THE MONOTONICITY OF SOME SEQUENCES RELATED TO HYPERHARMONIC NUMBERS

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Abstract
In this paper, we discuss the properties of hyperharmonic numbers $\{H_n^{[r]}\}_{n\geq 1}$, where r is a positive integer. We focus on the monotonicity of some sequences related to $H_n^{[r]}$. For example, we consider the monotonicity of the sequences such as $\{H_n^{[r]}/n\}_{n\geq 1}$, $\{\sqrt[n]{H_n^{[r]}}\}_{n\geq 1}$, $\{\sqrt[n]{H_{n+1}^{[r]}/H_n^{[r]}}\}_{n\geq 1}$, and $\{\sqrt[n+1]{H_{n+1}^{[r]}}/\sqrt[n]{H_n^{[r]}}\}_{n\geq 1}$.

1. Introduction

Let us first review some definitions in combinatorics.

Definition 1. Let $\{z_n\}_{n\geq 0}$ be a sequence of real numbers. We say that $\{z_n\}_{n\geq 0}$ is *convex* (or *concave*) if $2z_n \leq z_{n-1} + z_{n+1}$ (or $2z_n \geq z_{n-1} + z_{n+1}$) for all $n \geq 1$.

Definition 2. Let $\{z_n\}_{n\geq 0}$ be a sequence of positive numbers. We say that $\{z_n\}$ is called log-convex (log-concave) if $z_n^2 \leq z_{n-1}z_{n+1}(z_n^2 \geq z_{n-1}z_{n+1})$ for $n \geq 1$.

Log-concavity (log-convexity) and concavity (convexity) are important sources of inequalities. For a sequence of positive real numbers, it is easy to prove that its log-convexity implies its convexity and its concavity implies its log-concavity by the arithmetic-geometric mean inequality. It is obvious that a sequence $\{z_n\}_{n\geq 0}$ is log-convex (log-concave) if and only if its quotient sequence $\{\frac{z_{n+1}}{z_n}\}_{n\geq 0}$ is nondecreasing (nonincreasing). Sun [16] posed a series of conjectures on monotonicity of combinatorial sequences of the types $\{\sqrt[n]{z_n}\}$ and $\{\sqrt[n+1]{z_{n+1}}/\sqrt[n]{z_n}\}$, where $z_n > 0$. Many conjectures of Sun [16] have been confirmed in Chen et al. [4], Hou et al. [11], and Wang and Zhu [18]. Wang and Zhu [18] prove that the monotonicity of $\{\sqrt[n]{z_n}\}$ is associated with the log-convexity (log-concavity) of $\{z_n\}$ and the value of z_0. Hou et al. [11] discussed the monotonicity of some sequences related to a class of generalized harmonic numbers.

For mth order harmonic numbers $H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m}$, the monotonicity of $\{\sqrt[n+1]{H_{n+1}^{(m)}}/\sqrt[n]{H_n^{(m)}}\}_{n\geq 3}$ is studied in Hou et al. [11]. Here, we consider the monotonicity of some sequences involving hyperharmonic numbers. It is well-known that the harmonic numbers H_n are defined by

$$\begin{aligned} H_0 &= 0, \\ H_n &= \sum_{k=1}^{n} \frac{1}{k}, \quad n \geq 1. \end{aligned}$$

The harmonic numbers often arise in the solution to combinatorial problems and in the analysis of algorithms, and they have been generalized to many forms (See Adamchik [1], Benjamin et al. [2], Cheon and El-Mikkawy [5], Chu [6], Conway and Guy [7], Gertsch [10], and Santmyer [15]). In this paper, we are interested in a class of generalized harmonic numbers $H_n^{[r]}$ (r is a nonnegative integer), called hyperharmonic numbers by Conway and Guy [7]. The definition of $H_n^{[r]}$ is

$$\begin{aligned} H_0^{[r]} &= 0, \quad r \geq 0, \\ H_n^{[0]} &= 1/n, \quad n \geq 1, \\ H_n^{[r]} &= \sum_{k=1}^{n} H_k^{[r-1]}, \quad r \geq 1 \quad \textit{and} \quad n \geq 1. \end{aligned}$$

