REPRESENTATION OF NUMBERS BY SUMS OF SQUARES AND THE FORMS OF TYPE $X_1^2 + X_1X_2 + X_2^2$

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Abstract

In this paper, motivated from the work of Xia and Yao who obtained representation numbers of some octonary quadratic forms using theta function identities, we obtain representation numbers of certain quadratic forms in twelve variables with coefficients 1, 2 and 4 which are sums of squares and the forms of type $x_1^2+x_1x_2+x_2^2$. The method of the proof is due to the authors Alaca, Alaca and Williams. Firstly, we establish some new theta function identities using the (p,k)-parametrization of theta functions and Eisenstein series given by Alaca, Alaca and Williams, and then use them to obtain the mentioned formulae.

1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{C}$ denote the set of natural numbers, non-negative integers, integers and complex numbers respectively so that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout the paper, we always assume that $q \in \mathbb{C}$, $|q| < 1$.

For any positive integers k and n let $\sigma_k(n)$ be defined by

$$\sigma_k(n) = \sum_{\substack{d \in \mathbb{N} \\ d|n}} d^k, \tag{1}$$

and write $\sigma(n)$ for $\sigma_1(n)$.

The Jacobi one-dimensional theta function $\varphi(q)$ is defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{2}$$

Borwein, Borwein and Garvan, in their paper [7] on some cubic modular identities of Ramanujan, defined the two-dimensional theta function $a(q)$ as

$$a(q) := \sum_{(m,n)\in\mathbb{Z}^2} q^{m^2+mn+n^2}. \tag{3}$$

For k, l, r and $s \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we let $R(k, l, r, s; n)$ denote the number of representations of n by the form

$$\sum_{i=1}^{4k} x_i^2 + \sum_{i=1}^{l}(u_i^2 + u_i v_i + v_i^2) + 2\sum_{i=1}^{r}(e_i^2 + e_i f_i + f_i^2) + 4\sum_{i=1}^{s}(g_i^2 + g_i h_i + h_i^2), \tag{4}$$

with $2k + l + r + s = 6$. Clearly, $R(k, l, r, s; 0) = 1$ and also

$$\sum_{n=0}^{\infty} R(k, l, r, s; n) q^n = \varphi^{4k}(q) a^l(q) a^r(q^2) a^s(q^4). \tag{5}$$

Formula for the representation number $R(0, 6, 0, 0; n)$ was obtained by Lomadze [16]. In a recent publication Yao and Xia [21] obtained the following formula for $R(0, 6, 0, 0; n)$, as well as some others:

$$R(0, 6, 0, 0; n) = \frac{252}{13}\sigma_5(n) - \frac{6804}{13}\sigma_5(n/3) + \frac{216}{13}a(n), \tag{6}$$

where

$$\sum_{n=1}^{\infty} a\,(n)\, q^n = q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6. \tag{7}$$

The special case $k = 0$ of (4) for all possible l, r and s values was considered by Köklüce and Karatay [12]. In that study, the authors derived formulae for representation numbers of 21 quadratic forms by using theta function identities. Their list also contains the formula for $R(0, 6, 0, 0; n)$.

The case $k = 3$ clearly gives the representation numbers of integers by sums of twelve squares. A formula for $R(3, 0, 0, 0; n)$, in the case when n is even, was given by Liouville [14], and in the general case by Glaisher [9]. A simple proof of a formula for the number of representations of a positive integer as the sum of twelve squares was also given by Williams [18].

For $j \in \{1, 2, 3, 4, 5, 6, 7\}$ let $B_j(q)$ be defined by

$$B_1(q) \quad : \quad = q\prod_{n=1}^{\infty}(1-q^{2n})^{12}, \tag{8}$$

$$B_2(q) \quad : \quad = q\prod_{n=1}^{\infty}(1-q^n)^5(1-q^{2n})^5(1-q^{3n})(1-q^{6n}), \tag{9}$$

$$B_3(q) \quad : \quad = q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6, \tag{10}$$

$$B_4(q) \quad : \quad = q^2 \prod_{n=1}^{\infty}(1-q^{2n})^5(1-q^{4n})^5(1-q^{6n})(1-q^{12n}), \tag{11}$$

$$B_5(q) \quad : \quad = q^2 \prod_{n=1}^{\infty}(1-q^{n})(1-q^{3n})^5(1-q^{4n})^5(1-q^{12n}), \tag{12}$$

$$B_6(q) \quad : \quad = q^2 \prod_{n=1}^{\infty}(1-q^{n})(1-q^{2n})(1-q^{3n})^5(1-q^{6n})^5, \tag{13}$$

and

$$B_7(q) := q^3 \prod_{n=1}^{\infty}(1-q^{6n})^{12}. \tag{14}$$

For any $j \in \{1,2,3,4,5,6,7\}$ let $b_j(n)$ be defined by

$$B_j(q) := \sum_{n=1}^{\infty} b_j\,(n)\, q^n. \tag{15}$$

It is clear from (8), (9), (11) and (14) that $B_7(q) = B_1(q^3)$ and $B_4(q) = B_2(q^2)$. Thus from (15)

$$b_7(n) = b_1(\frac{n}{3}), b_4(n) = b_2(\frac{n}{2}). \tag{16}$$

Throughout the paper $\sigma_3(n), \sigma_5(n)$ and $b_j(n)$ are defined to be zero if n is not an integer.

