CONCERNING PARTITION REGULAR MATRICES

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Abstract

A finite or infinite matrix A with entries from $\mathbb{Q}$ is image partition regular provided that whenever $\mathbb{N}$ is finitely colored, there must be some $\vec{x}$ with entries from $\mathbb{N}$ such that all entries of $A\vec{x}$ are in some color class. In 2003, Hindman, Leader and Strauss studied centrally image partition regular matrices and extended many results of finite image partition regular matrices to infinite image partition regular matrices. It was shown that centrally image partition regular matrices are closed under diagonal sums. In the present paper, we show that the diagonal sum of two matrices, one of which comes from the class of all Milliken-Taylor matrices and the other from a suitable subclass of the class of all centrally image partition regular matrices, is also image partition regular. This will produce more image partition regular matrices. We also study the multiple structures within one cell of a finite partition of $\mathbb{N}$.

1. Introduction

In 1933, R. Rado [9] produced a computable characterization, called the columns condition, for the (finite) matrices with rational entries which are kernel partition regular. *Kernel partition regular matrices* are those matrices A which have the property that whenever $\mathbb{N}$ is finitely colored, there exists some $\vec{x}$ with monochrome entries such that $A\vec{x} = \vec{0}$. He also extended the result in his later paper [10] to cover other subsets of $\mathbb{R}$ (and even of $\mathbb{C}$).

Though several characterizations of (finite) image partition regular matrices were known, a reasonable characterization of image partition regular matrices was introduced by Hindman and Leader in 1993 [4]. A matrix A is said to be *image partition*

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regular if whenever $\mathbb{N}$ is finitely colored, there will be some $\vec{x}$ (with entries from $\mathbb{N}$) such that the entries of $A\vec{x}$ are monochrome. Image partition regular matrices generalize many of the classical theorems of Ramsey Theory. For example, Schur's Theorem [11] and van der Waerden's Theorem [12] are equivalent to saying that the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and for each $n \in \mathbb{N}$, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & n-1 \end{pmatrix}$, are image partition regular, respectively. Some of the characterizations of finite image partition regular matrices involve the notion of central sets. Central sets were introduced by Furstenberg and defined in terms of notions of topological dynamics. A nice characterization of central sets in terms of the algebraic structure of $\beta\mathbb{N}$, the Stone-$\check{C}$ech compactification of $\mathbb{N}$, is given in Definition 1.2 (a). Central sets are very rich in combinatorial properties. The basic fact about central sets is given by the Central Sets Theorem, which is due to Furstenberg [3, Proposition 8.21] for the case $S = \mathbb{Z}$.

Theorem 1.1. *(Central Sets Theorem) Let S be a commutative semigroup. Let τ be the set of sequences $\langle y_n \rangle_{n=1}^{\infty}$ in S. Let C be a subset of S which is central and let $F \in \mathcal{P}_f(\tau)$. Then there exist a sequence $\langle a_n \rangle_{n=1}^{\infty}$ in S and a sequence $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and for each $L \in \mathcal{P}_f(\mathbb{N})$ and each $f \in F$, $\sum_{n \in L}(a_n + \sum_{t \in H_n} f(t)) \in C$.*

We shall present this characterization below, after introducing the necessary background information.

Let $(S, \cdot)$ be an infinite discrete semigroup. Now the points of βS are taken to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. Given $A \subseteq S$ let us set $\bar{A} = \{p \in \beta S : A \in p\}$. Then the set $\{\bar{A} : A \subseteq S\}$ will become a basis for a topology on βS. The operation $\cdot$ on S can be extended to the Stone-$\check{C}$ech compactification βS of S so that $(\beta S, \cdot)$ is a compact right topological semigroup (meaning that for any $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological center (meaning that for any $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous). Given $p, q \in \beta S$ and $A \subseteq S, A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$.

A nonempty subset I of a semigroup $(T, \cdot)$ is called a left ideal of T if $T \cdot I \subseteq I$, a right ideal if $I \cdot T \subseteq I$, and a two-sided ideal (or simply an ideal) if it is both a left and a right ideal. A minimal left ideal is a left ideal that does not contain any proper left ideal. Similarly, we can define minimal right ideal and the smallest ideal. Any compact Hausdorff right topological semigroup $(T, \cdot)$ has the unique smallest two-sided ideal

$$\begin{aligned} K(T) &= \bigcup\{L : L \text{ is a minimal left ideal of } T\} \\ &= \bigcup\{R : R \text{ is a minimal right ideal of } T\}. \end{aligned}$$

Given a minimal left ideal L and a minimal right ideal R of T, $L \cap R$ is a group, and in particular $K(T)$ contains an idempotent. An idempotent that belongs to $K(T)$ is called a minimal idempotent. Given any subset $J \subseteq T$ we shall use the notation $E(J)$ to denote the set of all idempotents in J.

Definition 1.2. Let S be a semigroup and let $C \subseteq S$.
(a) C is called *central* in S if there is some idempotent $p \in K(\beta S)$ such that $C \in p$ [7, Definition 4.42].
(b) C is called *central** in S if $C \cap A \neq \emptyset$ for every cental set A in S [7, Definition 15.3].

Like kernel partition regular matrices, finite image partition regular matrices can also be described by a computable condition called the first entries condition. In the following theorem [5, theorem 2.10], we see that central sets characterize all finite image partition regular matrices.

Theorem 1.3. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then the following statements are equivalent.*
(a) A is image partition regular.
(b) For every additively central subset C of $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in C^u$.

It is an immediate consequence of Theorem 1.3 that whenever A and B are finite image partition regular matrices, so is $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where 0 represents a matrix of appropriate size with zero entries. However, it is a consequence of [2, Theorem 3.14] that the corresponding result is not true for infinite image partition regular matrices. In [6], it was shown that for an image partition regular matrix A, $I(A)$ (Definition 2.1) is a nonempty compact subset of $(\beta\mathbb{N}, +)$. It is also a sub-semigroup of $(\beta\mathbb{N}, +)$ if A is a finite image partition regular matrix.

In Section 2, we will investigate the multiplicative structure of $I(A)$ in $(\beta\mathbb{N}, \cdot)$ for finite and infinite matrix A. Using both additive and multiplicative structures of $\beta\mathbb{N}$ we show that the diagonal sum of two matrices, one of which is a Milliken-Taylor matrix and the other is a subtracted centrally image partition regular matrix (Definition 2.11), is also an image partition regular matrix.

One knows from Finite Sums Theorem [7, Corollary 5.10] that whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} E_i$ there exist $i \in \{1, 2, \ldots.., r\}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that

$FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq E_i$. It is also known that whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} E_i$ there exist $j \in \{1, 2, ..., r\}$ and a sequence $\langle y_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $FP(\langle y_n\rangle_{n=1}^{\infty}) \subseteq E_j$. The following theorem [7, Corollary 5.22] will allow us to take $i = j$.

