ON PARTITION CONFIGURATIONS OF ANDREWS-DEUTSCH

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Abstract

We consider *partition configurations* introduced recently by Andrews and Deutsch in connection with the Stanley-Elder theorems. Using a variation of Stanley's original technique we give a combinatorial proof of the equality of the number of times an integer k appears in all partitions and the number of partition configurations of length k. Then we establish new generalizations of the Elder and configuration theorems. We also consider a related result asserting the equality of the number of $2k$'s in partitions and the number of unrepeated multiples of k, providing a new proof and a generalization.

1. Introduction

A partition of a positive integer n is a representation of n as a sum of positive integers without regard to order. The summands are called parts, and n is the weight, of the partition [3][1]. A partition will be written as a nondecreasing sequence of its parts. Thus a partition λ of n into k parts will be expressed as

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k),\ 1 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k. \tag{1}$$

The number of parts function $g_k(n)$ is defined as the total number of k's appearing in all partitions of n.

We consider a general combinatorial technique for proving a wide class of partition identities of which the following results are prototypical examples.

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Stanley's Theorem. *The number of different parts in all partitions of n is equal to* $g_1(n)$.

Elder's Theorem. *The number of different parts repeated k or more times in all partitions of n is equal to* $g_k(n)$.

Andrews and Deutsch [2] recently devised a proof technique for such identities using "partition configurations" (defined below), and stated a parallel result based on the divisibility of parts. Gilbert [7] explored the origins of the Stanley and Elder theorems and indicated that the theorems were originally discovered by N. J. Fine [6, 5]. Dastidar and Gupta [4] considered certain generalizations of the theorems and developed Ramanujan-type congruence properties for $g_k(n)$. Further relevant work on this problem may be found in Knopfmacher and Munagi [8].

In this paper, we give combinatorial proofs of the main results in [2] and establish new generalizations. Our bijective proofs rely mostly on variations of Stanley's proof of Elder's theorem [10].

Definition. A *partition configuration*, A, is a finite nondecreasing sequence of non-negative integers containing 0. The weight of a partition configuration $A = (a_1, \ldots, a_k)$, of length k, is given by $w(A) = a_1 + a_2 + \cdots + a_k$.

Definition. A partition λ is said to contain a partition configuration $(a_1, \ldots, a_k)$ if there is a distinct subsequence of parts of λ of the form $a_1 + j, a_2 + j, \ldots, a_k + j$ for some integer $j > 0$.

For example, the partition $(1, 2, 2, 4, 4, 5, 8, 9, 9)$ contains an instance of $A = (0, 3, 6, 7)$ because the parts $2, 5, 8, 9$ exceed by 2 the successive entries of A.

The first main result in [2] is the configuration theorem:

Theorem 1. *(Andrews-Deutsch) Let A be a partition configuration of length k. The total number of configurations A in all partitions of n is equal to* $g_k(n - w(A))$.

The second main result concerns divisibility of parts:

Theorem 2. *(Andrews-Deutsch) Given $k \geq 1$, in each partition of n we count the number of times a part divisible by k appears uniquely (i.e. is not a repeated part); then sum these numbers over all the partitions of n. The result is equal to* $g_{2k}(n + k)$.

In Section 2 we state a reformulation of Theorem 1 and discuss the consequences and proofs. In Section 3 we present generalizations of Elder's theorem and Theorem 1. Section 4 is devoted to a combinatorial proof of Theorem 2. An extension of the theorem is proved using generating functions. Lastly, Section 5 provides additional properties of the function $g_k(n)$.

2. Combinatorial Proof of the Configuration Theorem

Theorem 1 depends on the weight $w(A)$ and length of a partition configuration A but not on specification of the parts. Since $0 \in A$, we can recover A from any of its occurrences in a partition λ. Thus if $(b_1, \ldots, b_k) \subseteq \lambda$ represents an occurrence of A, then $A = (0, b_2 - b_1, \ldots, b_k - b_1)$, an expression containing $k - \ell$ initial zeros and a partition of $w(A)$ into ℓ parts, $0 \leq \ell < k$. So Theorem 1 does not rely on the length k as a stringent defining property of A but rather as a preferred measure for traversing partition subsequences. Thus a configuration may be identified with the partition determined by its nonzero parts. The foregoing observations lead to the following definition.