It is clear that $H_n^{[1]} = H_n$. The generating function of $\{H_n^{[r]}\}$ is

$$\sum_{n=1}^{\infty} H_n^{[r]} z^n = -\frac{\ln(1-z)}{(1-z)^r}.$$

The initial values of $\{H_n^{[2]}\}$ are as follows:

n	0	1	2	3	4	5	6	7	8
$H_n^{[2]}$	0	1	$\frac{5}{2}$	$\frac{13}{3}$	$\frac{77}{12}$	$\frac{87}{10}$	$\frac{223}{20}$	$\frac{481}{35}$	$\frac{4609}{280}$

Hyperharmonic numbers have many combinatorial connections. For example, $H_n^{[r]}$ is related to the r-Stirling numbers $\left[{n \atop k}\right]_r$. For more properties of hyperharmonic numbers, see for instance Benjamin et al. [2] and Mezö [13, 14]. For $\left[{n \atop k}\right]_r$, Broder [3] gives the combinatorial definition:

Definition 3. The value of $\left[{n \atop k}\right]_r$ is the number of permutations of the set $\{1, 2, \cdots, n\}$ having k disjoint, non-empty cycle, in which the elements 1 through r are restricted to appear in different cycles.

We note that $\left[{n \atop k}\right]_r$ are the ordinary Stirling numbers $\left[{n \atop k}\right]$ when $r = 0$ or 1. Benjamin et al. [2] proved that

$$H_n^{[r]} = \frac{\left[{n+r \atop r+1}\right]_r}{n!}. \tag{1}$$

Identity (1) shows that hyperharmonic numbers can be expressed in terms of r-Stirling numbers. By proving the equivalent r-Stirling identities, Benjamin et al. [2] gave combinatorial proofs of many hyperharmonic identities. Hence the properties of hyperharmonic numbers deserve to be investigated. The main purpose of this paper is to study the monotonicity of some sequences related to hyperharmonic numbers such as $\{H_{n+1}^{[r]}/n\}_{n\geq 1}$, $\{\sqrt[n]{H_n^{[r]}}\}_{n\geq 1}$, $\{\sqrt[n]{H_n^{[r]}/H_n^{[r]}}\}_{n\geq 1}$, $\{\sqrt[n+1]{H_{n+1}^{[r]}}/\sqrt[n]{H_n^{[r]}}\}_{n\geq 1}$ and $\{\sqrt[j]{H_j^{[2]}/\binom{n}{j}}\}_{1\leq j\leq n}$ $(n \geq 4)$.

The following definition will be used later on.

Definition 4. (Merlini et al. [12]) Throughout, $[z^n]f(z)$ denotes the coefficient of z^n in $f(z)$, where

$$f(z) = \sum_{n=0}^{\infty} f_n z^n$$

is the generating function of the sequence $\{f_n\}_{n\geq 0}$.

2. The Monotonicity of Some Sequences Related to Hyperharmonic Numbers

In this section, we discuss the monotonicity of some sequences related to hyperharmonic numbers. We first give a lemma.

Lemma 1. *Let $\{z_n\}_{n\geq 0}$ be a sequence of real numbers. Assume that $\{z_n\}_{n\geq 0}$ is convex. If $z_0 \leq 0$, the sequence $\{\frac{z_n}{n}\}_{n\geq 1}$ is increasing.*