Theorem 1.4. *Suppose $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} E_i$. Then there exist $i \in \{1, 2, ..., r\}$ and sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $FS(\langle x_n\rangle_{n=1}^{\infty}) \bigcup FP(\langle y_n\rangle_{n=1}^{\infty}) \subseteq E_i$.*

We generalize this result in Section 3 in matrix version. We also use both additive and multiplicative structures of $\beta\mathbb{N}$ to show that multiple structures induced from matrices with rational co-efficients will be in one cell of a finite partition of $\mathbb{N}$.

2. Diagonal Sum of Matrices

It is natural that the additive structure of $\beta\mathbb{N}$ is useful to study the image partition regular matrices over $(\mathbb{N}, +)$. In [2], [5], [6] and [8], mostly the additive structure of $\beta\mathbb{N}$ is used to study the image partition regular matrices over $\beta\mathbb{N}$. In this section, we shall see that the multiplicative structure of $\beta\mathbb{N}$ is also helpful to study the image partition regular matrices over $(\mathbb{N}, +)$.

We start with the following definition [6, Definition 2.4].

Definition 2.1. Let A be a finite or infinite matrix with entries from $\mathbb{Q}$. Then $I(A) = \{p \in \beta\mathbb{N}$: for every $P \in p$, there exists $\vec{x}$ with entries from $\mathbb{N}$ such that all entries of $A\vec{x}$ are in $P\}$.

We also recall the following lemma [6, Lemma 2.5].

Lemma 2.2. *Let $u, v \in \mathbb{N} \cup \{\omega\}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$.*
(a) The set $I(A)$ is compact and $I(A) \neq \emptyset$ if and only if A is image partition regular.
(b) If A is finite and image partition regular, then $I(A)$ is a sub-semigroup of $(\beta\mathbb{N}, +)$.

Let us now investigate the multiplicative structure of $I(A)$. In the following lemma, we shall see that if A is a image partition regular matrix then $I(A)$ is a left ideal of $(\beta\mathbb{N}, \cdot)$. It is also a two-sided ideal of $(\beta\mathbb{N}, \cdot)$ provided A is a finite image partition regular matrix.

Lemma 2.3. *Let $u, v \in \mathbb{N} \cup \{\omega\}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$.*
(a) If A is image partition regular then $I(A)$ is a left ideal of $(\beta\mathbb{N}, \cdot)$.
(b) If A is finite and image partition regular, then $I(A)$ is a two-sided ideal of $(\beta\mathbb{N}, \cdot)$.

Proof. Assume that A is image partition regular. Let $p \in \beta\mathbb{N}$ and let $q \in I(A)$. We shall show that $p \cdot q \in I(A)$ and, if A is finite, then $q \cdot p \in I(A)$. Let $U \in p \cdot q$. Pick $a \in \mathbb{N}$ such that $a^{-1}U \in q$. Pick $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in (a^{-1}U)^u$. Then $a\vec{x} \in \mathbb{N}^v$ and $A(a\vec{x}) \in U^u$.

Now assume that A is finite and let $U \in q \cdot p$. Let $B = \{x \in \mathbb{N} : x^{-1}U \in p\}$. Pick $\vec{x} \in \mathbb{N}^v$ such that $\vec{y} = A\vec{x} \in B^u$. Let $C = \bigcap_{i=1}^{u} y_i^{-1}U$. Then $C \in p$ so pick $a \in C$. Then $\vec{x}a \in \mathbb{N}^v$ and $A(\vec{x}a) \in U^u$. □

The following theorem was proved in 2003 using combinatorics [6, Lemma 2.3]. We now provide an alternative proof of this theorem using the algebra of $(\beta\mathbb{N}, \cdot)$.

Theorem 2.4. *Let A and B be finite and infinite image partition regular matrices respectively (with rational co-efficients). Then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is image partition regular.*

Proof. Let $r \in \mathbb{N}$ be given and $\mathbb{N} = \bigcup_{i=1}^{r} E_i$. Suppose that A is a $u \times v$ matrix where $u, v \in \mathbb{N}$. Also let $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Now by Lemma 2.3(b), $I(A)$ is a two-sided ideal of $(\beta\mathbb{N}, \cdot)$. So $K(\beta\mathbb{N}, \cdot) \subseteq I(A)$. Also by Lemma 2.3(a), $I(B)$ is a left ideal of $(\beta\mathbb{N}, \cdot)$. Therefore $K(\beta\mathbb{N}, \cdot) \cap I(B) \neq \emptyset$. Hence $I(A) \cap I(B) \neq \emptyset$. Now choose $p \in I(A) \cap I(B)$. Since $\mathbb{N} = \bigcup_{i=1}^{r} E_i$, there exists $k \in \{1, 2, \ldots.., r\}$ such that $E_k \in p$. Thus by definition of $I(A)$ and $I(B)$, there exist $\vec{x} \in \mathbb{N}^v$ and $\vec{y} \in \mathbb{N}^\omega$ such that $A\vec{x} \in E_k^u$ and $B\vec{y} \in E_k^\omega$. Take $\vec{z} = \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}$. Then $M\vec{z} = \begin{pmatrix} A\vec{x} \\ B\vec{y} \end{pmatrix}$. So $M\vec{z} \in E_k^\omega$. Therefore $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is image partition regular. □

The following theorem is [6, Corollary 2.6]. We also provide an alternative proof of this corollary in the following theorem.

Theorem 2.5. *Let $\mathcal{F}$ denote the set of all finite image partition regular matrices with entries from $\mathbb{Q}$. If B is an image partition regular matrix then $I(B) \cap (\bigcap_{A \in \mathcal{F}} I(A)) \neq \emptyset$.*

Proof. Note that for each $A \in \mathcal{F}$, $I(A)$ is a two-sided ideal of $(\beta\mathbb{N}, \cdot)$ and therefore $K(\beta\mathbb{N}, \cdot) \subseteq I(A)$. Hence $K(\beta\mathbb{N}, \cdot) \subseteq \bigcap_{A \in \mathcal{F}} I(A)$. Also by Lemma 2.3(a), $I(B)$ is a left ideal of $(\beta\mathbb{N}, \cdot)$. Thus $I(B) \cap K(\beta\mathbb{N}, \cdot) \neq \emptyset$. Hence $I(B) \cap (\bigcap_{A \in \mathcal{F}} I(A)) \neq \emptyset$. □

We introduce the following definition to see that the analogous statements are also true for kernel partition matrices.

Definition 2.6. Let A be a finite or infinite matrix with entries from $\mathbb{Q}$. Then $J(A) = \{p \in \beta\mathbb{N} :$ for every $P \in p$, there exists $\vec{x}$ with entries from P such that $A\vec{x} = \vec{0}\}$.

Lemma 2.7. *Let $u, v \in \mathbb{N} \cup \{\omega\}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$.*
(a) The set $J(A)$ is compact and $J(A) \neq \emptyset$ if and only if A is kernel partition regular.
(b) If A is finite and kernel partition regular, then $J(A)$ is a sub-semigroup of $(\beta\mathbb{N}, +)$.