Definition. Given a positive integer k and a partition $\beta = (\beta_1, \ldots, \beta_\ell)$, $0 \leq \ell < k$, a *translate* of β, of length k, is any k-part partition of the form

$$A(\beta, k)_j = (j, \ldots, j, \beta_1 + j \ldots, \beta_\ell + j),$$

where j is a positive integer and appears with multiplicity $k - \ell > 0$ as a part.

We see that a partition λ contains a configuration

$$A = A(\beta, k)_0 = (0, \ldots, 0, \beta_1, \ldots, \beta_\ell)$$

if and only if the sequence of parts of λ contains a distinct translate of the underlying (possibly empty) partition β.

Theorem 1 then implies the following more inclusive statement.

Theorem 3. *Let n, m, k be positive integers with $k \leq n$, $0 \leq m < n$, and let β be a partition of m into less than k parts. The number of distinct translates of β, of length k, in all partitions of n is equal to the number of k's in all partitions of $n - m$.*

Note that Theorem 1 may be obtained from Theorem 3 by specifying β with k. For example if $m = 5$ and $k = 4$, the translates of each

$$\beta \in \{(5), (1,4), (2,3), (1,1,3), (1,2,2)\}$$

give the same number of 4's in partitions of $n \geq 9$, where $\beta = (5) \implies A = (0,0,0,5)$, $\beta = (1,4) \implies A = (0,0,1,4)$, and so forth.

We remark that the generating function proof of Theorem 1 given in [2] is sufficient to prove Theorem 3 since m is equal to the weight $w(A)$ of any partition configuration A with the given length. So we present only a bijective proof in this section.

We first note some proof applications of Theorem 3. The set of translates of $\beta = (\beta_1, \cdots, \beta_\ell)$ will be denoted by $A(\beta, k)$:

$$A(\beta, k) = \{(1, \ldots, 1, \beta_1 + 1 \ldots, \beta_\ell + 1), (2, \ldots, 2, \beta_1 + 2 \ldots, \beta_\ell + 2), \ldots\}.$$

Stanley's Theorem: Take $\beta = \emptyset$ and $k = 1$; thus $m = 0$ and

$$A(\emptyset, 1) = \{(1), (2), \dots\}.$$

Elder's Theorem: Take $\beta = \emptyset$ and $k \geq 1$; thus $m = 0$ and

$$A(\emptyset, k) = \{(1, \dots, 1), (2, \dots, 2), \dots\}.$$

The following result was discovered independently by Knopfmacher-Munagi [8] and Andrews-Deutsch [2]:

The number of sequences of elements of a multiset of k consecutive integers in all partitions of n is equal to $g_k(n - \binom{k}{2})$.

To prove the statement take $\beta = (1, 2, \dots, k-1)$ with $k \geq 1$; thus $m = \binom{k}{2}$ and $A_k(\beta, k) = \{(1, 2, \dots, k), (2, 3, \dots, k+1), \dots\}$.

Proof of Theorem 3. Let $T(n, A(\beta, k))$ denote the multiset of translates of β of length k in partitions of n, and let $G_k(n)$ be the multiset of k's in partitions of n, that is $|G_k(n)| = g_k(n)$.

We describe a bijection $\theta : T(n, A(\beta, k)) \to G_k(n - m)$ as follows.

If $\beta = (\beta_1, \dots, \beta_\ell) \vdash m$, then $(j, \dots, j, \beta_1 + j, \dots, \beta_\ell + j) \in T(n, A(\beta, k))$, and

$$\theta : \ (j, \dots, j, \beta_1 + j, \dots, \beta_\ell + j) \longmapsto \underbrace{k, \dots, k}_{j \text{ copies}}. \tag{2}$$

In practical terms, we write down all partitions λ of n containing a length k translate of β, writing down each partition λ of n as many times as there are length k translates of β within it. For each of these partitions take an instance of $(j, \dots, j, \beta_1 + j, \dots, \beta_\ell + j)$ within it, and remove these parts and replace them by j parts equal to k. This produces a list of partitions of $n - m$ in which each partition containing r parts equal to k occurs exactly r times.