Proof. For $n \geq 0$, let $d_n = z_{n+1} - z_n$. It is evident that a sequence $\{z_n\}_{n\geq 0}$ is convex if and only if its difference sequence $\{d_n\}_{n\geq 0}$ is increasing. Since $\{z_n\}_{n\geq 0}$ is convex, $\{d_n\}_{n\geq 0}$ is increasing. For $n \geq 1$, we note that

$$\begin{aligned} nz_{n+1} - (n+1)z_n &= nd_n - z_n \\ &= \sum_{j=0}^{n-1}(d_n - d_j) - z_0. \end{aligned}$$

Since $z_0 \leq 0$, $nz_{n+1} - (n+1)z_n \geq 0$. Then $\frac{z_n}{n} \leq \frac{z_{n+1}}{n}$ for $n \geq 1$. Hence the sequence $\{\frac{z_n}{n}\}_{n\geq 1}$ is increasing. □

Theorem 1. *Suppose that $r \geq 2$. For the hyperharmonic numbers $H_n^{[r]}$, the sequence $\{H_n^{[r]}/n\}_{n\geq 1}$ is increasing.*

Proof. We first show that the sequence $\{H_n^{[r]}\}_{n\geq 0}$ is convex. For $n \geq 0$, let $d_n = H_{n+1}^{[r]} - H_n^{[r]}$. It follows from the definition of $H_n^{[r]}$ that $d_0 = 1$ and $d_n = H_{n+1}^{[r-1]}$

$(n \geq 1)$. Since

$$d_1 - d_0 = 1 > 0, \quad d_{n+1} - d_n = H_{n+1}^{[r-2]} > 0 \quad (n \geq 1),$$

the sequence $\{d_n\}_{n\geq 0}$ is increasing. Then $\{H_n^{[r]}\}_{n\geq 0}$ is convex. We note that $H_0^{[r]} = 0$. It follows from (i) of Lemma 1 that the sequence $\{H_n^{[r]}/n\}_{n\geq 1}$ is increasing. $\square$

Before we give the other results of this paper, we recall some lemmas which will be used later on.

Lemma 2. *If the sequences $\{a_n\}$ and $\{b_n\}$ are log-concave, then so is their ordinary convolution $z_n = \sum_{k=0}^{n} a_k b_{n-k}$, $n = 0, 1, 2, \cdots$.*

This is a well-known result for the log-concavity of sequences. For its proof, see Wang and Yeh [17]

Lemma 3. (Wang and Zhu [18]) *Assume that $\{z_n\}_{n\geq 0}$ is log-concave and $z_0 \geq 1$. Then the sequence $\{\sqrt[n]{z_n}\}_{n\geq 1}$ is decreasing.*

Lemma 4. (Flajolet and Oldyzko [8] and Flajolet et al. [9]) *Let k be a nonzero integer, α be a real number with $\alpha \notin \mathbf{Z}_{\geq 0}$, and*

$$L(z) = \ln \frac{1}{1-z}.$$

Then

$$[z^n](1-z)^{\alpha}(L(z))^k \quad \sim \quad \frac{1}{\Gamma(-\alpha)} n^{-\alpha-1} \ln^k n, \quad n \to \infty.$$

Lemma 5. (Wang and Zhu [18]) *Let $\{z_n\}_{n\geq 0}$ be a sequence of positive numbers. Assume that $\{z_n\}_{n\geq N}$ is log-convex (log-concave) and $\sqrt[N]{z_N} < \sqrt[N+1]{z_{N+1}}$ ($\sqrt[N]{z_N} > \sqrt[N+1]{z_{N+1}}$) for some $N \geq 1$. Then $\{\sqrt[n]{z_n}\}_{n\geq N}$ is strictly increasing (decreasing).*

Theorem 2. *Suppose that $r \geq 2$ is fixed. For the hyperharmonic numbers $H_n^{[r]}$, we have the following results:*

(i) the sequence $\{\sqrt[n]{H_{n+1}^{[r]}}\}_{n\geq 1}$ is decreasing;