Proof. The proof is similar to the proof of [6, Lemma 2.5]. □

Lemma 2.8. *Let $u, v \in \mathbb{N} \cup \{\omega\}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$.*
(a) If A is kernel partition regular then $J(A)$ is a left ideal of $(\beta\mathbb{N}, \cdot)$.
(b) If A is finite and kernel partition regular, then $J(A)$ is a two-sided ideal of $(\beta\mathbb{N}, \cdot)$.

Proof. The proof is similar to the proof of Lemma 2.3. □

We now recall the following definition [8, Definition 1.6(a)].

Definition 2.9. Let A be a $\omega \times \omega$ matrix with entries from $\mathbb{Q}$. The matrix A is *centrally image partition regular* if for every central subset C of $(\mathbb{N}, +)$, there exists $\vec{x} \in \mathbb{N}^\omega$ such that $A\vec{x} \in C^\omega$.

We also recall the following definition [5, Definition 1.7].

Definition 2.10. Let A be a $u \times v$ matrix with rational entries. Then A is a *first entries matrix* if
(1) no row of A is $\vec{0}$;
(2) the first nonzero entry of each row is positive; and
(3) the first nonzero entries of any two rows are equal if they occur in the same column.

If A is a first entries matrix and d is the first nonzero entry of some row of A, then d is called a *first entry* of A. A first entries matrix A is said to be *monic* whenever all the first entries of A are 1.

Here we shall use both the additive and multiplicative structures of $\beta\mathbb{N}$ to show that the diagonal sum of two infinite image partition regular matrices, one of which comes from the class of all Milliken-Taylor matrices and the other from the class of all subtracted centrally image partition regular matrices, is also image partition regular. For this we introduce the following definition.

Definition 2.11. Let A be an $\omega \times \omega$ matrix with entries from $\mathbb{Q}$. The matrix A is *subtracted centrally image partition regular* if and only if
(1) no row of A is $\vec{0}$;
(2) for each $i \in \omega$, $\{j \in \omega : a_{ij} \neq 0\}$ is finite; and

(3) there exists $v \in \mathbb{N}$, an $\omega \times v$ matrix A_1, and an $\omega \times \omega$ matrix A_2 such that the rows of A_1 are the rows of a finite image partition regular matrix, A_2 is a centrally image partition regular matrix, and $A = \begin{pmatrix} A_1 & A_2 \end{pmatrix}$.

Example 2.12. *An example of a subtracted centrally image partition regular matrix is given below.*

$$\begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 0 \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 \cdots \\ 2 & 1 & 1 & 0 & 0 & 0 \cdots \\ 2 & 0 & 0 & 1 & 0 & 0 \cdots \\ 2 & 0 & 1 & 0 & 0 & 0 \cdots \\ 2 & 1 & 0 & 1 & 0 & 0 \cdots \\ 2 & 1 & 1 & 1 & 0 & 0 \cdots \\ 2 & 0 & 1 & 1 & 0 & 0 \cdots \\ 2 & 0 & 0 & 0 & 0 & 1 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

In the following theorem we show that subtracted centrally image partition regular matrices are centrally image partition regular.

Theorem 2.13. *Let A be a subtracted centrally image partition regular matrix. Then A is centrally image partition regular.*

Proof. Since A is a subtracted centrally image partition regular matrix, pick $u, v \in \mathbb{N}$, a $u \times v$ image partition regular matrix D, an $\omega \times v$ matrix A_1 whose rows are all rows of D, and an $\omega \times \omega$ centrally image partition regular matrix A_2 such that $A = \begin{pmatrix} A_1 & A_2 \end{pmatrix}$ as in Definition 2.11. Let C be a central subset of $(\mathbb{N}, +)$ and p be a minimal idempotent of $(\beta\mathbb{N}, +)$ such that $C \in p$. Let $B = \{x \in \mathbb{N} : -x + C \in p\}$. Then $B \in p$ and hence B is central in $(\mathbb{N}, +)$. Now by Theorem 1.3, pick $\vec{x}^{(1)} \in \mathbb{N}^v$ such that $D\vec{x}^{(1)} \in B^u$. If $y = A_1\vec{x}^{(1)}$ then $\{y_i : i \in \omega\}$ is finite so $E = \bigcap_{i<\omega}(-y_i + C) \in p$. Therefore E is central in $(\mathbb{N}, +)$. Choose $\vec{x}^{(2)} \in \mathbb{N}^\omega$ such that $\vec{w} = A_2\vec{x}^{(2)} \in E^\omega$. Let $\vec{x} = \begin{pmatrix} \vec{x}^{(1)} \\ \vec{x}^{(2)} \end{pmatrix}$. Then $\vec{x} \in \mathbb{N}^\omega$ and $A\vec{x} = A_1\vec{x}^{(1)} + A_2\vec{x}^{(2)} = \vec{y} + \vec{w} \in C^\omega$. Therefore A is centrally image partition regular. $\square$

The converse of Theorem 2.13 is not true. For example, finite sum matrix (i.e. the Milliken-Taylor matrix [Definition 2.16] generated by the compressed sequence $\langle 1 \rangle$) is centrally image partition regular but not subtracted centrally image partition regular.

We now recall the following definitions [8, Definition 2.1], [7, Definition 5.13(b)] and [2, Definition 2.4].

Definition 2.14. Let $v \in \mathbb{N} \cup \{\omega\}$ and let $\vec{x} \in \mathbb{Z}^v$. Then

(a) $d(\vec{x})$ is the sequence obtained by deleting all occurrence of 0 from $\vec{x}$.

(b) $c(\vec{x})$ is the sequence obtained by deleting every digit in $d(\vec{x})$ which is equal to its predecessor and

(c) $\vec{x}$ is a compressed sequence if and only if $\vec{x} = c(\vec{x})$.

Definition 2.15. Let $\langle x_t \rangle_{t=0}^{\infty}$ be a sequence in $\mathbb{N}$. A sequence $\langle y_t \rangle_{t=0}^{\infty}$ in $\mathbb{N}$ is said to be a $sum - subsystem$ of $\langle x_t \rangle_{t=0}^{\infty}$ if there exists a sequence $\langle H_t \rangle_{t=0}^{\infty}$ of finite subsets of $\mathbb{N}$ with $\max H_t < \min H_{t+1}$ for all $t \in \omega$ such that $y_t = \sum_{s \in H_t} x_s$.

Definition 2.16. An $\omega \times \omega$ matrix M with entries from $\mathbb{Z}$ is a $Milliken - Taylor$ matrix if and only if

(1) each row of M has only finitely many nonzero entries; and

(2) there exist $m \in \mathbb{N}$ and a finite compressed sequence $\vec{a} = \langle a_1, a_2, \ldots.., a_m \rangle \in (\mathbb{Z} \setminus \{0\})^m$ such that $a_m > 0$ and $\vec{r}$ is a row of M if and only if $c(\vec{r}) = \vec{a}$.

