1} - 1}{\lambda_j - 1}. \tag{10}$$

(iii) If M has the form $M = (3m^2-m)/2$ for some $m \in \mathbb{N}$, then $+1$ is an eigenvalue of H_M with algebraic multiplicity one. Denoting this eigenvalue by λ_1 the spectral representation

$$\sum_{n=1}^{N} \sigma(n) = N + 1 - M + \sum_{j=2}^{M} \frac{\lambda_j^{N+1} - 1}{\lambda_j - 1} \tag{11}$$

is valid.

Proof. Identity (8) follows from (6). Laplace expansion along the first row yields the characteristic function $\chi_M(\lambda) = \det(H_M - \lambda I_M)$ of H_M due to

$$\chi_M(\lambda) = (\mathtt{1}_{12}(1) - \lambda) \cdot (-\lambda)^{M-1} + \sum_{r=2}^{M-1} \mathtt{1}_{12}(r)\,(-1)^{r-1} \cdot \det(W_r) + \mathtt{1}_{12}(M) \cdot (-1)^{M-1}$$

where the rth submatrix W_r is a block diagonal matrix comprising an upper triangular matrix with diagonal entries $+1$ as well as a lower triangular matrix having diagonal entries $-\lambda$. The relation $\det(W_r) = (-\lambda)^{M-r}$ then implies

$$\chi_M(\lambda) = (-1)^M \lambda^M + (-1)^{M-1} \sum_{r=1}^{M} \mathtt{1}_{12}(r) \lambda^{M-r}, \quad \lambda \in \mathbb{C},$$

hence $+1$ is an eigenvalue of H_M if and only if $\sum_{r=1}^{M} \mathtt{1}_{12}(r) = 1$ holds. As the sequence $1, 1, -1, -1, 1, 1, -1, -1, \ldots$ generates the sequence $1, 2, 1, 0, 1, 2, 1, 0, \ldots$ of partial sums one finds

$$\sum_{r=1}^{M} \mathtt{1}_{12}(r) = \begin{cases} 2 \text{ or } 0 & \text{if } M = (3m^2+m)/2 \\ 1 & \text{if } M = (3m^2-m)/2. \end{cases}$$

Suppose $M = (3m^2 + m)/2$; then $\sum_{r=1}^{M} \mathtt{1}_{12}(r)$ is equal to 2 or 0, $+1$ is not an eigenvalue of H_M, and matrix $H_M - I_M$ is nonsingular. Setting

$$S_N = \sum_{n=1}^{N} H_M^n$$

one obtains

$$H_M^{N+1} - H_M = (H_M - I_M) \cdot S_N$$

as well as

$$S_N = (H_M - I_M)^{-1} \cdot (H_M^N - I_M) \cdot H_M,$$

which implies (9) in view of (8). The equation (10) is a consequence of (7) and

$$\sum_{n=1}^{N} \sigma(n) = \sum_{j=1}^{M} \sum_{n=1}^{N} \lambda_j^n = \sum_{j=1}^{M} \left(\frac{\lambda_j^{N+1} - 1}{\lambda_j - 1} - 1 \right).$$

On the other hand, suppose $M = (3m^2 - m)/2$; then $\sum_{r=1}^{M} \mathtt{1}_{12}(r)$ is equal to 1, and $+1$ is an eigenvalue of H_M. Denoting this eigenvalue by λ_1 differentiation of the characteristic function χ_M shows that the condition $\sum_{r=1}^{M} r\,\mathtt{1}_{12}(r) = 0$ is necessary for $\lambda_1 = 1$ to have an algebraic multiplicity of at least two. However, this condition is violated because of

$$\sum_{r=1}^{M} r\,\mathtt{1}_{12}(r) = \sum_{k=1}^{m} \frac{1}{2}(3k^2 - k)(-1)^{k-1} + \sum_{k=1}^{m-1} \frac{1}{2}(3k^2 + k)(-1)^{k-1} = m(-1)^{m-1}$$

which implies that the eigenvalues $\lambda_2, \ldots, \lambda_M$ are different from 1. This yields (11) according to

$$\sum_{n=1}^{N} \sigma(n) = \sum_{j=1}^{M} \sum_{n=1}^{N} \lambda_j^n = N + \sum_{j=2}^{M} \left(\frac{\lambda_j^{N+1} - 1}{\lambda_j - 1} - 1 \right).$$

$\square$

Analogous representations for the summatory function associated with the partition function are formulated in Theorem 2.2.