Conversely, if a partition $\gamma \vdash n - m$ contains r copies of k, then for each $j \in \{1, \dots, r\}$, we use (2) to map γ to a partition of n containing j translates of a fixed partition of m. So γ produces r partition pre-images.

This gives the asserted bijection. $\square$

The bijection is illustrated in Table 1 for $n = 11$, $\beta = (3)$ and $k = 3$. Using a larger value of n, say $n = 15$, then, for instance, $(4, 4, 7)$ with translate $(4, 4, 7)$ maps to $(3, 3, 3, 3)$ while $(1, 1, 2, 2, 4, 5)$ with translates $(1, 1, 4)$, $(2, 2, 5)$ maps to $(2, 2, 3, 5)$ and $(1, 1, 3, 3, 4)$ respectively. Conversely $(3, 3, 3, 3)$ has the following pre-images corresponding respectively to $1, 2, 3$ and 4 copies of 3: $(1, 1, 3, 3, 3, 4)$, $(2, 2, 3, 3, 5), (3, 3, 3, 6), (4, 4, 7)$, and so forth.

Note the following property of Table 1 which is analogously shared by all such tables:

$T(11, A((3), 3))$	translates	$G_3(8)$
(1, 1, 4, 5)	(1, 1, 4)	(3, 5)
(2, 2, 2, 5)	(2, 2, 5)	(2, 3, 3)
(1, 1, 1, 4, 4)	(1, 1, 4)	(1, 3, 4)
(1, 1, 2, 2, 5)	(2, 2, 5)	(1, 1, 3, 3)
(1, 1, 2, 3, 4)	(1, 1, 4)	(2, 3, 3)
(1, 1, 1, 1, 3, 4)	(1, 1, 4)	(1, 1, 3, 3)
(1, 1, 1, 2, 2, 4)	(1, 1, 4)	(1, 2, 2, 3)
(1, 1, 1, 1, 1, 2, 4)	(1, 1, 4)	(1, 1, 1, 2, 3)
(1, 1, 1, 1, 1, 1, 1, 4)	(1, 1, 4)	(1, 1, 1, 1, 1, 3)

Table 1: The bijection $T(11, A(\beta), 3) \to G_3(8)$, where $\beta = (3)$

"*Each partition in the first column appears as many times as the number of β-translates it contains and each partition in the third column appears as many times as the number of k's it contains*".

Remark 1. The map θ can be factored into a composition of two bijections as follows:
A de-configuration or leveling map: $\rho : T(n, A(\beta, k)) \to T(n - m, A(\emptyset, k))$, where

$$\rho : \ (a_1, \ldots, a_k) \longmapsto (a_1, \ldots, a_k) - A(\beta, k)_0 = (a_1, a_1, \ldots, a_1).$$

The Elder map: $\varepsilon : T(n, A(\emptyset, k)) \to G_k(n)$, where

$$\varepsilon : \ (a, \ldots, a) \longmapsto \underbrace{k, \ldots, k}_{a \text{ copies}}.$$

The bijection ε (strictly $\varepsilon = \theta_{|(j,\ldots,j)}$) was popularized by Richard Stanley [10] who used it to prove Elder's theorem.

Then we see that $\theta = \varepsilon\rho$.

3. Generalization of the Elder and Configuration Theorems

In this section we give natural extensions of Elder's theorem and Theorem 3.

Let $v_k(n, t)$ denote the number of multiples of k appearing at least t times in all partitions of n. Thus Elder's theorem takes the compact form

$$g_k(n) = v_1(n, k). \tag{3}$$

Theorem 4. *The number of multiples of k appearing at least t times in all partitions of n equals the number of tk's in all partitions of n:*

$$v_k(n,t) = g_{tk}(n), \quad t = 1, 2, \ldots.$$

Note that Theorem 4 becomes Elder's theorem when $k = 1$.