For Lemma 2.17 and Lemma 2.18, we need to view $\beta\mathbb{N}$ as a subset of $\beta\mathbb{Z}$. Given $p \in \beta\mathbb{Z}$, let $-p = \{-B : B \in p\}$. Then given $p, q \in \beta\mathbb{Z}$, $(-p) + (-q) = -(p+q)$. One can establish this fact by simple observation. Let $\nu : \mathbb{Z} \longrightarrow \mathbb{Z}$ be defined by $\nu(x) = -x$ and suppose $\tilde{\nu} : \beta\mathbb{Z} \longrightarrow \beta\mathbb{Z}$ is its continuous extension. Then $\tilde{\nu}(p) = -p$. Since ν is a homomorphism, so is $\tilde{\nu}$ by [7, Corollary 4.22] and from this, one can easily verify the fact.

Lemma 2.17. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ image partition regular matrix with entries from $\mathbb{Q}$, and let p be a minimal idempotent of $(\beta\mathbb{N}, +)$. For each $D \in p$ and each $n \in \mathbb{N}$, there exists $Q \subseteq \times_{j=1}^{v} (\mathbb{N} \setminus \{1, 2, ..., n\})$ such that for all $\vec{z} \in Q$, $A\vec{z} \in D^u$.*

Proof. Let $\bar{p} = (p, p, \ldots.., p) \in (\beta\mathbb{N})^u$. Define $f : \mathbb{N}^v \longrightarrow \mathbb{Z}^u$ by $f(x) = A\vec{x}$ and note that f is a homomorphism. Let $Y = \beta(\mathbb{N}^v)$ and let $\tilde{f} : Y \longrightarrow (\beta\mathbb{Z})^u$ be the continuous extension of f, so that $\tilde{f}$ is a homomorphism by [7, Corollary 4.22]. We claim that $\tilde{f}^{-1}[\{\bar{p}\}] \neq \emptyset$. To see this, note that $\{f^{-1}[B^u] : B \in p\}$ has the finite intersection property since $p \in I(A)$. Therefore $\bigcap_{B \in p} cl_Y f^{-1}[B^u] \neq \emptyset$. It is routine to verify that $\bigcap_{B \in p} cl_Y f^{-1}[B^u] \subseteq \tilde{f}^{-1}[\{\bar{p}\}]$. Since $\tilde{f}^{-1}[\{\bar{p}\}] \neq \emptyset$ and $\tilde{f}$ is a homomorphism, $\tilde{f}^{-1}[\{\bar{p}\}]$ is a subsemigroup of Y, so pick an idempotent $q \in \tilde{f}^{-1}[\{\bar{p}\}]$. Then for each $D \in p$ there is some $Q \in q$ such that $f[Q] \subseteq D^u$. So it suffices to show that for each $n \in \mathbb{N}$, $\times_{j=1}^{v} (\mathbb{N} \setminus \{1, 2, ..., n\}) \in q$. Since $\times_{j=1}^{v} (\mathbb{N} \setminus \{1, 2, ..., n\}) = \bigcap_{j=1}^{v} \pi_j^{-1}[\mathbb{N} \setminus \{1, 2, ..., n\}]$, it suffices to let $j \in \{1, 2, \ldots.., v\}$ and show that $\pi_j^{-1}[\mathbb{N} \setminus \{1, 2, \ldots.., n\}] \in q$. Suppose instead that $\bigcup_{t=1}^{n} \pi_j^{-1}[\{t\}] = \pi_j^{-1}[\{1, 2, \ldots..n\}] \in q$ and pick $t \in \{1, 2, \ldots.., n\}$ such that $\pi_j^{-1}[\{t\}] \in q$. If $\tilde{\pi}_j : Y \longrightarrow$

$(\beta\mathbb{N})^u$ is the continuous extension of π_j, $\tilde{\pi_j}(q) = t$. But $\tilde{\pi_j}$ is a homomorphism, so t is an idempotent, a contradiction. □

Lemma 2.18. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ image partition regular matrix with entries from $\mathbb{Q}$. Let $r \in \beta\mathbb{Z}$ and let p be a minimal idempotent of $(\beta\mathbb{N}, +)$. Then the following are true.*
(1) If $r \in I(A)$, then $r + p \in I(A)$.
(2) If $r \in -I(A)$, then $r + p \in I(A)$.
(3) If $r \in I(A)$, then $r + (-p) \in -I(A)$.
(4) If $r \in I(A)$, then $r + (-p) \in -I(A)$.

Proof. We already have (1) by Lemma 2.2 and Theorem 1.3. Statement (3) follows from statement (2) and statement (4) follows from statement (1). So it suffices to prove (2).
Assume $r \in -I(A)$. Let $B \in r + p$ and $C = \{x \in \mathbb{Z} : -x + B \in p\}$. Then $-C \in -r$ and $-r \in I(A)$ so pick $\vec{x} \in \mathbb{N}^v$ such that $\vec{y} = A\vec{x} \in (-C)^u$. Then for each $i \in \{1, 2, ...u\}$, $y_i + B \in p$. Let $n = \max\{x_i : i = 1, 2..., v\} + 1$ and let $D = \bigcap_{i=1}^{u} y_i + B$. Then $D \in p$ and so by Lemma 2.17, pick $\vec{z} \in (\mathbb{N} \setminus \{1, 2, ..., n\})^v$ such that $\vec{w} = A\vec{z} \in D^u$. Then $\vec{w} - \vec{y} = A(\vec{z} - \vec{x}) \in B^u$ and observe that $\vec{z} - \vec{x} \in \mathbb{N}^v$, which completes the proof. □

Theorem 2.19. *Let A be a subtracted centrally image partition regular matrix and let M be a Milliken-Taylor matrix. Then $\begin{pmatrix} A & 0 \\ 0 & M \end{pmatrix}$ is image partition regular.*