Theorem 2.2 *Let $M \in P_1 \cup P_2$ be a pentagonal number and N a given integer with $1 \le N \le M' - 1$. Then the following identities hold:*
(i) The summatory function of the partition function $p(n)$ satisfies

$$\sum_{n=1}^{N} p(n) = \sum_{n=1}^{N} (H_M^n)_{11}. \tag{12}$$

In case $N \le M$ one also has

$$\sum_{n=1}^{N} p(n) = \sum_{n=1}^{N} (H_M^N)_{n1}, \tag{13}$$

i.e., the summatory function can also be expressed as the sum of the first N entries of the first column of H_M^N.
(ii) If M has the form $M = (3m^2 + m)/2$ for some $m \in \mathbb{N}$ and the eigenvalues of H_M are pairwise distinct (and necessarily different from $+1$), then the summatory function admits the spectral representation

$$\sum_{n=1}^{N} p(n) = -1 + \sum_{j=1}^{M} \alpha_j \, \frac{\lambda_j^{N+1} - 1}{\lambda_j - 1}. \tag{14}$$

(iii) Assume that M has the form $M = (3m^2 - m)/2$ for some $m \in \mathbb{N}$. On the condition that the eigenvalues of H_M are pairwise distinct the spectral representation

$$\sum_{n=1}^{N} p(n) = \frac{N+1}{m}(-1)^{m-1} - 1 + \sum_{j=2}^{M} \alpha_j \, \frac{\lambda_j^{N+1} - 1}{\lambda_j - 1} \tag{15}$$

is valid where H_M has the eigenvalue $\lambda_1 = 1$ and the eigenvalues $\lambda_2, \ldots, \lambda_M$ which are different from $+1$.

Proof. Identity (3) implies (12), and (13) follows from (12) and the fact that the rows of a matrix are shifted downwards if the matrix is multiplied by H_M. Formulae (14), (15) can be derived by analogy with (10), (11); recall that $\alpha_1 + \alpha_2 + \ldots + \alpha_M = 1$. $\square$

Let k, m be nonnegative integers. Results akin to Theorem 2.1 and Theorem 2.2 can be established for summatory functions such as

$$\sum_{n=1}^{N} \sigma^k(n), \quad \sum_{n=1}^{N} p^m(n), \quad \text{or} \quad \sum_{n=1}^{N} \sigma^k(n) \cdot p^m(n)$$

involving powers of $\sigma(n)$ and $p(n)$.

3. Finite Representations of Ramanujan's τ-Function

Ramanujan's τ-function $\tau(n), n \in \mathbb{N}$, may be defined via the Fourier expansion of the discriminant modular form $\Delta(z)$ due to

$$\Delta(z) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n)\, e^{2\pi i n z},$$

where the complex arguments z are assumed to have positive imaginary parts, (see, e.g., [1], page 20). The τ-function obeys a recurrence relation and satisfies numerous identities expressing $\tau(n)$ in terms of other arithmetical functions such as divisor functions of higher order; a typical formula reads

$$\tau(n) = \frac{65}{756}\,\sigma_{11}(n) + \frac{691}{756}\,\sigma_5(n) - \frac{691}{3}\sum_{k=1}^{n-1} \sigma_5(k)\cdot\sigma_5(n-k)$$

([1], page 140). More identities can be found, e.g., in Lehmer [4] or Ewell [3].

Identities of this type cannot be regarded as closed-form representations because the τ-function is only formulated in terms of another arithmetical function, and functions such as divisor functions of higher order and their convolutions would require knowledge of the respective divisors of n, or must itself be computed by means of recursive relations. (Actually, the literature seems to be silent on finite explicit formulae for $\tau(n)$.) However, if $\tau(n)$ could be expressed solely in terms of the 'ordinary' divisor function $\sigma(n)$ - alternatively, in terms of the partition function $p(n)$ - then closed-form representations would be available upon inserting the corresponding matrix or spectral representations of $\sigma(n)$ or $p(n)$. In this vein, a formula due to Niebur [5] is applicable.