For $a, b, r, s, n \in \mathbb{N}$, we define the convolution sum $W_{a,b}^{r,s}(n)$ by

$$W_{a,b}^{r,s}(n) := \sum_{\substack{(l,m)\in\mathbb{N}^2 \\ al+bm=n}} \sigma_r(l)\sigma_s(m). \tag{17}$$

Recently, Köklüce [10] derived formulae for $W_{1,6}^{1,3}(n)$, $W_{2,3}^{1,3}(n)$, $W_{3,2}^{1,3}(n)$ and $W_{6,1}^{1,3}(n)$. He proved that

$$\begin{aligned} W_{6,1}^{1,3}(n) \quad = \quad & \frac{5}{2184}\sigma_5\,(n) + \frac{2}{273}\sigma_5\left(\frac{n}{2}\right) + \frac{27}{1456}\sigma_5\left(\frac{n}{3}\right) + \frac{27}{455}\sigma_5\left(\frac{n}{6}\right) \\ & - + \frac{2-n}{48}\sigma_3\,(n)\,\frac{1}{240}\sigma\left(\frac{n}{6}\right) - \frac{1}{4368}u_1\,(n)\,, \end{aligned} \tag{18}$$

$$\begin{aligned} W_{2,3}^{1,3}(n) \quad = \quad & \frac{1}{4368}\sigma_5\,(n) + \frac{1}{1365}\sigma_5\left(\frac{n}{2}\right) + \frac{15}{728}\sigma_5\left(\frac{n}{3}\right) + \frac{6}{91}\sigma_5\left(\frac{n}{6}\right) \\ & + \frac{2-3n}{48}\sigma_3\left(\frac{n}{3}\right) - \frac{1}{240}\sigma\left(\frac{n}{2}\right) + \frac{1}{8736}u_2\,(n)\,, \end{aligned} \tag{19}$$

$$W_{1,6}^{1,3}(n) = \frac{1}{21840}\sigma_5(n) + \frac{1}{1092}\sigma_5\left(\frac{n}{2}\right) + \frac{3}{728}\sigma_5\left(\frac{n}{3}\right) + \frac{15}{182}\sigma_5\left(\frac{n}{6}\right) + \frac{1-3n}{24}\sigma_3\left(\frac{n}{6}\right) - \frac{1}{240}\sigma(n) + \frac{1}{156}a(\frac{n}{2}) + \frac{1}{4368}u_3(n), \quad (20)$$

and

$$W_{3,2}^{1,3}(n) = \frac{1}{2184}\sigma_5(n) + \frac{5}{546}\sigma_5\left(\frac{n}{2}\right) + \frac{27}{7280}\sigma_5\left(\frac{n}{3}\right) + \frac{27}{364}\sigma_5\left(\frac{n}{6}\right) + \frac{1-n}{24}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{240}\sigma\left(\frac{n}{3}\right) - \frac{1}{8736}u_4(n) \quad (21)$$

where the $u_i(n)$'s (for $i = 1, 2, 3, 4$ and $n \in \mathbb{N}$) are

$$u_1(n) = -4b_2(n) + 105b_3(n) + 972b_6(n),$$
$$u_2(n) = -72b_2(n) + 70b_3(n) + 24b_6(n),$$
$$u_3(n) = 4b_2(n) + 14b_3(n) + 120b_6(n)$$

and

$$u_4(n) = -101b_2(n) + 105b_3(n) - 27b_6(n),$$

where $b_2(n)$, $b_3(n)$ and $b_6(n)$ are defined by the equations (9), (10), (13) and (15). As an application he used these evaluations to determine formulae for $R(0, 5, 1, 0; n)$, $R(0, 4, 2, 0; n)$ and $R(0, 3, 3, 0; n)$. In another work, Köklüce [11] used a method developed by Lomadze [15] to obtain explicit formulae for the number of representations of integers by quadratic forms in twelve and sixteen variables which are direct sums of binary quadratic forms with discriminant -23. Determination of representation number formulae for quadratic forms which are sums of binary quadratic forms of type $x_1^2 + x_1x_2 + x_2^2$ was considered before by many mathematicians. See for example [8, 19, 16].

In the present paper, we obtain some new theta function identities (see Theorem 4.1) and then use these identities to determine the number of representations of $n \in \mathbb{N}$ by each of the 16 quadratic forms in twelve variables. These formulae are given in terms of $\sigma_5(n)$ and the integers $b_j(n)$, $j \in \{1, 2, 3, 5, 6\}$. Since the integers $b_j(n)$, $j \in \{1, 2, 3, 4, 5, 6, 7\}$ defined in (8)-(15) are also used in [12] and [13], to make the notation consistent, we have used them in the same order.

Similar methods have been used before for deriving representation numbers of positive integers by quaternary, sextenary and octonary quadratic forms, see for example [6, 5], [3] and [20], respectively.

The rest of this paper is organized as follows. In Section 2, we state our main results. In Section 3, we give parametrizations of theta functions and Eisenstein series in terms of p and k. In Section 4, we state and prove a theorem which contains required theta function identities for the proof of the main theorem. In Section 5, we prove our main theorem.

2. Statement of the Main Theorem

Theorem 1. *If $n \in \mathbb{N}$, then the following hold:*

(i)

$$\begin{aligned} R(1,4,0,0;n) &= \frac{1280}{91}\sigma_5(n) - \frac{1200}{91}\sigma_5(\frac{n}{2}) + \frac{10368}{91}\sigma_5(\frac{n}{3}) - \frac{5120}{91}\sigma_5(\frac{n}{4}) \\ &\quad - \frac{9720}{91}\sigma_5(\frac{n}{6}) - \frac{41472}{91}\sigma_5(\frac{n}{12}) + \frac{2160}{91}b_1(n) - \frac{648}{91}b_1(\frac{n}{3}) \\ &\quad + \frac{9768}{91}b_2(n) + \frac{10752}{13}b_2(\frac{n}{2}) - \frac{792}{7}b_3(n) - \frac{52992}{91}b_5(n) \\ &\quad - \frac{36936}{91}b_6(n), \end{aligned} \tag{22}$$

(ii)

$$\begin{aligned} R(1,0,4,0;n) &= \frac{80}{91}\sigma_5(n) + \frac{648}{91}\sigma_5(\frac{n}{3}) - \frac{5120}{91}\sigma_5(\frac{n}{4}) - \frac{41472}{91}\sigma_5(\frac{n}{12}) \\ &\quad - \frac{24}{91}b_1(n) - \frac{648}{91}b_1(\frac{n}{3}) + \frac{636}{91}b_2(n) + \frac{768}{13}b_2(\frac{n}{2}) \\ &\quad + \frac{36}{91}b_3(n) - \frac{576}{91}b_5(n) + \frac{324}{91}b_6(n), \end{aligned} \tag{23}$$