Proof. Pick $u, v \in \mathbb{N}$, a $u \times v$ image partition regular matrix D, an $\omega \times v$ matrix A_1 whose rows are all rows of D, and an $\omega \times \omega$ centrally image partition regular matrix A_2 such that $A = \begin{pmatrix} A_1 & A_2 \end{pmatrix}$. Pick $m \in \mathbb{N}$ and a compressed sequence $\vec{a} = \langle a_i \rangle_{i=1}^{m}$ in $\mathbb{Z} \setminus \{0\}$ such that $a_m > 0$ and M is the Milliken-Taylor matrix determined by $\vec{a}$. Pick a minimal idempotent p in $(\beta\mathbb{N}, +)$ and note that by [7, Lemma 5.19.2], for any $b \in \mathbb{N}$, $b \cdot p$ is a minimal idempotent in $(\beta N, +)$. Let $q = a_1 \cdot p + a_2 \cdot p + ... + a_m \cdot p$ and $r = a_m \cdot p$. Note that $q = q + r$. It suffices to show that $I(A) \bigcap I(M) \neq \emptyset$. By [6, Corollary 3.6], $q + r \in I(M)$. So it suffices to show that $q + r \in I(A)$. To this end, let $V \in q + r$ and let $W = \{x \in \mathbb{N} : -x + V \in r\}$. Now for each $i \in \{1, 2, ..., m - 1\}$, if $a_i > 0$, then since $a_i \cdot p$ is a minimal idempotent in $(\beta\mathbb{N}, +)$, $a_i \cdot p \in I(D) = I(A_1)$. And if $a_i < 0$, then since $(-a_i) \cdot p$ is a minimal idempotent in $(\beta\mathbb{N}, +)$, $a_i \cdot p \in -I(A_1)$. Applying Lemma 2.18 repeatedly and using the fact that $a_m > 0$, we have that $q \in I(A_1)$. Since $W \in q$, pick $\vec{x}^{(1)} \in \mathbb{N}^v$ such that $\vec{y} = A\vec{x}^{(1)} \in W^\omega$. Then $\{y_i : i \in \omega\}$ is finite and so $Z = \bigcap_{i<\omega}(-y_i + V) \in r$. Since r ia a minimal idempotent in $(\beta\mathbb{N}, +)$, pick $\vec{x}^{(2)}\mathbb{N}^\omega$ such that $\vec{w} = A\vec{x}^{(2)} \in \mathbb{Z}^\omega$. Let $\vec{x} = \begin{pmatrix} \vec{x}^{(1)} \\ \vec{x}^{(2)} \end{pmatrix}$. Then $\vec{x} \in \mathbb{N}^\omega$ and $A\vec{x} = A_1\vec{x}^{(1)} + A_2\vec{x}^{(2)} = \vec{y} + \vec{w} \in V^\omega$. □

3. Combined Additive and Multiplicative Structures Induced by Matrices

In this section, we shall present some Ramsey theoretic properties induced from combined additive and multiplicative structures of $\mathbb{N}$. We start with the following theorem [7, Theorem 15.5].

Theorem 3.1. *Let (S,+) be an infinite commutative semigroup with identity* 0*, let* $u, v \in \mathbb{N}$ *and let* A *be a* $u \times v$ *matrix with entries from* ω *which satisfies the first entries condition. Let* C *be a central set in* S*. If for every first entry* c *of* A*,* cS *is a central* set then there exist sequences* $\langle x_{1,n}\rangle_{n=1}^{\infty}, \langle x_{2,n}\rangle_{n=1}^{\infty}, \ldots.., \langle x_{v,n}\rangle_{n=1}^{\infty}$ *in* S *such that for every* $F \in \mathcal{P}_f(\mathbb{N})$,

$$\vec{x}_F \in (S \setminus \{0\})^v \text{ and } A\vec{x}_F \in C^u, \text{ where } \vec{x_F} = \begin{pmatrix} \sum_{n\in F} x_{1,n} \\ \sum_{n\in F} x_{2,n} \\ \vdots \\ \sum_{n\in F} x_{v,n} \end{pmatrix}.$$

Let $u, v \in \mathbb{N} \cup \{\omega\}$ and A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Also assume that each row of A has all but finitely many nonzero entries. Given $\vec{x} \in \mathbb{N}^v$ and $\vec{y} \in \mathbb{N}^u$, we write $\vec{x}^A = \vec{y}$ to mean that for $i \in \{0, 1, \ldots.., u-1\}$, $\prod_{j=0}^{v-1} x_j^{a_{ij}} = y_i$, where $A = (a_{ij})$ and $\vec{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \end{pmatrix}$.

We now state the following [7, Theorem 15.20].

Theorem 3.2. *Let* $u, v \in \mathbb{N}$ *and* C *be a* $u \times v$ *matrix with entries from* $\mathbb{Z}$*. Then the following statements are equivalent.*
(a) The matrix C *is kernel partition regular over* $(\mathbb{N}, +)$.
(b) The matrix C *is kernel partition regular over* $(\mathbb{N}, \cdot)$*. That is whenever* $r \in \mathbb{N}$ *and* $\mathbb{N}\setminus\{1\} = \bigcup_{i=1}^{r} D_i$*, there exist* $i \in \{1, 2, \ldots.., r\}$ *and* $\vec{x} \in (D_i)^v$ *such that* $\vec{x}^C = \vec{1}$.
(c) The matrix C *satisfies the columns condition over* $\mathbb{Q}$.

We recall the following definition [8, Definition 3.1].

Definition 3.3. Let A be a $\omega\times\omega$ matrix with entries from $\mathbb{Q}$. Then A is a *segmented image partition regular matrix* if

(1) no row of A is $\vec{0}$;

(2) for each $i \in \omega$, $\{j \in \omega : a_{ij} \neq 0\}$ is finite; and

(3) there is an increasing sequence $\langle \alpha_n\rangle_{n=0}^{\infty}$ in ω such that $\alpha_0 = 0$ and for each $n \in \omega$, $\{(a_{i,\alpha_n}, a_{i,\alpha_{n+1}}, \ldots.., a_{i,\alpha_{n+1}-1}) : i \in \omega\} \setminus \{\vec{0}\}$ is empty or is the set of rows of a finite image partition regular matrix.

If each of these finite image partition regular matrices is a first entries matrix, A is called a *segmented first entries matrix*. If also the first nonzero entry of each $(a_{i,\alpha_n}, a_{i,\alpha_{n+1}}, ..., a_{i,\alpha_{n+1}-1})$ is 1, then A is a *monic segmented first entries matrix*.

Lemma 3.4. *Let A be a monic segmented first entries matrix with entries from ω and let C be a central subset of $(\mathbb{N}, \cdot)$. Then there exists $\vec{x} \in \mathbb{N}^\omega$ such that $\vec{x}^A \in C^\omega$.*