Proof. Let $V_k(n,t)$ be the set of objects enumerated by $v_k(n,t)$. Define the map $\varepsilon_{k,t} : V_k(n,t) \to G_{tk}(n)$ as follows. If $\lambda \vdash n$ contains $r \geq t$ copies of mk, replace t copies of mk by m copies of tk:

$$\varepsilon_{k,t} : \underbrace{mk, \ldots, mk}_{t \text{ copies}} \longmapsto \underbrace{tk, \ldots, tk}_{m \text{ copies}}.$$

Conversely, if $\lambda \vdash n$ contains r copies of tk, then for each $j \in \{1, 2, \ldots, r\}$ replace j copies of tk with t copies of jk:

$$\varepsilon_{k,t}^{-1} : \underbrace{tk, \ldots, tk}_{j \text{ copies}} \longmapsto \underbrace{jk, \ldots, jk}_{t \text{ copies}}.$$

Thus $\varepsilon_{k,t}$ is a bijection. Hence the result.

The bijection is illustrated in Table 2 for $n = 12, k = 3, t = 2$. □

$V_3(12,2)$	multiples of 3	$G_6(12)$
(6, 6)	6, 6	(6, 6)
(3, 3, 6)	3, 3	(6, 6)
(1, 3, 3, 5)	3, 3	(1, 5, 6)
(2, 3, 3, 4)	3, 3	(2, 4, 6)
(1, 1, 3, 3, 4)	3, 3	(1, 1, 4, 6)
(3, 3, 3, 3)	3, 3	(3, 3, 6)
(1, 2, 3, 3, 3)	3, 3	(1, 2, 3, 6)
(1, 1, 1, 3, 3, 3)	3, 3	(1, 1, 1, 3, 6)
(2, 2, 2, 3, 3)	3, 3	(2, 2, 2, 6)
(1, 1, 2, 2, 3, 3)	3, 3	(1, 1, 2, 2, 6)
(1, 1, 1, 1, 2, 3, 3)	3, 3	(1, 1, 1, 1, 2, 6)
(1, 1, 1, 1, 1, 1, 3, 3)	3, 3	(1, 1, 1, 1, 1, 1, 6)

Table 2: The bijection $V_k(12,t) \to G_{tk}(12)$ for $k = 3$, $t = 2$.

Remark 2. Theorem 4 implies the symmetry property: $v_k(n,t) = v_t(n,k)$.

Definition. Given positive integers k, t and a partition $\beta = (\beta_1, \ldots, \beta_\ell)$, $0 \le \ell < t$, a *k-translate* of β, of length t, is any t-part partition of the form

$$A_k(\beta, t)_{jk} = (jk, \ldots, jk, \beta_1 + jk \ldots, \beta_\ell + jk),$$

where j is a positive integer and jk appears with multiplicity $t - \ell > 0$ as a part.

Thus $A_1(\beta, t)_j = A(\beta, t)_j$. We now state a generalization of Theorem 3.

Theorem 5. *Let n, m, k, t be positive integers with $t \le n$, $0 \le m < n$, and let β be a partition of m into less than t parts. The number of distinct k-translates of β, of length t, in all partitions of n is equal to the number of tk's in all partitions of $n - m$.*

Note that Theorem 5 reduces to Theorem 3 when $k = 1$.

Proof. We define a bijection $\theta_{k,t} : T(n, A_k(\beta, t)) \to G_{tk}(n - m)$. Since $\varepsilon = \varepsilon_{1,t}$ and $\theta = \theta_{1,t}$, it is clear that Remark 1 and the proof of Theorem 4 imply the definition

$$\theta_{k,t} = \varepsilon_{k,t}\rho.$$

This shows that $\theta_{k,t}$ may be realized as a composition of two bijections. □

As an illustration suppose that $k = 2, t = 5$ and $\beta = (1, 1, 4)$, then
$A_2(\beta, 5) = \{(2, 2, 3, 3, 6), (4, 4, 5, 5, 8), (6, 6, 7, 7, 10), \ldots\}$.
So if $n \ge 16$ and $(2, 2, 3, 3, 6) \in T(n, A_2(\beta, 5))$, then

$$(2, 2, 3, 3, 6) \xrightarrow{\rho} (2, 2, 2, 2, 2) \xrightarrow{\varepsilon_{2,5}} (10),$$

where $(10) \in G_{10}(n - 6)$. Similarly,

$$(4, 4, 5, 5, 8) \xrightarrow{\rho} (4, 4, 4, 4, 4) \xrightarrow{\varepsilon_{2,5}} (10, 10);$$

and so forth.