(iii)

$$\begin{aligned} R(1,0,0,4;n) &= \frac{5}{91}\sigma_5(n) + \frac{75}{91}\sigma_5(\frac{n}{2}) + \frac{81}{182}\sigma_5(\frac{n}{3}) - \frac{5120}{91}\sigma_5(\frac{n}{4}) \\ &\quad + \frac{1215}{182}\sigma_5(\frac{n}{6}) - \frac{41472}{91}\sigma_5(\frac{n}{12}) + \frac{522}{91}b_1(n) \\ &\quad + \frac{20817}{182}b_1(\frac{n}{3}) - \frac{276}{91}b_2(n) - \frac{480}{13}b_2(\frac{n}{2}) + \frac{477}{91}b_3(n) \\ &\quad + \frac{5976}{91}b_5(n) + \frac{810}{91}b_6(n), \end{aligned} \tag{24}$$

(iv)

$$\begin{aligned} R(1,3,1,0;n) &= \frac{656}{91}\sigma_5(n) - \frac{738}{91}\sigma_5(\frac{n}{2}) - \frac{6480}{91}\sigma_5(\frac{n}{3}) + \frac{5248}{91}\sigma_5(\frac{n}{4}) \\ &\quad + \frac{7290}{91}\sigma_5(\frac{n}{6}) - \frac{51840}{91}\sigma_5(\frac{n}{12}) - \frac{30}{91}b_1(n) - \frac{810}{91}b_1(\frac{n}{3}) \\ &\quad - \frac{6054}{91}b_2(n) - \frac{6528}{13}b_2(\frac{n}{2}) + \frac{7794}{91}b_3(n) + \frac{38592}{91}b_5(n) \\ &\quad + \frac{4050}{13}b_6(n), \end{aligned} \tag{25}$$

(v)

$$\begin{aligned} R(1,3,0,1;n) &= \frac{320}{91}\sigma_5(n) - \frac{240}{91}\sigma_5(\frac{n}{2}) + \frac{2592}{91}\sigma_5(\frac{n}{3}) - \frac{5120}{91}\sigma_5(\frac{n}{4}) \\ &\quad -\frac{1944}{91}\sigma_5(\frac{n}{6}) - \frac{41472}{91}\sigma_5(\frac{n}{12}) + \frac{5436}{91}b_1(n) \\ &\quad +\frac{176256}{91}b_1(\frac{n}{3}) + \frac{5307}{182}b_2(n) - \frac{480}{13}b_2(\frac{n}{2}) - \frac{12087}{182}b_3(n) \\ &\quad +\frac{64944}{91}b_5(n) - \frac{139563}{182}b_6(n), \end{aligned} \tag{26}$$

(vi)

$$\begin{aligned} R(1,2,2,0;n) &= \frac{320}{91}\sigma_5(n) - \frac{240}{91}\sigma_5(\frac{n}{2}) + \frac{2592}{91}\sigma_5(\frac{n}{3}) - \frac{5120}{91}\sigma_5(\frac{n}{4}) \\ &\quad -\frac{1944}{91}\sigma_5(\frac{n}{6}) - \frac{41472}{91}\sigma_5(\frac{n}{12}) - \frac{24}{91}b_1(n) - \frac{648}{91}b_1(\frac{n}{3}) \\ &\quad +\frac{3063}{91}b_2(n) + \frac{3264}{13}b_2(\frac{n}{2}) - \frac{1539}{91}b_3(n) - \frac{26784}{91}b_5(n) \\ &\quad +\frac{14985}{91}b_6(n), \end{aligned} \tag{27}$$

(vii)

$$\begin{aligned} R(1,2,1,1;n) &= \frac{164}{91}\sigma_5(n) - \frac{246}{91}\sigma_5(\frac{n}{2}) - \frac{1620}{91}\sigma_5(\frac{n}{3}) + \frac{5248}{91}\sigma_5(\frac{n}{4}) \\ &\quad +\frac{2430}{91}\sigma_5(\frac{n}{6}) - \frac{51840}{91}\sigma_5(\frac{n}{12}) + \frac{3246}{91}b_1(n) \\ &\quad +\frac{87642}{91}b_1(\frac{n}{3}) - \frac{5037}{91}b_2(n) - \frac{6528}{13}b_2(\frac{n}{2}) + \frac{3447}{91}b_3(n) \\ &\quad +\frac{77904}{91}b_5(n) - \frac{27135}{91}b_6(n), \end{aligned} \tag{28}$$

(viii)

$$\begin{aligned} R(1,2,0,2;n) &\quad \frac{80}{91}\sigma_5(n) + \frac{648}{91}\sigma_5(\frac{n}{3}) - \frac{5120}{91}\sigma_5(\frac{n}{4}) - \frac{41472}{91}\sigma_5(\frac{n}{12}) \\ &\quad +\frac{2160}{91}b_1(n) + \frac{87804}{91}b_1(\frac{n}{3}) + \frac{1182}{91}b_2(n) - \frac{480}{13}b_2(\frac{n}{2}) \\ &\quad -\frac{1602}{91}b_3(n) + \frac{25632}{91}b_5(n) - \frac{14418}{91}b_6(n), \end{aligned} \tag{29}$$

(ix)

$$\begin{aligned} R(1,1,2,1;n) &= \frac{80}{91}\sigma_5(n) + \frac{648}{91}\sigma_5(\frac{n}{3}) - \frac{5120}{91}\sigma_5(\frac{n}{4}) - \frac{41472}{91}\sigma_5(\frac{n}{12}) \\ &\quad +\frac{1614}{91}b_1(n) + \frac{43578}{91}b_1(\frac{n}{3}) + \frac{363}{91}b_2(n) - \frac{480}{13}b_2(\frac{n}{2}) \\ &\quad -\frac{783}{91}b_3(n) + \frac{12528}{91}b_5(n) - \frac{7047}{91}b_6(n), \end{aligned} \tag{30}$$

(x)

$$\begin{aligned} R(1,1,1,2;n) \quad = \quad & \frac{41}{91}\sigma_5(n) - \frac{123}{91}\sigma_5(\frac{n}{2}) - \frac{405}{91}\sigma_5(\frac{n}{3}) + \frac{5248}{91}\sigma_5(\frac{n}{4}) \\ & + \frac{1215}{91}\sigma_5(\frac{n}{6}) - \frac{51840}{91}\sigma_5(\frac{n}{12}) + \frac{1608}{91}b_1(n) \\ & + \frac{43416}{91}b_1(\frac{n}{3}) - \frac{303}{13}b_2(n) - \frac{2784}{13}b_2(\frac{n}{2}) \\ & + \frac{1746}{91}b_3(n) + \frac{38592}{91}b_5(n) - \frac{13365}{91}b_6(n), \qquad (31) \end{aligned}$$