Proof. Let $\vec{c_0}, \vec{c_1}, \vec{c_2}, ...$ denote the columns of A. Let $\langle \alpha_n \rangle_{n=0}^\infty$ be an increasing sequence as in the definition of a monic segmented first entries matrix [Definition 3.3]. For $n \in \omega$, let A_n be the matrix whose columns are $\vec{c}_{\alpha_n}, \vec{c}_{\alpha_n+1}, \vec{c}_{\alpha_n+2}, ..., \vec{c}_{\alpha_{n+1}-1}$. Then the set of nonzero rows of A_n is finite and, if nonempty, is the set of rows of a finite monic first entries matrix. (A_n may contain infinitely many nonzero rows but only finitely many rows are distinct.) Let B_n=$(A_0\ A_1\ .....\ A_n)$. Let C be a central subset of $(\mathbb{N}, \cdot)$ and p be a minimal idempotent in $(\beta\mathbb{N}, \cdot)$ such that $C \in p$. Let $C^* = \{n \in C : n^{-1}C \in p\}$. Then $C^* \in p$ and for every $n \in C^*$, $n^{-1}C^* \in p$ (by Lemma 4.14, [7]). Now by Theorem 3.1, we can choose $\vec{x}^{(0)} \in \mathbb{N}^{\alpha_1-\alpha_0}$ such that if $\vec{y} = (\vec{x}^{(0)})^{A_0}$ then $y_i \in C^*$ for every $i \in \omega$ for which the i^{th} row of A_0 is nonzero. We now make the induction assumption that, for some $m \in \omega$, we have chosen $\vec{x}^{(0)}$, $\vec{x}^{(1)}$,.....,$\vec{x}^{(m)}$ such that if $i \in \{0, 1, 2, ..., m\}$ then $\vec{x}^{(i)} \in \mathbb{N}^{\alpha_{i+1}-\alpha_i}$, and if $\vec{z} = \begin{pmatrix} \vec{x}^{(0)} \\ \vec{x}^{(1)} \\ \vdots \\ \vec{x}^{(m)} \end{pmatrix}$ and $\vec{y} = \vec{z}^{B_m}$ then $y_j \in C^*$ for every $j \in \omega$ for which the j^{th} row of B_m is nonzero. Let $D = \{j \in \omega : j^{th}$ row of B_m is nonzero$\}$ and note that for each $j \in \omega$, we have $y_j^{-1}C^* \in p$. By Theorem 3.1, we can choose $\vec{x}^{(m+1)} \in \mathbb{N}^{\alpha_{m+2}-\alpha_{m+1}}$ such that if $\vec{z} = (\vec{x}^{(m+1)})^{A_{m+1}}$ then $z_j \in C^* \cap (\bigcap_{t \in D} y_t^{-1}C^*)$ for every $j \in \omega$ for which the j^{th} row of A_{m+1} is nonzero and is equal to 1 otherwise. Thus we can choose an infinite sequence $\langle \vec{x}^{(i)} \rangle_{i=0}^\infty$ such that if $i \in \omega$ then $\vec{x}^{(i)} \in \mathbb{N}^{\alpha_{i+1}-\alpha_i}$, and if $\vec{z} = \begin{pmatrix} \vec{x}^{(0)} \\ \vec{x}^{(1)} \\ \vdots \\ \vec{x}^{(i)} \end{pmatrix}$ and $\vec{y} = \vec{z}^{B_i}$ then $y_j \in C^*$ for every $j \in \omega$ for which the j^{th} row of B_m is nonzero. Let $\vec{x} = \begin{pmatrix} \vec{x}^{(0)} \\ \vec{x}^{(1)} \\ \vec{x}^{(2)} \\ \vdots \end{pmatrix}$ and let $\vec{y} = \vec{x}^A$. We note that for every $j \in \omega$, there exists $m \in \omega$ such that y_j is the j^{th} entry of $\vec{z} = \begin{pmatrix} \vec{x}^{(0)} \\ \vec{x}^{(1)} \\ \vdots \\ \vec{x}^{(i)} \end{pmatrix}$ and $\vec{y} = \vec{z}^{B_i}$ whenever

$i > m$. Thus, all the entries of $\vec{y}$ are in C^*. □

Theorem 3.5. *Let A be a $\omega \times \omega$ monic segmented first entries matrix with entries from ω. Let $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} E_i$ be an r-coloring of $\mathbb{N}$. Then there exist $i \in \{1, 2, ..., r\}$ and vectors $\vec{x}, \vec{y} \in \mathbb{N}^{\omega}$ such that all the elements of $A\vec{x}$ and $\vec{y}^A$ are in E_i.*

Proof. Since $I(A)$ is a left ideal of $(\beta\mathbb{N}, \cdot)$, we can choose $p \cdot p = p \in K(\beta\mathbb{N}, \cdot) \bigcap I(A)$. Also since $\mathbb{N} = \bigcup_{i=1}^{n} E_i$, there exists $i \in \{1, 2, ..., r\}$ such that $E_i \in p$. As $E_i \in p$ and $p \in I(A)$, we can choose $\vec{x} \in \mathbb{N}^{\omega}$ such that $A\vec{x} \in E_i^{\omega}$. Also the facts that $E_i \in p$ and $p \in K(\beta\mathbb{N}, \cdot)$ imply that E_i is multiplicatively central in $(\mathbb{N}, \cdot)$. Now by the previous lemma, we can find $\vec{y} \in \mathbb{N}^{\omega}$ such that $\vec{y}^A \in E_i^{\omega}$. So our claim is proved. □

We note that the Theorem 1.4 follows as a corollary of the above theorem.

Theorem 3.6. *Let $u, v \in \mathbb{N}$ and A be a $u \times v$ monic first entries matrix with entries from ω. Let $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} E_i$ be an r-coloring of $\mathbb{N}$. Then there exist $i \in \{1, 2, ..., r\}$ and $\vec{x}, \vec{y} \in \mathbb{N}^{v}$ such that $A\vec{x} \in E_i^{u}$ and $\vec{y}^A \in E_i^{u}$.*

Proof. The proof is almost the same as that of Theorem 3.5. □

We now raise the following question.

Question 3.7. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ monic first entries matrix with entries from ω. Let $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} E_i$ be an r-coloring of $\mathbb{N}$. Does there exist $\vec{x} \in \mathbb{N}^{u}$ such that $A\vec{x} \in E_i^{u}$ and $\vec{x}^A \in E_i^{u}$ for some $i \in \{1, 2, ..., r\}$?*

If the question 3.7 is true, one may extend it by taking $u, v \in \mathbb{N} \cup \{\omega\}$ by considering A to be a monic segmented first entries matrix.

The following theorem is the kernel partition regular version of Theorem 3.7.

Theorem 3.8. *Let $u, v, r \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ satisfying the columns condition over $\mathbb{Z}$. If $\mathbb{N} = \bigcup_{i=1}^{r} E_i$ then there exist $i \in \{1, 2, ..., r\}$ and $\vec{x}, \vec{y} \in E_i^{u}$ such that $A\vec{x} = \vec{0}$ and $\vec{y}^A = \vec{1}$.*

Proof. Imitate the proof of Theorem 3.5 using Lemma 2.8 and [7, Theorem 15.16(a)]. □

We devote the remaining portion of this section to investigate some Ramsey theoretic properties concerning product of sums or sum of products arising from matrices. Henceforth, all the matrices under consideration are with rational entries.

Note that $I(A)$ contains all minimal idempotents of $(\beta\mathbb{N}, +)$ for all $A \in \mathcal{F}$ where $\mathcal{F}$ denotes the set of all finite image partition regular matrices over $\mathbb{Q}$. Hence

$I=\bigcap_{A\in\mathcal{F}} I(A)$ contains all the minimal idempotents of $(\beta\mathbb{N},+)$. In Theorem 3.16, we show that multiple structures induced by image partition regular matrices are contained in one cell of a finite coloring of $\mathbb{N}$.

Definition 3.9. Let $m\in\mathbb{N}$ be given and $\vec{y}^{(i)}\in\mathbb{N}^{u_i}$, $u_i\in\mathbb{N}\cup\{\omega\}$ for $i\in\{1,\cdots,m\}$. Also let $\langle x_t\rangle_{t=1}^{\infty}$ be a sequence in $\mathbb{N}$. Then

(a) $P(\vec{y}^{(1)},\vec{y}^{(2)},.....,\vec{y}^{(m)})=\{\prod_{i=1}^{m} y_i : y_i$ is an entry of $\vec{y}^{(i)}$ for all $i=1,2,....,m\}$.