Theorem 3.1 *Let M be an arbitrary pentagonal number with successor M'. Then Ramanujan's τ-function obeys the following matrix representations for all $1 \le n \le M' - 1$:*

$$\tau(n) = n^4 \cdot \mathrm{tr}(H_M^n) - 24 \sum_{k=1}^{n-1} (35k^4 - 60n^2k^2 + 13n^4) \cdot \mathrm{tr}(H_M^k) \cdot \mathrm{tr}(H_M^{n-k}). \quad (16)$$

In terms of the quantities

$$\Lambda_{j,l}^{(n)}(r) = \sum_{k=1}^{n-1} k^r\, \lambda_j^k \lambda_l^{n-k}, \quad r = 0, 1, 2, 3, 4, \quad (17)$$

depending upon n and the (not necessarily distinct) eigenvalues $\lambda_1, \ldots, \lambda_M$ of H_M the τ-function admits the spectral representations

$$\tau(n) = n^4 \cdot \sum_{j=1}^{M} \lambda_j^n - 24 \sum_{j,l=1}^{M} \left(35\Lambda_{j,l}^{(n)}(4) - 60n^2\Lambda_{j,l}^{(n)}(2) + 13n^4\Lambda_{j,l}^{(n)}(0)\right) \quad (18)$$

and

$$\tau(n) = (13n^4 - 40n^3 + 28n) \cdot \sum_{j=1}^{M} \lambda_j^n - 24 \cdot \sum_{\substack{j,l=1 \\ j<l}}^{M} \Big(70\, \Lambda_{j,l}^{(n)}(4) - 140n\, \Lambda_{j,l}^{(n)}(3) + 90n^2\, \Lambda_{j,l}^{(n)}(2) - 20n^3\, \Lambda_{j,l}^{(n)}(1) + n^4\, \Lambda_{j,l}^{(n)}(0)\Big). \quad (19)$$

Proof. (i) $\tau(n)$ can be expressed solely in terms of the divisor function $\sigma(n)$ according to

$$\tau(n) = n^4 \cdot \sigma(n) - 24 \left(35 \sum_{k=1}^{n-1} k^4 \sigma(k)\sigma(n-k) - 52n \sum_{k=1}^{n-1} k^3 \sigma(k)\sigma(n-k) + 18n^2 \sum_{k=1}^{n-1} k^2 \sigma(k)\sigma(n-k)\right)$$

(see [5]). Upon replacing the summation index k by $n-k$ the simplification

$$\sum_{k=1}^{n-1} k^3 \sigma(k)\sigma(n-k) = \frac{3}{2} n \sum_{k=1}^{n-1} k^2 \sigma(k)\sigma(n-k) - \frac{1}{4} n^3 \sum_{k=1}^{n-1} \sigma(k)\sigma(n-k)$$

implies

$$\tau(n) = n^4 \cdot \sigma(n) - 24 \left(35 \sum_{k=1}^{n-1} k^4 \sigma(k)\sigma(n-k) - 60n^2 \sum_{k=1}^{n-1} k^2 \sigma(k)\sigma(n-k) + 13n^4 \sum_{k=1}^{n-1} \sigma(k)\sigma(n-k)\right), \quad (20)$$

and (16) follows from the matrix representation (6).
(ii) Combining the spectral representation (7) of $\sigma(n)$ with (20) one obtains (18) because of

$$\sum_{k=1}^{n-1} k^r \sigma(k)\sigma(n-k) = \sum_{j,l=1}^{M} \Lambda_{j,l}^{(n)}(r) \quad \text{for } r = 0, 2, 4.$$

(iii) Setting

$$s_{j,l}^{(n)} = 35\Lambda_{j,l}^{(n)}(4) - 60n^2\Lambda_{j,l}^{(n)}(2) + 13n^4\Lambda_{j,l}^{(n)}(0),$$

equation (18) becomes

$$\tau(n) = n^4 \cdot \sum_{j=1}^{M} \lambda_j^n - 24 \left(\sum_{j=1}^{M} s_{j,j}^{(n)} + \sum_{\substack{j,l=1 \\ j<l}}^{M} (\, s_{j,l}^{(n)} + s_{l,j}^{(n)}\,)\right).$$