(xi)

$$\begin{aligned} R(2,2,0,0;n) \quad = \quad & \frac{976}{91}\sigma_5(n) - \frac{1220}{91}\sigma_5(\frac{n}{2}) - \frac{3888}{91}\sigma_5(\frac{n}{3}) + \frac{15616}{91}\sigma_5(\frac{n}{4}) \\ & + \frac{4860}{91}\sigma_5(\frac{n}{6}) - \frac{62208}{91}\sigma_5(\frac{n}{12}) + \frac{2148}{91}b_1(n) \\ & - \frac{972}{91}b_1(\frac{n}{3}) - \frac{1296}{13}b_2(n) - \frac{8832}{13}b_2(\frac{n}{2}) + \frac{8496}{91}b_3(n) \\ & + \frac{77760}{91}b_5(n) - \frac{10368}{91}b_6(n), \qquad (32) \end{aligned}$$

(xii)

$$\begin{aligned} R(2,1,1,0;n) \quad = \quad & \frac{484}{91}\sigma_5(n) - \frac{242}{91}\sigma_5(\frac{n}{2}) + \frac{972}{91}\sigma_5(\frac{n}{3}) - \frac{15488}{91}\sigma_5(\frac{n}{4}) \\ & - \frac{486}{91}\sigma_5(\frac{n}{6}) - \frac{31104}{91}\sigma_5(\frac{n}{12}) - \frac{18}{91}b_1(n) - \frac{486}{91}b_1(\frac{n}{3}) \\ & + \frac{3336}{91}b_2(n) + \frac{3072}{13}b_2(\frac{n}{2}) - \frac{1800}{91}b_3(n) - \frac{39744}{91}b_5(n) \\ & + \frac{27864}{91}b_6(n), \qquad (33) \end{aligned}$$

(xiii)

$$\begin{aligned} R(2,1,0,1;n) \quad & \frac{244}{91}\sigma_5(n) - \frac{488}{91}\sigma_5(\frac{n}{2}) - \frac{972}{91}\sigma_5(\frac{n}{3}) + \frac{15616}{91}\sigma_5(\frac{n}{4}) \\ & + \frac{1944}{91}\sigma_5(\frac{n}{6}) - \frac{62208}{91}\sigma_5(\frac{n}{12}) + \frac{4878}{91}b_1(n) \\ & + \frac{131706}{91}b_1(\frac{n}{3}) - \frac{5568}{91}b_2(n) - \frac{7584}{13}b_2(\frac{n}{2}) + \frac{2448}{91}b_3(n) \\ & + \frac{117072}{91}b_5(n) - \frac{9396}{13}b_6(n), \qquad (34) \end{aligned}$$

(xiv)

$$\begin{aligned} R(2,0,2,0;n) &= \frac{244}{91}\sigma_5(n)-\frac{488}{91}\sigma_5(\frac{n}{2})-\frac{972}{91}\sigma_5(\frac{n}{3})+\frac{15616}{91}\sigma_5(\frac{n}{4})\\ &+\frac{1944}{91}\sigma_5(\frac{n}{6})-\frac{62208}{91}\sigma_5(\frac{n}{12})-\frac{36}{91}b_1(n)-\frac{972}{91}b_1(\frac{n}{3})\\ &-\frac{1200}{91}b_2(n)-\frac{1344}{13}b_2(\frac{n}{2})+\frac{2448}{91}b_3(n)-\frac{864}{91}b_5(n)\\ &+\frac{3240}{13}b_6(n), \end{aligned} \tag{35}$$

(xv)

$$\begin{aligned} R(2,0,1,1;n) &= \frac{121}{91}\sigma_5(n)+\frac{121}{91}\sigma_5(\frac{n}{2})+\frac{243}{91}\sigma_5(\frac{n}{3})-\frac{15488}{91}\sigma_5(\frac{n}{4})\\ &+\frac{243}{91}\sigma_5(\frac{n}{6})-\frac{31104}{91}\sigma_5(\frac{n}{12})+\frac{2439}{91}b_1(n)\\ &+\frac{65853}{91}b_1(\frac{n}{3})-\frac{816}{91}b_2(n)-\frac{1920}{13}b_2(\frac{n}{2})-\frac{288}{91}b_3(n)\\ &+\frac{38880}{91}b_5(n)-\frac{24624}{91}b_6(n), \end{aligned} \tag{36}$$

(xvi)

$$\begin{aligned} R(2,0,0,2;n) &= \frac{61}{91}\sigma_5(n)-\frac{305}{91}\sigma_5(\frac{n}{2})-\frac{243}{91}\sigma_5(\frac{n}{3})+\frac{15616}{91}\sigma_5(\frac{n}{4})\\ &+\frac{1215}{91}\sigma_5(\frac{n}{6})-\frac{62208}{91}\sigma_5(\frac{n}{12})+\frac{1875}{91}b_1(n)\\ &+\frac{65367}{91}b_1(\frac{n}{3})-\frac{1416}{91}b_2(n)-\frac{2592}{13}b_2(\frac{n}{2})\\ &+\frac{72}{7}b_3(n)+\frac{38448}{91}b_5(n)-\frac{13284}{91}b_6(n). \end{aligned} \tag{37}$$

3. The (p, k)-Parametrization of Theta Functions and Eisenstein Series

In his second notebook Ramanujan [17] gives the definitions of Eisenstein series $L(q)$, $M(q)$ and $N(q)$ by

$$L(q) := 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \tag{38}$$

$$M(q) := 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} \tag{39}$$

and

$$N(q) := 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}\cdot \tag{40}$$

It can be easily seen that

$$L(q) := 1 - 24\sum_{n=1}^{\infty}\sigma\left(n\right)q^n, \tag{41}$$

$$M(q) := 1 + 240\sum_{n=1}^{\infty}\sigma_3\left(n\right)q^n \tag{42}$$

and

$$N(q) := 1 - 504\sum_{n=1}^{\infty}\sigma_5\left(n\right)q^n. \tag{43}$$

Alaca, Alaca and Williams [1] defined p and k respectively by

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)} \tag{44}$$

and

$$k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}\cdot \tag{45}$$

The (p, k)-parametrization of $\varphi(q)$ is given in [4] as

$$\varphi(q) = (1 + 2p)^{3/4}k^{1/2}. \tag{46}$$

Alaca, Alaca and Williams [1] derived formulae for the representations of $a(q)$, $a(q^2)$ and $a(q^4)$ in terms of p and k. They have proved that