An alternative proof of Theorem 4 may be deduced from Theorem 5 as follows: take $\beta = \emptyset$, $k \ge 1$ and $t \ge 1$ so that $A_k(\beta, t) = \{(k, \ldots, k), (2k, \ldots, 2k), \ldots\}$.

4. Divisibility of Parts

This section is devoted to the proof of Theorem 2. We first establish a related result.

Theorem 6. *We have*

$$g_k(n) = g_{2k}(n) + g_{2k}(n + k).$$

Proof. The theorem is a special case ($t = 2$) of Theorem 9 below. But we give a full bijection here: $G_{2k}(n) \cup G_{2k}(n+k) \longrightarrow G_k(n)$.

If $2k \in \lambda$, then

$$2k \mapsto \begin{cases} k, k & \text{if } 2k \in G_{2k}(n), \\ k & \text{if } 2k \in G_{2k}(n+k). \end{cases}$$

The rule for $r > 1$ copies of $2k$, denoted by $(2k)^r$, is as follows:
(I) If $(2k)^r \in G_{2k}(n)$, then for each $j \in \{1, \ldots, r\}$ replace j copies of $2k$ with $2j$ copies of k.
(II) If $(2k)^r \in G(n+k)$, then for each $j \in \{1, \ldots, r\}$ replace j copies of $2k$ with $2j - 1$ copies of k.
Note that if $(2k)^r \in \lambda$, then λ begets r image partitions for $G_k(n)$ in either case.

For the inverse map let λ have $r \geq 1$ parts equal to k. Then we map λ to $\lfloor \frac{r}{2} \rfloor$ partitions of n by replacing $2j$ parts k by j parts $2k$ ($1 \leq j \leq \frac{r}{2}$), and also to $\lceil \frac{r}{2} \rceil$ partitions of $n + k$ by replacing $2j + 1$ parts k by $j + 1$ parts $2k$ for $0 \leq j < \frac{r}{2}$.

Hence the bijection.

The bijection is illustrated in Table 3. Note that if a partition λ contains r copies of k then λ appears $\lfloor \frac{r}{2} \rfloor$ times as an image of members of $G_{2k}(n)$, and $\lfloor \frac{r+1}{2} \rfloor$ times as an image of members of $G_{2k}(n+k)$. ∎

$G_4(7) \cup G_4(9)$	$\longrightarrow$	$G_2(7)$
(3,4)	$\mapsto$	(2,2,3)
(1,2,4)	$\mapsto$	(1,2,2,2)
(1,1,1,4)	$\mapsto$	(1,1,1,2,2)
(4, 5)	$\mapsto$	(2,5)
(1, 4, 4)	$\mapsto$	(1,2,4)
(1, 4, 4)	$\mapsto$	(1,2,2,2)
(2,3,4)	$\mapsto$	(2,2,3)
(1,1,3,4)	$\mapsto$	(1,1,2,3)
(1,2,2,4)	$\mapsto$	(1,2,2,2)
(1,1,1,2,4)	$\mapsto$	(1,1,1,2,2)
(1,1,1,1,1,4)	$\mapsto$	(1,1,1,1,1,2)

Table 3: The map $G_{2k}(n) \cup G_{2k}(n+k) \to G_k(n)$ for $n = 7$, $k = 2$.