Using well-known formulae for the sums $\sum_{k=1}^{n-1} k^r$ the expression $s_{j,j}^{(n)}$ is found to be

$$s_{j,j}^{(n)} = \frac{1}{6}n(n-1)(-3n^2+7n+7)\cdot\lambda_j^n.$$

On the other hand, for indexes $j < l$ the relations

$$\begin{aligned}
&\Lambda_{j,l}^{(n)}(4) + \Lambda_{l,j}^{(n)}(4) = 2\Lambda_{j,l}^{(n)}(4) - 4n\Lambda_{j,l}^{(n)}(3) + 6n^2\Lambda_{j,l}^{(n)}(2) - 4n^3\Lambda_{j,l}^{(n)}(1) + n^4\Lambda_{j,l}^{(n)}(0),\\
&\Lambda_{j,l}^{(n)}(2) + \Lambda_{l,j}^{(n)}(2) = 2\Lambda_{j,l}^{(n)}(2) - 2n\Lambda_{j,l}^{(n)}(1) + n^2\Lambda_{j,l}^{(n)}(0),\\
&\Lambda_{j,l}^{(n)}(0) + \Lambda_{l,j}^{(n)}(0) = 2\Lambda_{j,l}^{(n)}(0)
\end{aligned}$$

yield (19) in view of

$$s_{j,l}^{(n)} + s_{l,j}^{(n)} = 70\,\Lambda_{j,l}^{(n)}(4) - 140n\,\Lambda_{j,l}^{(n)}(3) + 90n^2\,\Lambda_{j,l}^{(n)}(2) - 20n^3\,\Lambda_{j,l}^{(n)}(1) + n^4\,\Lambda_{j,l}^{(n)}(0).$$

□

Quantities $\Lambda_{j,l}^{(n)}(r)$, cf. equation (17), satisfy the recursion

$$\Lambda_{j,l}^{(n+1)}(r) = \lambda_l\cdot\Lambda_{j,l}^{(n)}(r) + n^r\lambda_j^n\lambda_l, \quad n\ge 1$$

provided r and j,l are fixed. Being modified geometric series, $\Lambda_{j,l}^{(n)}(r)$ can also be expressed directly in terms of the eigenvalues without recourse to summation (see the Appendix).

According to (6) the divisor function can be viewed as the trace of a matrix. This is also true for Ramanujan's τ-function; if E denotes the square $M\times M$ matrix with entry $E_{11}=1$ and zero entries otherwise, then (16) implies

$$\tau(n) = \operatorname{tr}\left(n^4\cdot H_M^n\otimes E - 24\sum_{k=1}^{n-1}(35k^4 - 60n^2k^2 + 13n^4)\cdot H_M^k\otimes H_M^{n-k}\right),$$

i.e., $\tau(n)$ is the trace of a linear combination of appropriate Kronecker products.

As mentioned above, the spectral representations of both the partition and divisor function show that these oscillating functions are just superpositions of sinusoids. The erratic behaviour of Ramanujan's τ-function may now be explained by the corresponding spectral representations of Theorem 3.1. The first term in (18) is an inflated divisor function with ordinary sinusoids λ_j^n but the second term comprises inflated 'cross sinusoids' $\lambda_j^k\lambda_l^{n-k}, k=1,2,\ldots,n-1$. This illustrates that the nature of $\tau(n)$ is much more complicated than the nature of $\sigma(n)$ and $p(n)$.

4. Spectral Representations of Certain Divisor Functions of Higher Order

Matrix or spectral representations for certain other arithmetical functions can be derived by analogy with Theorem 3.1. In this subsection, M is always a fixed pentagonal number, and n is an integer with $1 \leq n \leq M' - 1$. For example, combining the well-known identity

$$5\,\sigma_3(n) = (6n-1)\sigma(n) + 12\sum_{k=1}^{n-1} \sigma(k)\,\sigma(n-k),$$

which goes back to Ramanujan [6], with the spectral representation (7), one obtains the corresponding spectral version

$$5\,\sigma_3(n) = (6n-1)\sum_{j=1}^{M} \lambda_j^n + 12\sum_{j,l=1}^{M} \Lambda_{j,l}^{(n)}(0) \tag{21}$$

of the divisor function $\sigma_3(n)$ where

$$\Lambda_{j,l}^{(n)}(0) = \begin{cases} (n-1)\cdot\lambda_j^n & \text{if } \lambda_j = \lambda_l \\ \dfrac{\lambda_j\lambda_l}{(\lambda_j-\lambda_l)}(\lambda_j^{n-1} - \lambda_l^{n-1}) & \text{if } \lambda_j \neq \lambda_l \end{cases}$$