$$a(q) = (1 + 4p + p^2)k, \tag{47}$$

$$a(q^2) = (1 + p + p^2)k \tag{48}$$

and

$$a(q^4) = (1 + p - \frac{1}{2}p^2)k. \tag{49}$$

Formulae for the series $N(q)$, $N(q^2)$, $N(q^3)$, $N(q^4)$, $N(q^6)$ and $N(q^{12})$ in terms of p and k are found by the same authors [2]. Equations (3.22)-(3.27) in [2] are respectively as follows

$$\begin{aligned} N(q) &= (1 - 246p - 5532p^2 - 38614p^3 - 135369p^4 - 276084p^5 \\ &\quad -348024p^6 - 276084p^7 - 135369p^8 - 38614p^9 - 5532p^{10} \\ &\quad -246p^{11} + p^{12})k^6, \end{aligned} \tag{50}$$

$$\begin{aligned} N(q^2) &= (1 + 6p - 114p^2 - 625p^3 - \frac{4059}{2}p^4 - 4302p^5 - 5556p^6 \\ &\quad -4302p^7 - \frac{4059}{2}p^8 - 625p^9 - 114p^{10} + 6p^{11} + p^{12})k^6, \end{aligned} \tag{51}$$

$$\begin{aligned} N(q^3) &= (1 + 6p + 12p^2 - 58p^3 - 297p^4 - 396p^5 - 264p^6 - 396p^7 \\ &\quad -297p^8 - 58p^9 + 12p^{10} + 6p^{11} + p^{12})k^6, \end{aligned} \tag{52}$$

$$\begin{aligned} N(q^4) &= (1 + 6p + 12p^2 + 5p^3 - 45p^4 - 144p^5 - \frac{1167}{8}p^6 + \frac{171}{8}p^7 \\ &\quad +\frac{2151}{32}p^8 - \frac{739}{16}p^9 - \frac{345}{8}p^{10} + \frac{129}{32}p^{11} + \frac{1}{64}p^{12})k^6, \end{aligned} \tag{53}$$

$$\begin{aligned} N(q^6) &= (1 + 6p + 12p^2 + 5p^3 - \frac{27}{2}p^4 - 18p^5 - 12p^6 - 18p^7 - \frac{27}{2}p^8 \\ &\quad +5p^9 + 12p^{10} + 6p^{11} + p^{12})k^6, \end{aligned} \tag{54}$$

and

$$\begin{aligned} N(q^{12}) &= (1 + 6p + 12p^2 + 5p^3 - \frac{27}{2}p^4 - 18p^5 - \frac{33}{8}p^6 + \frac{45}{8}p^7 + \frac{135}{32}p^8 \\ &\quad +\frac{17}{16}p^9 + \frac{3}{16}p^{10} + \frac{3}{32}p^{11} + \frac{1}{64}p^{12})k^6. \end{aligned} \tag{55}$$

In that study, Alaca, Alaca and Williams also give formulae for $\triangle(q^i)$ ($i = 1, 2, 3, 4, 6$ and 12) in equations (3.28)-(3.33). Alaca and Williams [6] solved those equations for

$$\prod_{n=1}^{\infty}(1 - q^{in}), (i = 1, 2, 3, 4, 6 \text{ and } 12) \tag{56}$$

and derived the following equations:

$$\prod_{n=1}^{\infty}(1-q^n)=q^{-\frac{1}{24}}2^{-\frac{1}{6}}p^{\frac{1}{24}}(1-p)^{\frac{1}{2}}(1+p)^{\frac{1}{6}}(1+2p)^{\frac{1}{8}}(2+p)^{\frac{1}{8}}k^{\frac{1}{2}}, \tag{57}$$

$$\prod_{n=1}^{\infty}(1-q^{2n})=q^{-\frac{1}{12}}2^{-\frac{1}{3}}p^{\frac{1}{12}}(1-p)^{\frac{1}{4}}(1+p)^{\frac{1}{12}}(1+2p)^{\frac{1}{4}}(2+p)^{\frac{1}{4}}k^{\frac{1}{2}}, \tag{58}$$

$$\prod_{n=1}^{\infty}(1-q^{3n})=q^{-\frac{1}{8}}2^{-\frac{1}{6}}p^{\frac{1}{8}}(1-p)^{\frac{1}{6}}(1+p)^{\frac{1}{2}}(1+2p)^{\frac{1}{24}}(2+p)^{\frac{1}{24}}k^{\frac{1}{2}}, \tag{59}$$

$$\prod_{n=1}^{\infty}(1-q^{4n})=q^{-\frac{1}{6}}2^{-\frac{2}{3}}p^{\frac{1}{6}}(1-p)^{\frac{1}{8}}(1+p)^{\frac{1}{24}}(1+2p)^{\frac{1}{8}}(2+p)^{\frac{1}{2}}k^{\frac{1}{2}}, \tag{60}$$

$$\prod_{n=1}^{\infty}(1-q^{6n})=q^{-\frac{1}{4}}2^{-\frac{1}{3}}p^{\frac{1}{4}}(1-p)^{\frac{1}{12}}(1+p)^{\frac{1}{4}}(1+2p)^{\frac{1}{12}}(2+p)^{\frac{1}{12}}k^{\frac{1}{2}} \tag{61}$$

and

$$\prod_{n=1}^{\infty}(1-q^{12n})=q^{-\frac{1}{2}}2^{-\frac{2}{3}}p^{\frac{1}{2}}(1-p)^{\frac{1}{24}}(1+p)^{\frac{1}{8}}(1+2p)^{\frac{1}{24}}(2+p)^{\frac{1}{6}}k^{\frac{1}{2}}. \tag{62}$$