(ii) there exists an unique positive integer $N_{1,r}$ such that the sequence $\{\sqrt[n]{H_n^{[r]}}\}_{n\geq N_{1,r}}$ is decreasing, and $\lim_{n\to+\infty} \sqrt[n]{H_n^{[r]}} = 1$;

(iii) there exists a positive integer $N_{2,r} \geq 3$ such that the sequence $\{\sqrt[n]{nH_n^{[r]}}\}_{n\geq N_{2,r}}$ is decreasing;

(iv) the sequence $\{\sqrt[n]{H_{n+1}^{[r]}/H_n^{[r]}}\}_{n\geq 1}$ is decreasing;

(v) there exists a positive integer $N_{3,r}$ such that the sequence $\{\sqrt[n+1]{H_{n+1}^{[r]}}/\sqrt[n]{H_n^{[r]}}\}_{n\geq N_{3,r}}$ is increasing.

Proof. (i) We first prove that $\{H_n^{[1]}\}_{n\geq 1} = \{H_n\}_{n\geq 1}$ is log-concave. When $k \geq 2$,

$$\begin{aligned} H_k^2 - H_{k-1}H_{k+1} &= \frac{1+H_k}{k(k+1)} \\ &> 0. \end{aligned}$$

Then $\{H_n\}_{n\geq 1}$ is log-concave. It follows from Lemma 2 that $\{H_n^{[r]}\}_{n\geq 1}$ $(r \geq 2)$ is log-concave. For $n \geq 0$, let $z_n = H_{n+1}^{[r]}$. Clearly, $z_0 = 1$. It follows from Lemma 3 that $\{\sqrt[n]{z_n}\}_{n\geq 1} = \{\sqrt[n]{H_{n+1}^{[r]}}\}_{n\geq 1}$ is decreasing.

(ii) For $n \geq 1$, it is clear that

$$\begin{aligned} (n+1)\ln H_n^{[r]} - n\ln H_{n+1}^{[r]} &= (n+1)\ln H_n^{[r]} - n\ln(H_n^{[r]} + H_{n+1}^{[r-1]}) \\ &= \ln H_n^{[r]} - n\ln\left(1 + \frac{H_{n+1}^{[r-1]}}{H_n^{[r]}}\right). \end{aligned}$$

By applying the following inequality

$$\ln(1+x) < x \quad (x > 0), \tag{2}$$

we get

$$(n+1)\ln H_n^{[r]} - n\ln(H_n^{[r]} + H_{n+1}^{[r-1]}) > \ln H_n^{[r]} - \frac{nH_{n+1}^{[r-1]}}{H_n^{[r]}}.$$

By using Lemma 4, we have

$$\begin{aligned} H_n^{[r]} &\sim \frac{n^{r-1}\ln n}{\Gamma(r)}, \quad (n \to +\infty), \qquad (3) \\ \lim_{n\to+\infty} \frac{nH_{n+1}^{[r-1]}}{H_n^{[r]}} &= r-1. \end{aligned}$$

This means that

$$\lim_{n\to+\infty}\left[(n+1)\ln H_n^{[r]} - n\ln(H_n^{[r]} + H_{n+1}^{[r-1]})\right] = +\infty.$$

Then, there exists a positive integer $N_{1,r}$ such that

$$(n+1)\ln H_n^{[r]} - n\ln(H_n^{[r]} + H_{n+1}^{[r-1]}) > 0, \quad n \geq N_{1,r}.$$

Since $\sqrt[n]{H_n^{[r]}} > \sqrt[n+1]{H_{n+1}^{[r]}}$ if and only if $(n+1)\ln H_n^{[r]} - n\ln(H_n^{[r]} + H_{n+1}^{[r-1]}) > 0$, the sequence $\{\sqrt[n]{H_n^{[r]}}\}_{n\geq N_{1,r}}$ is decreasing.