(see formula (29) in the appendix). Likewise, identity

$$\frac{21}{10}\,\sigma_5(n) = (3n-1)\sigma_3(n) + \sigma(n) + 24\sum_{k=1}^{n-1} \sigma_3(k)\,\sigma(n-k),$$

see again Ramanujan [6], in conjunction with (7) and the spectral representation (21) of $\sigma_3(n)$ gives

$$\begin{aligned} 7\,\sigma_5(n) = (12n^2 - 6n + 1)\sum_{j=1}^{M} \lambda_j^n + 24(n-1)\sum_{j,l=1}^{M} \Lambda_{j,l}^{(n)}(0) \\ + 96\sum_{j,l=1}^{M} \Lambda_{j,l}^{(n)}(1) + 192\sum_{j,l,m=1}^{M} \Lambda_{j,l,m}^{(n)}(0) \end{aligned} \tag{22}$$

with

$$\Lambda_{j,l,m}^{(n)}(0) = \begin{cases} \Lambda_{j,m}^{(n)}(1) - \Lambda_{j,m}^{(n)}(0) & \text{if } \lambda_j = \lambda_l \\ \dfrac{\lambda_l}{(\lambda_j-\lambda_l)}\Lambda_{j,m}^{(n)}(0) - \dfrac{\lambda_j}{(\lambda_j-\lambda_l)}\Lambda_{l,m}^{(n)}(0) & \text{if } \lambda_j \neq \lambda_l. \end{cases}$$

This procedure can be extended to odd indexes $k \geq 7$: if spectral representations have already been established for $\sigma(n), \sigma_3(n), \ldots, \sigma_{k-2}(n)$ then identities such as the formulae 3. - 9. given in table IV of Ramanujan [6] would yield the spectral representation of $\sigma_k(n)$.

5. A Numerical Example

The pentagonal number $M = 15$ has successor $M' = 22$, and the associated truncated companion matrix $H_M = H_{15}$ has pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_{15}$ with $\max_j |\lambda_j| = 1.2769$ and $\min_j |\lambda_j| = 0.8475$; three eigenvalues are real. By the results discussed in the preceding sections, matrix H_{15} determines the values of the partition function, the divisor function, their corresponding summatory functions, Ramanujan's τ-function, and several other arithmetical functions for arguments less than $M' - 1 = 21$. As an illustration, consider the argument 20. The matrix H_{15}^{20} has main diagonal

$$(627, 137, -248, -248, -248, -72, -72, 29, 29, 29, 29, 29, 7, 7, 7).$$

By (3), (6) one finds

$$p(20) = 627 \quad \text{and} \quad \sigma(20) = 627 + 137 \mp \ldots + 7 = 42,$$

and the same result is obtained using (4), (7). Regarding the summatory functions, the main diagonal of $(H_{15} - I_{15})^{-1} \cdot (H_{15}^{20} - I_{15}) \cdot H_{15}$ is given by

$$(2713, 626, -971, -971, -971, -287, -287, 86, 86, 86, 86, 86, 19, 19, 19).$$

Note that the inverse matrix exists because M has the form $M = (3m^2 + m)/2$ with $m = 3$. Therefore, Theorems 2.1 and 2.2 guarantee

$$\sum_{n=1}^{20} p(n) = 2713 \quad \text{and} \quad \sum_{n=1}^{20} \sigma(n) = 2713 + 626 \mp \ldots + 19 = 339.$$

Of course, formulae (10), (14) yield the same values. In order to determine $\tau(20)$, an application of relation (16) of Theorem 3.1 gives

$$\tau(20) = -7, 109, 760.$$

If $\tau(20)$ is calculated via eigenvalues one may insert

$$\sum_{j=1}^{15} \lambda_j^{20} = 42, \quad \sum_{j,l=1}^{15} \Lambda_{j,l}^{(20)}(0) = 3, 416,$$

$$\sum_{j,l=1}^{15} \Lambda_{j,l}^{(20)}(2) = 404, 600, \quad \sum_{j,l=1}^{15} \Lambda_{j,l}^{(20)}(4) = 74, 448, 464$$

into equation (18) of Theorem 3.1 to arrive at the same result for $\tau(20)$.