(b) $S(\vec{y}^{(1)},\vec{y}^{(2)},.....,\vec{y}^{(m)})=\{\sum_{i=1}^{m} y_i : y_i$ is an entry of $\vec{y}^{(i)}$ for all $i=1,2,....,m\}$.

(c) $PS_m(\langle x_t\rangle_{t=1}^{\infty})=\{\prod_{i=1}^{m}\sum\limits_{t\in F_i} x_t : F_1,F_2,.....,F_m\in\mathcal{P}_f(\mathbb{N})$ and $\max F_i<\min F_{i+1}$ for all $i=1,....,m-1\}$.

By $y\in\vec{y}$, we mean y is an entry of $\vec{y}$.

Lemma 3.10. *Let $m\in\mathbb{N}$. For each $i\in\{1,2,.....,m\}$, let A_i be a $u_i\times v_i$ image partition regular matrix over $\mathbb{N}$ where $u_i,v_i\in\mathbb{N}$. If $U\in p^m$ where $p\in I=\bigcap_{A\in\mathcal{F}} I(A)$, there exists $\vec{x}^{(i)}\in\mathbb{N}^{u_i}$, $1\le i\le m$, such that $P(A_1\vec{x}^{(1)},A_2\vec{x}^{(2)},.....,A_m\vec{x}^{(m)})\subseteq U$.*

Proof. We shall prove this theorem by induction on m. Clearly, by definition of I, the theorem is true for $m=1$. Let the theorem be true for $m=n$. Let $U\in p^{n+1}$ and for each $i\in\{1,2,.....,n+1\}$, let A_i be a $u_i\times v_i$ image partition regular matrix over $\mathbb{N}$. Now $U\in p^{n+1}$ implies that $\{x\in\mathbb{N}: x^{-1}U\in p\}\in p^n$. Thus by induction hypothesis, there exists $\vec{x}^{(i)}\in\mathbb{N}^{v_i}$ for $1\le i\le n$, such that $P(A_1\vec{x}^{(1)},A_2\vec{x}^{(2)},.....,A_n\vec{x}^{(n)})\subseteq\{x\in\mathbb{N}: x^{-1}U\in p\}$. Let $\vec{y}^{(i)}=A_i\vec{x}^{(i)}$ for $1\le i\le n$. Then $(\prod_{i=1}^{n} y_i)^{-1}U\in p$ whenever $y_i\in\vec{y}^{(i)}$ for $1\le i\le n$. Let $Y=\{\prod_{i=1}^{n} y_i : y_i\in\vec{y}^{(i)}$ for $1\le i\le n\}$. Since $\vec{y}^{(i)}$ is finite for each $i\in\{1,2,.....,n\}$, Y is also finite. Thus we have $\bigcap_{y\in Y} y^{-1}U\in p$. Now $I\subseteq I(A_{n+1})$. Hence $p\in I(A_{n+1})$. Also since $\bigcap_{y\in Y} y^{-1}U\in p$, there exists $\vec{x}^{(n+1)}\in\mathbb{N}^{u_{n+1}}$ such that $y_{n+1}\in\bigcap_{y\in Y} y^{-1}U$ for all $y_{n+1}\in\vec{y}^{(n+1)}$ where $\vec{y}^{(n+1)}=A\vec{x}^{(n+1)}$. It is easy to see that $\prod_{i=1}^{n+1} y_i\in U$ for all $y_i\in\vec{y}^{(i)}$ for $1\le i\le n+1$. So $P(\vec{y}^{(1)},\vec{y}^{(2)},.....,\vec{y}^{(n+1)})\subseteq U$. Therefore $P(A_1\vec{x}^{(1)},A_2\vec{x}^{(2)},.....,A_{n+1}\vec{x}^{(n+1)})\subseteq U$. This completes the proof. $\square$

Note that in above theorem, we need not assume p to be a minimal idempotent of $(\beta\mathbb{N},+)$. We now prove a similar version of Lemma 3.10 by replacing one of the finite image partition regular matrices by an infinite image partition regular matrix.

Lemma 3.11. *Let $m\in\mathbb{N}$ and let for each $i\in\{1,2,.....,m\}$, A_i be a $u_i\times v_i$ image partition regular matrix where $u_i,v_i\in\mathbb{N}$. Also let B be any infinite image partition regular matrix. If $U\in p^m\cdot q$ where $p\in I=\bigcap_{A\in\mathcal{F}} I(A)$ and $q\in I(B)$ then there exist $\vec{x}^{(i)}\in\mathbb{N}^{u_i}$ for each $i\in\{1,2,.....,m\}$ and $\vec{x}^{(m+1)}\in\mathbb{N}^{\omega}$ such that $P(A_1\vec{x}^{(1)},A_2\vec{x}^{(2)},.....,A_m\vec{x}^{(m)},B\vec{x}^{(m+1)})\subseteq U$.*

Proof. Let $U\in p^m\cdot q$. Then $\{x\in\mathbb{N}: x^{-1}U\in q\}\in p^m$. By Lemma 3.10, there exists $\vec{x}^{(i)}\in\mathbb{N}^{u_i}$, $1\le i\le m$, such that $P(A_1\vec{x}^{(1)},A_2\vec{x}^{(2)},.....,A_m\vec{x}^{(m)})\subseteq\{x\in\mathbb{N}: x^{-1}U\in q\}$. Now let for each $i\in\{1,2,.....,m\}$, $y_i=A\vec{x}^{(i)}$. For simplicity, let

$Y = P(A_1\vec{x}^{(1)}, A_2\vec{x}^{(2)}, ..., A_m\vec{x}^{(m)})$. Since A_i is a finite image partition regular matrix for each $i \in \{1, 2, ..., m\}$, Y is finite. Thus we have $\bigcap_{y\in Y} y^{-1}U \in q$. Also since $q \in I(B)$, there exists $\vec{x}^{(m+1)} \in \mathbb{N}^\omega$ such that $y_{m+1} \in \bigcap_{y\in Y} y^{-1}U$ for all $y_{m+1} \in \vec{y}^{(m+1)}$ where $\vec{y}^{(m+1)} = B\vec{x}^{(m+1)}$. Hence $yy_{m+1} \in U$ for all $y \in Y$ and $y_{m+1} \in \vec{y}^{(m+1)}$. Therefore $P(A_1\vec{x}^{(1)}, A_2\vec{x}^{(2)}, ..., A_m\vec{x}^{(m)}, B\vec{x}^{(m+1)}) \subseteq U$. □

Corollary 3.12. *Let $r, m \in \mathbb{N}$. For each $i \in \{1, 2, ..., m\}$, let A_i be a $u_i \times v_i$ image partition regular matrix over $\mathbb{N}$; $u_i, v_i \in \mathbb{N}$. If $\mathbb{N} = \bigcup_{i=1}^r E_i$ be an r-coloring of $\mathbb{N}$ then there exist $k \in \{1, 2, ..., r\}$ and $\vec{x}^{(i)} \in \mathbb{N}^{v_i}$ for $1 \le i \le m$, such that $P(A_1\vec{x}^{(1)}, A_2\vec{x}^{(2)}, ..., A_m\vec{x}^{(m)}) \subseteq E_k$.*

Proof. Let p be a minimal idempotent of $(\beta\mathbb{N}, +)$. Choose $k \in \{1, 2, ..., r\}$ such that $E_k \in p^m$ and use Lemma 3.10. □

Theorem 3.13 is an analogous version of Corollary 3.12 for the sums of products induced by a certain class of image partition regular matrices.