Now we prove that the positive integer $N_{1,r}$ is unique when r is fixed. For $n \geq 1$, let

$$f(n) = (n+1)\ln H_n^{[r]} - n \ln H_{n+1}^{[r]}.$$

For $n \geq 1$, we have

$$\begin{aligned} f(n+1) - f(n) &= 2(n+1)\ln H_{n+1}^{[r]} - (n+1)\ln H_n^{[r]} - (n+1)\ln H_{n+2}^{[r]} \\ &= (n+1)\ln \frac{(H_{n+1}^{[r]})^2}{H_n^{[r]} H_{n+2}^{[r]}}. \end{aligned}$$

Since $\{H_n^{[r]}\}_{n\geq 1}$ $(r \geq 2)$ is log-concave, $f(n+1) - f(n) > 0$ and $\{f(n)\}_{n\geq 1}$ is increasing. Hence the positive integer $N_{1,r}$ is unique.

From (3), we obtain

$$\lim_{n\to+\infty} \frac{\ln H_n^{[r]}}{n} = 0.$$

Hence, we have

$$\lim_{n\to+\infty} \sqrt[n]{H_n^{[r]}} = 1.$$

(iii) We note that there exists a positive integer $N_{1,r}$ such that the sequence $\{\sqrt[n]{H_n^{[r]}}\}_{n\geq N_{1,r}}$ is decreasing. On the other hand, the sequence $\{\sqrt[n]{n}\}_{n\geq 3}$ is decreasing. Therefore, there exists a positive integer $N_{2,r} \geq 3$ such that $\{\sqrt[n]{nH_n^{[r]}}\}_{n\geq N_{2,r}}$ is decreasing.

(iv) For $n \geq 1$, it is clear that $\sqrt[n]{H_{n+1}^{[r]}/H_n^{[r]}} > \sqrt[n+1]{H_{n+2}^{[r]}/H_{n+1}^{[r]}}$ if and only if $(2n+1)\ln H_{n+1}^{[r]} - (n+1)\ln H_n^{[r]} - n\ln H_{n+2}^{[r]} > 0$. It is evident that

$$\begin{aligned} (2n+1)\ln H_{n+1}^{[r]} - (n+1)\ln H_n^{[r]} - n\ln H_{n+2}^{[r]} &= (n+1)\ln \frac{H_{n+1}^{[r]}}{H_n^{[r]}} - n\ln \frac{H_{n+2}^{[r]}}{H_{n+1}^{[r]}} \\ &= \ln \frac{H_{n+1}^{[r]}}{H_n^{[r]}} + n\ln \frac{(H_{n+1}^{[r]})^2}{H_n^{[r]} H_{n+2}^{[r]}}. \end{aligned}$$

We note that $\frac{H_{n+1}^{[r]}}{H_n^{[r]}} > 1$ and the sequence $\{H_n^{[r]}\}_{n\geq 1}$ $(r \geq 2)$ is log-concave. Then the sequence $\{\sqrt[n]{H_{n+1}^{[r]}/H_n^{[r]}}\}_{n\geq 1}$ is decreasing.

(v) It is well-known that $\sqrt[n]{H_n^{[r]}}/\sqrt[n-1]{H_{n-1}^{[r]}} \leq \sqrt[n+1]{H_{n+1}^{[r]}}/\sqrt[n]{H_n^{[r]}}$ if and only $2(n^2-1)\ln H_n^{[r]} - n(n+1)\ln H_{n-1}^{[r]} - n(n-1)\ln H_{n+1}^{[r]} \leq 0$. We need prove that there exists a positive integer $N_{3,r}$ such that

$$2(n^2-1)\ln H_n^{[r]} - n(n+1)\ln H_{n-1}^{[r]} - n(n-1)\ln H_{n+1}^{[r]} \leq 0, \quad (n \geq N_{3,r}).$$