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References

[1] T.M. Apostol, *Modular Functions and Dirichlet Series in Number Theory (2nd Edition)*, Springer, New York, 1990.

[2] J.H. Bruinier and K. Ono, Algebraic formulas for the coefficients of half-integral weight harmonic weak Maass forms, *Adv. Math.* **246** (2013), 198-219.

[3] J. A. Ewell, New representations of Ramanujan's τ-function, *Proc. Amer. Math. Soc.* **128** (2000), 723-726.

[4] D. H. Lehmer, *Some Functions of Ramanujan in Selected Papers of D. H. Lehmer (Vol. II)*, Charles Babbage research center, St. Pierre, Manitoba, 1981.

[5] D. Niebur, A formula for Ramanujan's τ-function, *Illinois J. Math.* **19** (1975), 448-449.

[6] S. Ramanujan, On certain arithmetical functions, *Trans. Cambridge Philos. Soc.* **XXII** (1916), 159-184.

[7] M. Weba, A finite spectral representation of the partition function, *Ramanujan J.* **42** (2017), 29-42.

Appendix: Formulae for the Quantities $\Lambda_{j,l}^{(n)}(r)$

The quantities $\Lambda_{j,l}^{(n)}(r)$ in (19) admit the closed-form representations

$$\Lambda_{j,l}^{(n)}(r) = \begin{cases} A^{(n)}(r) \cdot \lambda_j^n & \text{if } \lambda_j = \lambda_l \\ B_{j,l}^{(n)}(r) \cdot \dfrac{\lambda_j \lambda_l}{(\lambda_j - \lambda_l)^{r+1}} & \text{if } \lambda_j \neq \lambda_l \end{cases} \tag{23}$$

for $r = 0, 1, 2, 3, 4$ where

$$A^{(n)}(0) = n-1, \quad A^{(n)}(1) = \frac{1}{2}(n-1)n, \quad A^{(n)}(2) = \frac{1}{6}(n-1)n(2n-1),$$

$$A^{(n)}(3) = \frac{1}{4}(n-1)^2 n^2, \quad A^{(n)}(4) = \frac{1}{30}(n-1)n(2n-1)(3n^2-3n-1);$$

and

$$B_{j,l}^{(n)}(0) = \lambda_j^{n-1} - \lambda_l^{n-1}, \quad B_{j,l}^{(n)}(1) = (n-1)\lambda_j^n - n\lambda_j^{n-1} + \lambda_l^n,$$

$$B_{j,l}^{(n)}(2) = (n-1)^2 \lambda_j^{n+1} - (2n^2-2n-1)\lambda_j^n \lambda_l + n^2 \lambda_j^{n-1} \lambda_l^2 - \lambda_j \lambda_l^n - \lambda_l^{n+1},$$

$$\begin{aligned} B_{j,l}^{(n)}(3) = {} & (n-1)^3 \lambda_j^{n+2} - (3n^3-6n^2+4)\lambda_j^{n+1}\lambda_l + (3n^3-3n^2-3n-1)\lambda_j^n \lambda_l^2 \\ & - n^3 \lambda_j^{n-1}\lambda_l^3 + \lambda_j^2 \lambda_l^n + 4\lambda_j \lambda_l^{n+1} + \lambda_l^{n+2}, \end{aligned}$$

$$\begin{aligned} B_{j,l}^{(n)}(4) = {} & (n-1)^4 \lambda_j^{n+3} - (4n^4-12n^3+6n^2+12n-11)\lambda_j^{n+2}\lambda_l \\ & + (6n^4-12n^3-6n^2+12n+11)\lambda_j^{n+1}\lambda_l^2 \\ & - (4n^4-4n^3-6n^2-4n-1)\lambda_j^n \lambda_l^3 \\ & + n^4 \lambda_j^{n-1}\lambda_l^4 - \lambda_j^3 \lambda_l^n - 11\lambda_j^2 \lambda_l^{n+1} - 11\lambda_j \lambda_l^{n+2} - \lambda_l^{n+3}. \end{aligned}$$