From (57)-(62) it can be deduced that

$$q\prod_{n=1}^{\infty}(1-q^{2n})^{12}=\frac{1}{16}p(1-p)^3(1+p)(1+2p)^3(2+p)^3k^6, \tag{63}$$

$$q\prod_{n=1}^{\infty}(1-q^n)^5(1-q^{2n})^5(1-q^{3n})(1-q^{6n})=\frac{1}{8}p(1-p)^4(1+p)^2(1+2p)^2(2+p)^2k^6, \tag{64}$$

$$q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6=\frac{1}{4}p(1-p)^4(1+p)^4(1+2p)(2+p)k^6, \tag{65}$$

$$q^2\prod_{n=1}^{\infty}(1-q^{2n})^5(1-q^{4n})^5(1-q^{6n})(1-q^{12n})=\frac{1}{64}p^2(1-p)^2(1+p)(1+2p)^2(2+p)^4k^6, \tag{66}$$

$$q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{3n})^5(1-q^{4n})^5(1-q^{12n})=\frac{1}{32}p^2(1-p)^2(1+p)^3(1+2p)(2+p)^3k^6, \tag{67}$$

$$q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})(1-q^{3n})^5(1-q^{6n})^5=\frac{1}{8}p^2(1-p)^2(1+p)^4(1+2p)(2+p)k^6 \quad (68)$$

and

$$q^3\prod_{n=1}^{\infty}(1-q^{6n})^{12}=\frac{1}{16}p^3(1-p)(1+p)^3(1+2p)(2+p)k^6. \quad (69)$$

4. Theta Function Identities

In this section we establish some necessary theta function identities. We prove that the generating functions of the quadratic forms considered in this paper can all be expressed as a linear combination of $N(q^i)$ ($i = 1, 2, 3, 4, 6$ and 12) and $B_j(q)$ ($j = 1, 2, ...7$). There are no non-trivial linear relationships between series $N(q^i)$ and $B_j(q)$($i = 1, 2, 3, 4, 6$ and 12, $j = 1, 2, ...7$). Using (16) we have reduced the cusp forms used in formulae to 5.

Theorem 2. *The following hold:*

(i)

$$\begin{aligned}\varphi^4(q)a^4(q) \quad = \quad & \frac{-160}{5733}N(q)+\frac{50}{1911}N(q^2)-\frac{144}{637}N(q^3)+\frac{640}{5733}N(q^4)\\ &+\frac{135}{637}N(q^6)+\frac{576}{637}N(q^{12})+\frac{2160}{91}B_1(q)-\frac{648}{91}B_1(q^3)\\ &+\frac{9768}{91}B_2(q)+\frac{10752}{13}B_2(q^2)-\frac{792}{7}B_3(q)-\frac{52992}{91}B_5(q)\\ &-\frac{36936}{91}B_6(q),\end{aligned}$$

(ii)

$$\begin{aligned}\varphi^4(q)a^4(q^2) \quad = \quad & \frac{-10}{5733}N(q)-\frac{9}{637}N(q^3)+\frac{640}{5733}N(q^4)+\frac{576}{637}N(q^{12})\\ &-\frac{24}{91}B_1(q)-\frac{648}{91}B_1(q^3)+\frac{636}{91}B_2(q)+\frac{768}{13}B_2(q^2)\\ &+\frac{36}{91}B_3(q)-\frac{576}{91}B_5(q)+\frac{324}{91}B_6(q),\end{aligned}$$

(iii)

$$\begin{aligned}\varphi^4(q)a^4(q^4) \quad = \quad & \frac{-5}{45864}N(q) - \frac{25}{15288}N(q^2) - \frac{9}{10192}N(q^3) + \frac{640}{5733}N(q^4) \\ & -\frac{135}{10192}N(q^6) + \frac{576}{637}N(q^{12}) + \frac{522}{91}B_1(q) + \frac{20817}{182}B_1(q^3) \\ & -\frac{276}{91}B_2(q) - \frac{480}{13}B_2(q^2) + \frac{477}{91}B_3(q) + \frac{5976}{91}B_5(q) \\ & +\frac{810}{91}B_6(q),\end{aligned}$$

(iv)

$$\begin{aligned}\varphi^4(q)a^3(q)a(q^2) \quad = \quad & \frac{-82}{5733}N(q) + \frac{41}{2548}N(q^2) + \frac{90}{637}N(q^3) - \frac{656}{5733}N(q^4) \\ & -\frac{405}{2548}N(q^6) + \frac{720}{637}N(q^{12}) - \frac{30}{91}B_1(q) - \frac{810}{91}B_1(q^3) \\ & -\frac{6054}{91}B_2(q) - \frac{6528}{13}B_2(q^2) + \frac{7794}{91}B_3(q) \\ & +\frac{38592}{91}B_5(q) + \frac{4050}{13}B_6(q),\end{aligned}$$

(v)

$$\begin{aligned}\varphi^4(q)a^3(q)a(q^4) \quad = \quad & \frac{-40}{5733}N(q) + \frac{10}{1911}N(q^2) - \frac{36}{637}N(q^3) + \frac{640}{5733}N(q^4) \\ & +\frac{27}{637}N(q^6) + \frac{576}{637}N(q^{12}) + \frac{5436}{91}B_1(q) \\ & +\frac{176256}{91}B_1(q^3) + \frac{5307}{182}B_2(q) - \frac{480}{13}B_2(q^2) \\ & -\frac{12087}{182}B_3(q) + \frac{64944}{91}B_5(q) - \frac{139563}{182}B_6(q),\end{aligned}$$

(vi)

$$\begin{aligned}\varphi^4(q)a^2(q)a^2(q^2) \quad = \quad & \frac{-40}{5733}N(q) + \frac{10}{1911}N(q^2) - \frac{36}{637}N(q^3) + \frac{640}{5733}N(q^4) \\ & +\frac{27}{637}N(q^6) + \frac{576}{637}N(q^{12}) - \frac{24}{91}B_1(q) - \frac{648}{91}B_1(q^3) \\ & +\frac{3063}{91}B_2(q) + \frac{3264}{13}B_2(q^2) - \frac{1539}{91}B_3(q) \\ & -\frac{26784}{91}B_5(q) + \frac{14985}{91}B_6(q),\end{aligned}$$