4.1. Proof and Extension of Theorem 2

Define $f_k(n)$ as the number of times a multiple of k appears uniquely in all partitions of n. Then Theorem 2 takes the form

$$f_k(n) = g_{2k}(n+k). \tag{4}$$

The proof is deduced from Theorems 6 and 4:

$$g_{2k}(n+k) = g_k(n) - g_{2k}(n) = v_k(n,1) - v_k(n,2) = f_k(n).$$

□

Now consider the function

$f_k(n,s) :=$ number of multiples of k appearing exactly s times in all partitions of n.

Thus $f_k(n,1) = f_k(n)$. By definition we have

$$f_k(n,s) = v_k(n,s) - v_k(n,s+1).$$

Hence from Theorem 4 we obtain:

Theorem 7. *We have*

$$f_k(n,s) = g_{sk}(n) - g_{(s+1)k}(n).$$

Note that Theorem 7 is a generalization of Theorem 2 since Equation (4) may be stated as

$$f_k(n,1) = g_k(n) - g_{2k}(n).$$

Proof. The generating function for $g_{sk}(n)$ is given by

$$\sum_{n=0}^{\infty} g_{sk}(n)q^n = (q^{sk} + 2q^{2sk} + 3q^{3sk} + \cdots) \prod_{\substack{n=1 \\ n\neq sk}}^{\infty} (1 + q^n + q^{2n} + q^{3n} + \cdots)$$

$$= \frac{q^{sk}}{(1-q^{sk})^2} \prod_{\substack{n=1 \\ n\neq sk}}^{\infty} \frac{1}{1-q^n} = \frac{q^{sk}}{1-q^{sk}} \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

Therefore,

$$\begin{aligned}
\sum_{n=0}^{\infty}(g_{sk}(n) - g_{(s+1)k}(n))q^n &= \left(\frac{q^{sk}}{1-q^{sk}} - \frac{q^{(s+1)k}}{1-q^{(s+1)k}}\right) \prod_{n=1}^{\infty} \frac{1}{1-q^n} \\
&= \prod_{n=1}^{\infty} \frac{1}{1-q^n} \sum_{j=1}^{\infty} q^{skj}(1-q^{kj}) \\
&= \sum_{j=1}^{\infty} \frac{q^{skj}}{\prod_{\substack{n=1 \\ n\neq kj}}^{\infty}(1-q^n)} \\
&= \sum_{n=0}^{\infty} f_k(n,s)q^n.
\end{aligned}$$

Equating the coefficients of q^n on both sides gives the theorem. □

Remark 3. The function $f_1(n,s)$ has been tabulated in Sloane [9, A197126] under the description "number of cliques of size s in all partitions of $n \geq 1$" (where a "clique" refers to all parts in a partition with the same value). If we designate this as the sequence of 1-cliques, then $f_k(n,s)$ assumes the definition: "number of k-cliques of size s in all partitions of $n \geq 1$" (where a "k-clique" refers to all copies of a fixed multiple of k in a partition). Small values of some of the sequences $f_k(n,s)$ are shown in Table 4.

$f_1(n,s)$

n \ s	1	2	3	4	5
1	1	0	0	0	0
2	1	1	0	0	0
3	3	0	1	0	0
4	4	2	0	1	0
5	8	2	1	0	1
6	11	4	2	1	0
7	19	5	3	1	1
8	26	10	3	3	1
9	41	11	7	3	2
10	56	20	8	5	3

$f_2(n,s)$

n \ s	1	2	3	4	5
1	0	0	0	0	0
2	1	0	0	0	0
3	1	0	0	0	0
4	2	1	0	0	0
5	3	1	0	0	0
6	6	1	1	0	0
7	8	2	1	0	0
8	13	4	1	1	0
9	18	5	2	1	0
10	28	8	3	1	1

$f_3(n,s)$

n \ s	1	2	3
2	0	0	0
3	1	0	0
4	1	0	0
5	2	0	0
6	3	1	0
7	5	1	0
8	7	2	0
9	12	2	1
10	16	4	1
11	24	5	2

$f_4(n,s)$

n \ s	1	2	3
3	0	0	0
4	1	0	0
5	1	0	0
6	2	0	0
7	3	0	0
8	5	1	0
9	7	1	0
10	11	2	0
11	15	3	0
12	23	4	1

$f_5(n,s)$

n \ s	1	2	3
4	0	0	0
5	1	0	0
6	1	0	0
7	2	0	0
8	3	0	0
9	5	0	0
10	7	1	0
11	11	1	0
12	15	2	0
13	22	3	0

Table 4: Small values of $f_k(n,s)$ for $k = 1, 2, 3, 4$ and 5

5. Further Properties of $g_k(n)$

The summation of Theorem 7 over s results in the following identity.