For $n \geq 2$, let

$$T_n = (H_n^{[r-1]})^2 - H_{n-1}^{[r]} H_{n+1}^{[r-2]}.$$

By means of the definition of $H_n^{[r]}$, we compute that

$$(H_n^{[r]})^2 - H_{n-1}^{[r]} H_{n+1}^{[r]} = T_n, \quad n \geq 2.$$

Then we obtain

$$\begin{aligned} &2(n^2-1)\ln H_n^{[r]} - n(n+1)\ln H_{n-1}^{[r]} - n(n-1)\ln H_{n+1}^{[r]} \\ &= (n^2-1)\ln[H_{n-1}^{[r]} H_{n+1}^{[r]} + T_n] - n(n+1)\ln H_{n-1}^{[r]} - n(n-1)\ln H_{n+1}^{[r]} \\ &= -(n+1)\ln H_{n-1}^{[r]} + (n-1)\ln H_{n+1}^{[r]} + (n^2-1)\ln\left(1 + \frac{T_n}{H_{n-1}^{[r]} H_{n+1}^{[r]}}\right). \end{aligned}$$

It follows from the definition of $H_n^{[r]}$ and the inequality (2) that

$$\begin{aligned} &2(n^2-1)\ln H_n^{[r]} - n(n+1)\ln H_{n-1}^{[r]} - n(n-1)\ln H_{n+1}^{[r]} \\ &= (n-1)\ln\left(1 + \frac{H_n^{[r-1]} + H_{n+1}^{[r-1]}}{H_{n-1}^{[r]}}\right) - 2\ln H_{n-1}^{[r]} + (n^2-1)\ln\left(1 + \frac{T_n}{H_{n-1}^{[r]} H_{n+1}^{[r]}}\right) \\ &\leq \frac{(n-1)(H_n^{[r-1]} + H_{n+1}^{[r-1]})}{H_{n-1}^{[r]}} - 2\ln H_{n-1}^{[r]} + \frac{(n^2-1)T_n}{H_{n-1}^{[r]} H_{n+1}^{[r]}}. \end{aligned}$$

Due to (3), we have

$$\begin{aligned} &\lim_{n\to+\infty} \frac{(n-1)(H_n^{[r-1]} + H_{n+1}^{[r-1]})}{H_{n-1}^{[r]}} = 2(r-1), \quad \lim_{n\to+\infty} \ln H_{n-1}^{[r]} = +\infty, \\ &\lim_{n\to+\infty} \frac{(n^2-1)T_n}{H_{n-1}^{[r]} H_{n+1}^{[r]}} = r-1. \end{aligned}$$

This implies that

$$\lim_{n\to+\infty} [2(n^2-1)\ln H_n^{[r]} - n(n+1)\ln H_{n-1}^{[r]} - n(n-1)\ln H_{n+1}^{[r]}] = -\infty,$$

and there exists a positive integer $N_{3,r}$ such that

$$2(n^2-1)\ln H_n^{[r]} - n(n+1)\ln H_{n-1}^{[r]} - n(n-1)\ln H_{n+1}^{[r]} < 0, \quad n \geq N_{3,r}.$$

Hence the sequence $\{\sqrt[n+1]{H_{n+1}^{[r]}}/\sqrt[n]{H_n^{[r]}}\}_{n \geq N_{3,r}}$ is increasing. □

For Theorem 2, we note that the value of $N_{1,r}$ is related to r. We prove by induction that

$$H_2^{[r]} = r + \frac{1}{2}, \quad H_3^{[r]} = \frac{r^2}{2} + r + \frac{1}{3}, \quad H_4^{[r]} = \frac{r(r+1)(2r+1)}{12} + \frac{r(r+1)}{2} + \frac{r}{3} + \frac{1}{4}.$$

It is clear that $N_{1,r} \geq 2$. In fact, we observe that $\sqrt[3]{H_3^{[2]}} > \sqrt[4]{H_4^{[2]}}$ and $\sqrt[3]{H_3^{[3]}} > \sqrt[4]{H_4^{[3]}}$. Owing to Lemma 5, the sequences $\{\sqrt[n]{H_n^{[2]}}\}_{n\geq 3}$ and $\{\sqrt[n]{H_n^{[3]}}\}_{n\geq 3}$ are decreasing. For $r \geq 2$, let

$$g(r) = \left(\frac{r^2}{2} + r + \frac{1}{3}\right)^4 - \left[\frac{r(r+1)(2r+1)}{12} + \frac{r(r+1)}{2} + \frac{r}{3} + \frac{1}{4}\right]^3.$$

Through computation, we have $g(8) < 0$. Then $N_{1,8} \geq 4$.