(vii)

$$\begin{aligned}\varphi^4(q)a^2(q)a(q^2)a(q^4) &= \frac{-41}{11466}N(q)+\frac{41}{7644}N(q^2)+\frac{45}{1274}N(q^3)\\ &-\frac{656}{5733}N(q^4)-\frac{135}{2548}N(q^6)+\frac{720}{637}N(q^{12})\\ &+\frac{3246}{91}B_1(q)+\frac{87642}{91}B_1(q^3)-\frac{5037}{91}B_2(q)\\ &-\frac{6528}{13}B_2(q^2)+\frac{3447}{91}B_3(q)+\frac{77904}{91}B_5(q)\\ &-\frac{27135}{91}B_6(q),\end{aligned}$$

(viii)

$$\begin{aligned}\varphi^4(q)a^2(q)a^2(q^4) &= \frac{-10}{5733}N(q)-\frac{9}{637}N(q^3)+\frac{640}{5733}N(q^4)+\frac{576}{637}N(q^{12})\\ &+\frac{2160}{91}B_1(q)+\frac{87804}{91}B_1(q^3)+\frac{1182}{91}B_2(q)\\ &-\frac{480}{13}B_2(q^2)-\frac{1602}{91}B_3(q)+\frac{25632}{91}B_5(q)\\ &-\frac{14418}{91}B_6(q)\end{aligned}$$

(ix)

$$\begin{aligned}\varphi^4(q)a(q)a^2(q^2)a(q^4) &= \frac{-10}{5733}N(q)-\frac{9}{637}N(q^3)+\frac{640}{5733}N(q^4)\\ &+\frac{576}{637}N(q^{12})+\frac{1614}{91}B_1(q)+\frac{43578}{91}B_1(q^3)\\ &+\frac{363}{91}B_2(q)-\frac{480}{13}B_2(q^2)-\frac{783}{91}B_3(q)\\ &+\frac{12528}{91}B_5(q)-\frac{7047}{91}B_6(q),\end{aligned}$$

(x)

$$\begin{aligned}\varphi^4(q)a(q)a(q^2)a^2(q^4) &= \frac{-41}{45864}N(q)+\frac{41}{15288}N(q^2)+\frac{45}{5096}N(q^3)\\ &-\frac{656}{5733}N(q^4)-\frac{135}{5096}N(q^6)+\frac{720}{637}N(q^{12})\\ &+\frac{1608}{91}B_1(q)+\frac{43416}{91}B_1(q^3)-\frac{303}{13}B_2(q)\\ &-\frac{2784}{13}B_2(q^2)+\frac{1746}{91}B_3(q)+\frac{38592}{91}B_5(q)\\ &-\frac{13365}{91}B_6(q),\end{aligned}$$

(xi)

$$\begin{aligned}\varphi^8(q)a^2(q) \quad = \quad & \frac{-122}{5733}N(q)+\frac{305}{11466}N(q^2)+\frac{54}{637}N(q^3)-\frac{1952}{5733}N(q^4)\\ & -\frac{135}{1274}N(q^6)+\frac{864}{637}N(q^{12})+\frac{2148}{91}B_1(q)-\frac{972}{91}B_1(q^3)\\ & -\frac{1296}{13}B_2(q)-\frac{8832}{13}B_2(q^2)+\frac{8496}{91}B_3(q)+\frac{77760}{91}B_5(q)\\ & -\frac{10368}{91}B_6(q),\end{aligned}$$

(xii)

$$\begin{aligned}\varphi^8(q)a(q)a(q^2) \quad = \quad & \frac{-121}{11466}N(q)+\frac{121}{22932}N(q^2)-\frac{27}{1274}N(q^3)+\frac{1936}{5733}N(q^4)\\ & +\frac{27}{2548}N(q^6)+\frac{432}{637}N(q^{12})-\frac{18}{91}B_1(q)-\frac{486}{91}B_1(q^3)\\ & +\frac{3336}{91}B_2(q)+\frac{3072}{13}B_2(q^2)-\frac{1800}{91}B_3(q)\\ & -\frac{39744}{91}B_5(q)+\frac{27864}{91}B_6(q),\end{aligned}$$

(xiii)

$$\begin{aligned}\varphi^8(q)a(q)a(q^4) \quad = \quad & \frac{-61}{11466}N(q)+\frac{61}{5733}N(q^2)+\frac{27}{1274}N(q^3)-\frac{1952}{5733}N(q^4)\\ & -\frac{27}{637}N(q^6)+\frac{864}{637}N(q^{12})+\frac{4878}{91}B_1(q)+\frac{131706}{91}B_1(q^3)\\ & -\frac{5568}{91}B_2(q)-\frac{7584}{13}B_2(q^2)+\frac{2448}{91}B_3(q)\\ & +\frac{117072}{91}B_5(q)-\frac{9396}{13}B_6(q),\end{aligned}$$

(xiv)

$$\begin{aligned}\varphi^8(q)a^2(q^2) \quad = \quad & \frac{-61}{11466}N(q)+\frac{61}{5733}N(q^2)+\frac{27}{1274}N(q^3)-\frac{1952}{5733}N(q^4)\\ & -\frac{27}{637}N(q^6)+\frac{864}{637}N(q^{12})-\frac{36}{91}B_1(q)-\frac{972}{91}B_1(q^3)\\ & -\frac{1200}{91}B_2(q)-\frac{1344}{13}B_2(q^2)+\frac{2448}{91}B_3(q)-\frac{864}{91}B_5(q)\\ & +\frac{3240}{13}B_6(q),\end{aligned}$$

(xv)

$$\begin{aligned}\varphi^8(q)a(q^2)a(q^4) &= \frac{-121}{45864}N(q) - \frac{121}{45864}N(q^2) - \frac{27}{5096}N(q^3) + \frac{1936}{5733}N(q^4) \\ &- \frac{27}{5096}N(q^6) + \frac{432}{637}N(q^{12}) + \frac{2439}{91}B_1(q) + \frac{65853}{91}B_1(q^3) \\ &- \frac{816}{91}B_2(q) - \frac{1920}{13}B_2(q^2) - \frac{288}{91}B_3(q) + \frac{38880}{91}B_5(q) \\ &- \frac{24624}{91}B_6(q),\end{aligned}$$

(xvi)

$$\begin{aligned}\varphi^8(q)a^2(q^4) &= \frac{-61}{45864}N(q) + \frac{305}{45864}N(q^2) + \frac{27}{5096}N(q^3) - \frac{1952}{5733}N(q^4) \\ &- \frac{135}{5096}N(q^6) + \frac{864}{637}N(q^{12}) + \frac{1875}{91}B_1(q) + \frac{65367}{91}B_1(q^3) \\ &- \frac{1416}{91}B_2(q) - \frac{2592}{13}B_2(q^2) + \frac{72}{7}B_3(q) + \frac{38448}{91}B_5(q) \\ &- \frac{13284}{91}B_6(q).\end{aligned}$$

We just prove part (i). The rest can be proved similarly.