Theorem 3.13. *Let $m, r \in \mathbb{N}$ be given and for each $i \in \{1, 2, ..., m\}$, let A_i be a $u_i \times v_i$ monic first entries matrix; $u_i, v_i \in \mathbb{N}$. If $\mathbb{N} = \bigcup_{i=1}^r E_i$ then there exists $\vec{x}^{(i)} \in \mathbb{N}^{u_i}$, $1 \le i \le m$, such that $S((\vec{x}^{(1)})^{A_1}, (\vec{x}^{(2)})^{A_2}, ..., (\vec{x}^{(m)})^{A_m}) \subseteq E_i$ for some $i \in \{1, 2, ..., m\}$.*

Proof. See proof of Lemma 3.10 and Corollary 3.12. □

Theorem 3.14. *Let $p+p = p \in \beta\mathbb{N}$. Let $m \in \mathbb{N}$ and let $U \in p^m$. Then there exists a sequence $\langle x_t\rangle_{t=1}^\infty$ in $\mathbb{N}$ such that $PS_m(\langle x_t\rangle_{t=1}^\infty) \subseteq U$.*

Proof. Imitate the proof of [7, Theorem 17.24]. □

In the following theorem one does not require p to be an additive idempotent.

Theorem 3.15. *Let $\langle x_t\rangle_{t=1}^\infty$ be a sequence in $\mathbb{N}$. If $p \in \bigcap_{k=1}^\infty \overline{FS(\langle x_t\rangle_{t=k}^\infty)}$, then for all n and k in $\mathbb{N}$, $PS_n(\langle x_t\rangle_{t=k}^\infty) \in p^n$.*

Proof. Imitate the proof of [7, Theorem 17.26]. □

Theorem 3.16. *Let $r, m \in \mathbb{N}$ be given and $\mathbb{N} = \bigcup_{i=1}^n E_i$ be an r-coloring of $\mathbb{N}$. Suppose for each $i \in \{1, 2, ..., m\}$, A_i is a $u_i \times v_i$ image partition regular matrix over $(\mathbb{N}, +)$ where $u_i, v_i \in \mathbb{N}$. Then there exist $k \in \{1, 2, ..., m\}$; $\vec{x}^{(i)}, \vec{y}^{(i)} \in \mathbb{N}^v$ for $i \in \{1, 2, ..., m\}$ and $\langle z_t\rangle_{t=1}^\infty$ such that $A_i\vec{x}^{(i)} \in E_k^{u_i}$ for $i \in \{1, 2, ..., m\}$ and $P(A_1\vec{y}^{(1)}, A_2\vec{y}^{(2)}, ..., A_m\vec{y}^{(m)})$, $PS_m(\langle z_t\rangle_{t=1}^\infty) \subseteq E_k$.*

Proof. For each $i \in \{1, 2, ..., m\}$, A_i is a finite image partition regular matrix over $(\mathbb{N}, +)$ and hence by Lemma 2.2, $I(A_i) \neq \emptyset$ and is a compact sub-semigroup of $(\beta\mathbb{N}, +)$ for each $i \in \{1, 2, ..., m\}$. Also observe that $E(K(\beta\mathbb{N}, +)) \subseteq I(A_i)$ for each $i \in \{1, 2, ..., m\}$. Therefore $\bigcap_{i=1}^m I(A_i) \neq \emptyset$ and is a sub-semigroup of $(\beta\mathbb{N}, +)$.

Now choose an idempotent p of $(\beta\mathbb{N},+)$ such that $p \in \bigcap_{i=1}^{m} I(A_i)$. Also $\bigcap_{i=1}^{m} I(A_i)$ is a two-sided ideal of $(\beta\mathbb{N},\cdot)$ because $I(A_i)$ is a two-sided ideal of $(\beta\mathbb{N},\cdot)$ for each $i \in \{1,.....,m\}$ by Lemma 2.3(b). Thus $p^m \in \bigcap_{i=1}^{m} I(A_i)$. As $\mathbb{N} = \bigcup_{i=1}^{n} E_i$, choose $E_k \in p^m$ for some $k \in \{1,2,.....,r\}$. Then
(a) For each $i \in \{1,2,.....,m\}$ there exists $\vec{x}^{(i)} \in \mathbb{N}^v$ such that $A_i\vec{x}^{(i)} \in E_k^{u_i}$, because $E_k \in p^m$ and $p^m \in \bigcap_{i=1}^{m} I(A_i)$.
(b) Since $E_k \in p^m$, by Lemma 3.11, for each $i \in \{1,2,.....,m\}$ there exists $\vec{y_i} \in \mathbb{N}_i^v$ such that $P(A_1\vec{y}^{(1)}, A_2\vec{y}^{(2)}, ..., A_m\vec{y}^{(m)}) \subseteq E_k$.
(c) As $E_k \in p^m$, by Theorem 3.14, there exists a sequence $\langle z_t\rangle_{t=1}^{\infty}$ such that $PS_m(\langle z_t\rangle_{t=1}^{\infty}) \subseteq E_k$. This completes the proof. $\square$

A similar result will also be true if we replace $P(A_1\vec{y}^{(1)}, A_2\vec{y}^{(2)}, ..., A_m\vec{y}^{(m)})$ by $S((\vec{y}^{(1)})^{A_1}, (\vec{y}^{(2)})^{A_2}, ..., (\vec{y}^{(m)})^{A_m})$ and $PS_m(\langle z_t\rangle_{t=1}^{\infty})$ by $SP_m(\langle z_t\rangle_{t=1}^{\infty})$ in the previous theorem.

At the end of our paper we raise the following question.

Question 3.17. *Let $m \in \mathbb{N} \setminus \{1\}$. Does there exist a finite partition $\mathcal{R}$ of $\mathbb{N}$ such that given any $A \in \mathcal{R}$ there do not exist one-to-one sequences $\langle x_t\rangle_{t=1}^{\infty}$ and $\langle y_t\rangle_{t=1}^{\infty}$ with $PS_m(\langle x_t\rangle_{t=1}^{\infty}) \subseteq A$ and $PS_1(\langle y_t\rangle_{t=1}^{\infty}) \subseteq A$?*

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