ON THE COMPLEXITY AND TOPOLOGY OF SCORING GAMES: OF PIRATES AND TREASURE

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Abstract

The combinatorial game Pirates and Treasure is played between two players, Left and Right, on a finite, simple, undirected weighted graph. The vertices of the graph correspond to islands, and a weight function on the vertices indicates the amount of treasure the island has. Left player has n ships, Right player has m ships, in pre-defined vertices. Each turn, the current player moves one of his ships into an adjacent, non-visited vertex, and the weight of the vertex is added to the current points of the player. If a player in his turn cannot move, the game ends. Left moves first. The player who collects more treasures (points) than his opponent wins the game. It is shown that it is **PSPACE**-complete to decide whether or not Left has a winning strategy in the Pirates and Treasure game. Moreover, it is also **PSPACE**-complete to decide whether or not Left has a winning strategy when we assume that the two players are moving in different components of the graph. For a fixed graph G, topological and convexity properties of weightings are analyzed. Among other things it is shown that the winning space of Left is connected, but not always convex.

1. Introduction

In the past few decades the direction of the examination of combinatorial games has slightly changed. As a great mathematical invention of the 20th century the theory of "misère" and "partizan" type games were developed. In these games the last move decides the outcome of the game. The books and monographs (e.g., [1], [2]) on this subject have became a part of the classical mathematical knowledge. However, there are some other types of games that were not examined so extensively. These are the so-called "scoring games" in which the players aim to collect the most points. Although the first few papers about scoring games were published in the middle of the last century ([8], [4]), no deep theory was developed for them.

After several attempts to classify the scoring games, the two classes of "well tempered" and "dicot" games were introduced [9]. A theory of well-tempered scoring games was developed in [6]. There, algebraic and topological (ordering) aspects were investigated and applied to several games, like knot games [5]. In [3] a real-valued metric was defined for positional games, and it was proved that a particular class of games is a topological semigroup. Then a separation property was defined that connects these games to closely related Conway games.

Another attempt to generalize the previous works of Berlekamp, Conway and Guy ([1], [2]) was made to the theory for scoring play games [11]. Their types of games show nice behavior for the usual game operations, but the theory is fairly restrictive. It was shown that scoring play games do not form a group, there is no non-trivial identity among them, and that the comparison of two games in the usual sense is impossible. However, the paper also states that scoring play games are ordered under the disjunctive sum, and we can form equivalence classes among them using a canonical form. Unfortunately, this theory does not involve several types of games, as in the case of the Pirates and Treasure game.

The combinatorial game Pirates and Treasure was introduced by F. Stewart in [10]. The game is played between two players, Left and Right, on a finite, simple, undirected graph, where each vertex has a weight. The vertices of the graph correspond to islands, and a weight function on the vertices indicates the amount of treasure the island has. Left player has n ships, Right player has m ships, in pre-defined vertices. In each turn, the current player moves one of his ships into an adjacent, non-visited vertex, and the weight of the vertex is added to the current points of the player. If a player in his turn cannot move any of his ships to an adjacent, non-visited vertex, the game ends. Left moves first. The player who collects more treasures (points) than his opponent, wins the game. In this paper we will always assume that $n = m = 1$. All of our results apply easily for the general case as well.

In [10] it is shown that it is **NP**-hard to determine which player wins the game. It was also conjectured that this problem is, in fact, **PSPACE**-complete. In this paper we prove this conjecture. More precisely, we show that deciding whether Left has a winning strategy is **PSPACE**-complete. We reduce the quantified Boolean formula (QBF) problem to the Pirates and Treasures game. With a small modification we present a construction where the two players move in different components of the graph, and it remains still **PSPACE**-complete to decide, who wins.

Furthermore, in the third section we investigate topological properties, such as openness, connectivity and convexity of the winning (non-losing) strategies of Left and Right according to the weight function on a fixed graph, with fixed initial positions and a single ship for both players.

2. PSPACE-completeness

In [10] it was proved that it is **NP**-hard to decide which player has a winning strategy in the Pirates and Treasure game. The problem was stated as follows.

Problem 2.1 (F. Stewart, [10]). *Input: A graph $G = (V, E)$, the initial positions of the players Left and Right (denoted by L and R, respectively) and a $W : V \setminus \{L, R\} \to \mathbb{Z}^+$ weight function.*

Question: Does Left have a winning strategy in the Pirates and Treasure game corresponding to the graph G and the weight function W?

It is also conjectured in [10] that Problem 2.1 is in fact **PSPACE**-complete. In this section we prove this conjecture. For convenience throughout this paper we will assume that the weights are in $\mathbb{R}_0^+$ instead of $\mathbb{Z}^+$. It is easy to see that this assumption does not make much difference.

Theorem 2.2. *The following problem is* **PSPACE***-complete.*

Problem 2.3 (F. Stewart, [10]). *Input: A graph $G = (V, E)$, the initial positions of the players Left and Right (denoted by L and R, respectively) and a $W : V \setminus \{L, R\} \to \mathbb{R}_0^+$ weight function.*

Question: Does Left have a winning strategy in the "Pirates and Treasure" game corresponding to the graph G and the weight function W?