Corollary 8. *The number of k-cliques in all partitions of n equals the number of k's in all partitions of n:*

$$\sum_{s \geq 1} f_k(n,s) = g_k(n).$$

A bijective proof of Corollary 8 is given by $(mk)^s \longleftrightarrow (k)^{sm}$. This bijection is equivalent to $\varepsilon\sigma_k$, where $\sigma_k : mk \mapsto (m)^k$.

If an integer k occurs as a part of $\lambda \vdash n$, one can delete k from λ to obtain an arbitrary partition of $n-k$. So the number of partitions of n containing at least j copies of k is $p(n-jk)$; the number containing exactly j copies of k is $p(n-jk)-p(n-(j+1)k)$. Therefore

$$g_k(n) = \sum_{j\geq 1} j(p(n-jk)-p(n-(j+1)k)) = \sum_{j\geq 1} p(n-jk). \tag{5}$$

An immediate consequence is

$$p(n) = g_k(n+k) - g_k(n), \tag{6}$$

since the right-hand side of (6) is equal to $\sum_{j\geq 0} p(n-jk) - \sum_{j\geq 1} p(n-jk) = p(n)$.

We will need the following extension of Equation (6) (replace k by tk, then n by $n-ik$):

$$p(n-ik) = g_{tk}(n+(t-i)k) - g_{tk}(n-ik), \tag{7}$$

where i, t are integers. From (5) and (7) we have

$$g_k(n) = \sum_{j\geq 1} g_{tk}(n+(t-j)k) - \sum_{j\geq 1} g_{tk}(n-jk).$$

Group the summations into pairs of t summands, then isolate the first pair:

$$\begin{aligned}
g_k(n) &= \sum_{i\geq 0}\left(\sum_{j=it+1}^{(i+1)t} g_{tk}(n+(t-j)k) - \sum_{j=it+1}^{(i+1)t} g_{tk}(n-jk)\right)\\
&= \sum_{j=1}^{t} g_{tk}(n+(t-j)k) - \sum_{j=1}^{t} g_{tk}(n-jk)\\
&\qquad + \sum_{i\geq 0}\sum_{j=(i+1)t+1}^{(i+2)t} g_{tk}(n+(t-j)k) - \sum_{i\geq 1}\sum_{j=it+1}^{(i+1)t} g_{tk}(n-jk)\\
&= \sum_{j=1}^{t} g_{tk}(n+(t-j)k) + \left(-\sum_{j=1}^{t} g_{tk}(n-jk) + \sum_{j=t+1}^{2t} g_{tk}(n+(t-j)k)\right)\\
&\qquad + \sum_{i\geq 1}\left(\sum_{j=(i+1)t+1}^{(i+2)t} g_{tk}(n+(t-j)k) - \sum_{j=it+1}^{(i+1)t} g_{tk}(n-jk)\right)
\end{aligned} \tag{8}$$

The two summations inside either pair of parentheses are identical with opposite signs. So only the first summation survives. Reversing the order of summation in the latter we obtain the next result (also stated in [4]).

Theorem 9. *The following identity holds for all integers $n, k, t > 0$:*

$$g_k(n) = \sum_{j=0}^{t-1} g_{tk}(n + jk).$$

Note that Theorem 9 may also be proved bijectively by extending the proof of Theorem 6. The relevant bijection $\bigcup_j G_{tk}(n+jk) \longrightarrow G_k(n)$ is obtained as follows: if $(tk)^v \in G_{tk}(n + jk)$, then for each $i \in \{1, \ldots, v\}$ replace i copies of tk with $it - j$ copies of k. The reverse transformation may be deduced analogously.

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