Proof. From (8)-(15), (46), (47) and (50)-(55), we see that

$$\begin{aligned}& \frac{-160}{5733}N(q) + \frac{50}{1911}N(q^2) - \frac{144}{637}N(q^3) + \frac{640}{5733}N(q^4) + \frac{135}{637}N(q^6) \\ &+ \frac{576}{637}N(q^{12}) + \frac{2160}{91}B_1(q) - \frac{648}{91}B_1(q^3) + \frac{9768}{91}B_2(q) + \frac{10752}{13}B_2(q^2) \\ &- \frac{792}{7}B_3(q) - \frac{52992}{91}B_5(q) - \frac{36936}{91}B_6(q) \\ =\ & (1 + 22p + 208p^2 + 1104p^3 + 3606p^4 + 7476p^5 + 9804p^6 + 7896p^7 + 3729p^8 \\ & +998p^9 + 140p^{10} + 8p^{11})k^6 \\ =\ & (1+2p)^3(1+4p+p^2)^4k^6 \\ =\ & \varphi^4(q)a^4(q).\end{aligned}$$

□

5. Proof of Main Theorem

We just prove part (i) as the remaining parts can be proved similarly.

Proof. From (5), (40), (46), (47) and Theorem 4.1 we have

$$\begin{aligned}
1+\sum_{n=1}^{\infty} R(1,4,0,0;n)q^n &= \varphi^4(q)a^4(q) = (1+2p)^3(1+4p+p^2)^4k^6\\
&= \frac{-160}{5733}N(q)+\frac{50}{1911}N(q^2)-\frac{144}{637}N(q^3)+\frac{640}{5733}N(q^4)\\
&\quad +\frac{135}{637}N(q^6)+\frac{576}{637}N(q^{12})\\
&\quad +\frac{2160}{91}q\prod_{n=1}^{\infty}(1-q^{2n})^{12}\\
&\quad -\frac{648}{91}q^3\prod_{n=1}^{\infty}(1-q^{6n})^{12}\\
&\quad +\frac{9768}{91}q\prod_{n=1}^{\infty}(1-q^n)^5(1-q^{2n})^5(1-q^{3n})(1-q^{6n})\\
&\quad +\frac{10752}{13}q^2\prod_{n=1}^{\infty}(1-q^{2n})^5(1-q^{4n})^5(1-q^{6n})(1-q^{12n})\\
&\quad -\frac{792}{7}q\prod_{n=1}^{\infty}(1-q^n)^6(1-q^{3n})^6\\
&\quad -\frac{52992}{91}q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{3n})^5(1-q^{4n})^5(1-q^{12n})\\
&\quad -\frac{36936}{91}q^2\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})(1-q^{3n})^5(1-q^{6n})^5
\end{aligned}$$

$$\begin{aligned}
= &\frac{-160}{5733}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)\,q^n\right)+\frac{50}{1911}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)\,q^{2n}\right)\\
&-\frac{144}{637}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)\,q^{3n}\right)+\frac{640}{5733}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)\,q^{4n}\right)\\
&+\frac{135}{637}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)\,q^{6n}\right)+\frac{576}{637}\left(1-504\sum_{n=1}^{\infty}\sigma_5(n)\,q^{12n}\right)\\
&+\frac{2160}{91}\sum_{n=1}^{\infty}b_1(n)q^n-\frac{648}{91}\sum_{n=1}^{\infty}b_1(\frac{n}{3})q^n+\frac{9768}{91}\sum_{n=1}^{\infty}b_2(n)q^n\\
&+\frac{10752}{13}\sum_{n=1}^{\infty}b_2(\frac{n}{2})q^n-\frac{792}{7}\sum_{n=1}^{\infty}b_3(n)q^n-\frac{52992}{91}\sum_{n=1}^{\infty}b_5(n)q^n\\
&-\frac{36936}{91}\sum_{n=1}^{\infty}b_6(n)q^n
\end{aligned}$$

$$= \quad 1+\sum_{n=1}^{\infty}(\frac{1280}{91}\sigma_5(n)-\frac{1200}{91}\sigma_5(\frac{n}{2})+\frac{10368}{91}\sigma_5(\frac{n}{3})-\frac{5120}{91}\sigma_5(\frac{n}{4})$$
$$-\frac{9720}{91}\sigma_5(\frac{n}{6})-\frac{41472}{91}\sigma_5(\frac{n}{12})+\frac{2160}{91}b_1(n)-\frac{648}{91}b_1(\frac{n}{3})+\frac{9768}{91}b_2(n)$$
$$+\frac{10752}{13}b_2(\frac{n}{2})-\frac{792}{7}b_3(n)-\frac{52992}{91}b_5(n)-\frac{36936}{91}b_6(n))q^n.$$

For $n \in \mathbb{N}$, equating the coefficients of q^n on both sides in the above equations, we obtain (22). □

Denoting the right-hand side of (22) by $S(1,4,0,0;n)$ we give the first ten values of $R(1,4,0,0;n)$, and $S(1,4,0,0;n)$ in Table 1 to illustrate the equation.

n	$R(1,4,0,0;n)$	$\sigma_5(n)$	$b_1(n)$	$b_2(n)$	$b_3(n)$	$b_5(n)$	$b_6(n)$	$S(1,4,0,0;n)$
1	32	1	1	1	1	0	0	32
2	432	33	0	−5	−6	1	1	432
3	3224	244	−12	0	9	−1	−1	3224
4	14832	1057	0	34	4	−1	−2	14832
5	45888	3126	54	−30	6	−5	−4	45888
6	109080	8052	0	−81	−54	0	5	109080
7	234880	16808	−88	68	−40	11	12	234880
8	475632	33825	0	100	168	10	−4	475632
9	846848	59293	99	81	81	21	0	846848
10	1391904	103158	0	−174	−36	−30	−10	1391904

Table 1. The first ten values of $R(1,4,0,0;n)$ and $S(1,4,0,0;n)$.

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