Proof. It is obvious that Problem 2.3 is in **PSPACE**, because if we are given a graph G on n vertices, then any game will end in at most n steps, hence we can check every outcome of the corresponding Pirates and Treasure game in polynomial space, from which we can determine the player who has a winning or a non-losing strategy.

In order to prove the **PSPACE**-hardness of Problem 2.3 we do a polynomial time reduction from the QBF problem. The QBF problem is the following.

Problem 2.4. *Input:*

A formula of the form

$$\Psi(x_1, y_1, \ldots, x_n, y_n) = \forall x_1 \exists y_1 \forall x_2 \exists y_2 \ldots \forall x_n \exists y_n \Phi(x_1, y_1, \ldots, x_n, y_n),$$

where Φ is a conjunctive normal form, i.e, it is of the form $\Phi = \bigwedge_{i=1}^{k} C_i$, where for all $i = 1, 2, \ldots, k$, the formula C_i is of the form $C_i = \bigvee_{j=1}^{t_i} l_j$, where each l_j denotes a variable x_i or y_i or their negations.

Question: Is Ψ true?

It is well-known that Problem 2.4 is **PSPACE**-complete, so from the following reduction it follows that Problem 2.3 is also **PSPACE**-hard, and therefore **PSPACE**-complete, as well.

Suppose that we are given a formula $\Psi(x_1, y_1, \ldots, x_n, y_n)$ of the above form. Now, we construct from the formula Ψ a graph G and a weight function W on the vertices of G in the following way. Each $l \in \{x_i, y_i\}$ literal will correspond to two vertices of G: one vertex for l, the other one for $\bar{l}$. These vertices will be denoted by the corresponding variables or their negations, x_i or $\bar{x}_i$, or y_i or $\bar{y}_i$. Left and Right alternately choose values for variables $x_1, y_1, \ldots, x_n, y_n$ of Ψ by moving to the corresponding vertices. In the first step Left assigns a value to x_1 by moving to the vertex x_1 or $\bar{x}_1$. Then Right chooses a value for y_1 by moving to the vertex y_1 or $\bar{y}_1$. Then Left proceeds for x_2, and so on. After walking through the literals, Left chooses a clause by moving to the corresponding vertex and Right is allowed to choose an unvisited literal from that clause. If such a literal exists in the clause picked by Left, then Right wins. If Left can pick a clause with all literals visited before, then Left wins. Hence if Right has no chance to visit the literals wisely, than Left has a winning strategy. As we play on undirected graphs, both Left and Right might choose to "turn back" or alter their regular directions, hence we have to carefully choose the weight function, and in a few places we introduce gadgets connected to critical vertices. These extra gadgets are constructed in a way that if any of the two players "reroutes", the other one would have a shortcut, a short path of vertices with large weights, and would win the game after stopping at the end of the path. In general, Left will have shorter paths with larger weights and Right will have longer paths with smaller weights. Hence, if Right makes an illegal move, Left will have the chance to step on a vertex of high value, then stop the game.

For the construction let $N := n^2 + k^2$. First of all let us consider the vertices

$$L_1, L'_1, L_2, L'_2, \ldots, L_n, L'_n, L_{n+1}, x_1, \bar{x}_1, \ldots, x_n, \bar{x_n},$$

and connect the following pairs of vertices:

$$\{L_i, x_i\}, \{L_i, \bar{x_i}\}, \{L'_i, x_i\}, \{L'_i, \bar{x_i}\}, \{L'_i, L_{i+1}\}.$$

Let G_1 denote the graph obtained this way. Let us define the graph G_2 similarly with vertices

$$R'_0, R_1, R'_1, R_2, R'_2, \ldots, R_n, R'_n, y_1, \bar{y}_1, \ldots, y_n, \bar{y}_n.$$

The initial positions of Left and Right will be $L := L_0$ and $R := R'_0$, respectively. Let N^2 be the weight of all the vertices (except for the vertices L and R). These two graphs can be seen on the top of Figure 1. Note that there is one extra vertex for Right at the beginning, and one extra vertex for Left at the end. These will ensure that Right will have to follow the implicit instructions of Left.

Now, let us define the graph H_1 as follows. Let us consider the disjoint paths of length $2N$:

$$P_1, P_2, P_3, \ldots, P_k.$$

Note that k was the number of clauses in Φ. The path P_i will correspond to the clause C_i. Let V_i denote one of the endpoints of P_i for $i = 1, 2, \ldots, k$. Let us connect the vertices $V_1, V_2, \ldots, V_k$ with a new vertex V_0, and add new paths Q_i of length i starting from the vertices V_i. The paths Q_i will play the role of gadgets. The graph obtained this way will be H_1, and can be seen on the bottom left part of Figure 1. When Left passes through the "literal" part, he will choose a clause C_i by moving from V_0 to V_i.