Theorem 3. *For the sequence $\{H_n^{[2]}\}_{n\geq 1}$, we have the following statements:*

(i) $\{\sqrt[n]{n!H_n^{[2]}}\}_{n\geq 1}$ is increasing;

(ii) $\{\sqrt[j]{H_j^{[2]}/\binom{n}{j}}\}_{1\leq j\leq n}$ $(n \geq 4\ )$ is increasing for j when n is fixed;

(iii) $\{\sqrt[n]{H_n^{[2]}/n}\}_{n\geq 3}$ is decreasing.

Proof. (i) We first show that the sequence $\{n!H_n^{[2]}\}_{n\geq 1}$ is log-convex. In order to prove the log-convexity of $\{n!H_n^{[2]}\}_{n\geq 1}$, we only need verify that

$$n(H_n^{[2]})^2 - (n+1)H_{n-1}^{[2]}H_{n+1}^{[2]} \leq 0 \quad (n \geq 2).$$

For $n \geq 2$,

$$\begin{aligned} & n(H_n^{[2]})^2 - (n+1)H_{n-1}^{[2]}H_{n+1}^{[2]} \\ &= n(H_n^{[2]})^2 - (n+1)(H_n^{[2]} - H_n)\left(H_n^{[2]} + H_n + \frac{1}{n+1}\right) \\ &= -(H_n^{[2]})^2 + (n+1)H_n^2 - H_n^{[2]} + H_n. \end{aligned}$$

By using the definitions of H_n and $H_n^{[2]}$, we derive

$$H_n^{[2]} = (n+1)H_n - n. \tag{4}$$

It follows from (4) that

$$\begin{aligned} n(H_n^{[2]})^2 - (n+1)H_{n-1}^{[2]}H_{n+1}^{[2]} &= -[(n+1)H_n - n]^2 + (n+1)H_n^2 - nH_n + n \\ &= -n^2(H_n - 1)^2 - n(H_n^2 - H_n - 1) \\ &< 0. \end{aligned}$$

Then the sequence $\{n!H_n^{[2]}\}_{n\geq 1}$ is log-convex. On the other hand, we note that $H_1^{[2]} < \sqrt{2H_2^{[2]}}$. By Lemma 5, we prove that the sequence $\{\sqrt[n]{n!H_n^{[2]}}\}_{n\geq 1}$ is increasing.

(ii) We first show that $\{\sqrt[j]{H_j^{[2]}/\binom{n}{j}}\}_{1\le j\le n}$ is log-convex. To show the log-convexity of the sequence $\{H_j^{[2]}/\binom{n}{j}\}_{1\le j\le n}$, it suffices to prove that

$$j(n-j)(H_j^{[2]})^2-(n-j+1)(j+1)H_{j-1}^{[2]}H_{j+1}^{[2]}\le 0,\quad 2\le j\le n-1.$$

For $n\ge 4$ and $2\le j\le n-1$, it follows from (4) that

$$\begin{aligned}
&j(n-j)(H_j^{[2]})^2-(n-j+1)(j+1)H_{j-1}^{[2]}H_{j+1}^{[2]}\\
&=j(n-j)[(H_j^{[2]})^2-H_{j-1}^{[2]}H_{j+1}^{[2]}]-(n+1)H_{j-1}^{[2]}H_{j+1}^{[2]}\\
&=j(n-j)H_j^2-\frac{j(n-j)(H_j^{[2]}-H_j)}{j+1}-(n+1)\left[(H_j^{[2]})^2-H_j^2+\frac{H_j^{[2]}-H_j}{j+1}\right]\\
&<j(n-j)H_j^2-(n+1)[(H_j^{[2]})^2-H_j^2]\\
&=j(n-j)H_j^2-(n+1)[(j^2+2j)H_J^2-2j(j+1)H_j+j^2]\\
&=-jH_j[(n+nj+2j+2)H_j-2(j+1)]-(n+1)j^2\\
&<0.
\end{aligned}$$

Then the sequence $\{H_j^{[2]}/\binom{n}{j}\}_{1\le j\le n}$ is log-convex. It is clear that $\frac{H_1^{[2]}}{n}<\sqrt{H_2^{[2]}/\binom{n}{2}}$. Due to Lemma 5, the sequence $\{\sqrt[j]{H_j^{[2]}/\binom{n}{j}}\}_{1\le j\le n}$ is increasing for j when n is fixed.

(iii) For $n\ge 2$, it follows from the definition of hyperharmonic numbers and (4) that

$$\begin{aligned}
&\frac{(H_n^{[2]})^2}{n^2}-\frac{H_{n-1}^{[2]}H_{n+1}^{[2]}}{n^2-1}\\
&=\frac{(n^2-1)(H_n^{[2]})^2-n^2H_{n-1}^{[2]}H_{n+1}^{[2]}}{n^2(n^2-1)}\\
&=\frac{(n+1)n^2H_n^2-n^2(H_n^{[2]}-H_n)-(n+1)(H_n^{[2]})^2}{(n+1)n^2(n^2-1)}\\
&=\frac{(n+1)n^2H_n^2-n^2(H_n^{[2]}-H_n)-(n+1)[(n+1)H_n-n]^2}{(n+1)n^2(n^2-1)}\\
&=\frac{H_n[n(n^2+1-2nH_n)+(n-H_n)(3n+1)]+n^3(H_n-1)}{(n+1)n^2(n^2-1)}.
\end{aligned}$$

We compute that

$$\frac{(H_2^{[2]})^2}{2^2}-\frac{H_1^{[2]}H_3^{[2]}}{2^2-1}>0.$$

For $n\ge 3$, let $U_n=n^2+1-2nH_n$. Since $H_n\le\frac{2n-7}{6}$ $(n\ge 3)$, $W_n\ge\frac{n^2+7n+3}{3}>0$. Then

$$\frac{(H_n^{[2]})^2}{n^2}-\frac{H_{n-1}^{[2]}H_{n+1}^{[2]}}{n^2-1}>0\ \ (n\ge 3),$$

and the sequence $\{H_n^{[2]}/n\}_{n\geq 2}$ is log-concave. On the other hand, we observe that $\sqrt[3]{H_3^{[2]}/3} > \sqrt[4]{H_4^{[2]}/4}$. It follows from Lemma 5 that the sequence $\{\sqrt[n]{H_n^{[2]}/n}\}_{n\geq 3}$ is decreasing. □

3. Conclusions

For hyperharmonic numbers $\{H_n^{[r]}\}$, we have discussed the monotonicity of some sequences related to $\{H_n^{[r]}\}$. For example, we have investigated the monotonicity of sequences such as $\{H_n^{[r]}/n\}_{n\geq 1}$, $\{\sqrt[n]{H_n^{[r]}}\}_{n\geq 1}$, $\{\sqrt[n+1]{H_{n+1}^{[r]}}/\sqrt[n]{H_n^{[r]}}\}_{n\geq 1}$ and $\{\sqrt[n]{H_{n+1}^{[r]}/H_n^{[r]}}\}_{n\geq 1}$; see Theorems 1–3. For Theorem 2, we are unable to give the minimum value of $N_{3,r}$, and this may be our future work.

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