We proceed with a similar construction for Right as above, except that in this case the length of the paths will be k instead of $2N$. Let us call this graph H_2. For convenience, denote the corresponding paths by $P'_1, P'_2, \ldots, P'_k$ and $Q'_1, Q'_2, \ldots, Q'_k$. Let the length of P'_i be k, let the length of Q'_i be i, and for $i > 0$ let the vertex V'_i be an endpoint of P'_i. Then connect each V'_i, $i > 0$ to V'_0. This graph can be seen on the bottom right part of Figure 1.

Let us connect V_0 to L_{n+1} and V'_0 to R'_n by an edge.

Let C_i denote the last (kth) vertex of the path P'_i (in the graph H_2) for all $i = 1, 2, \ldots, k$. We will make these vertices correspond to the clauses in Ψ. Connect each literal $l \in x_i, \bar{x}_i, y_i, \bar{y}_i$ and clause C_i by a path of length N if and only if the clause C_i contains the *negation* of the literal l. Later we will refer to this path as $S_{C_i,l}$.

The graph we obtained this way will be denoted by G. The final construction can be seen in Figure 1. Clearly the graph G can be computed from the formula Ψ in polynomial time.

We define the weight function W on $H_1 \cup H_2$ as follows. Let the weight of the vertices V_0 and V'_0 be N^3; let the weight of the vertices on the paths $P_1, P_2, \ldots, P_k$, $P'_1, P'_2, \ldots, P'_k$ be $2k^2$; and for all $i = 1, 2, \ldots, k$ let the weight of the remaining vertices on the paths Q_i and Q'_i be $2k^2 + k + 1 - i$ except for the endpoint of the path Q'_i, where we set the weight to be $2k^2 + k + 2 - i$.

Let the weight function be identically 0 on the paths $S_{C_i,l}$.

We claim that Left has a winning strategy in the Pirates and Treasure game corresponding to the the graph G and the weight function W if and only if the formula Ψ is false. If we prove this, then this will provide us a polynomial time reduction from the QBF problem to Problem 2.3, which will imply the **PSPACE**-hardness of Problem 2.3. We show this in the following several steps.

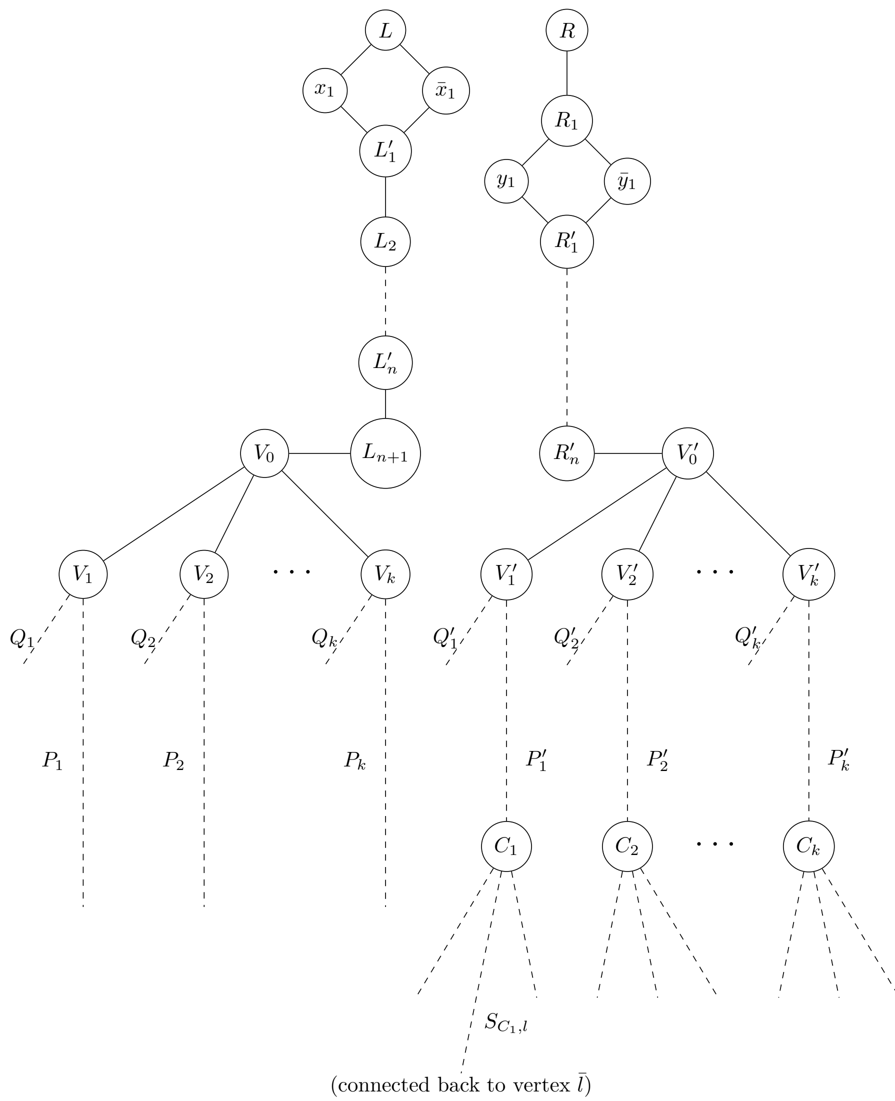

Figure 1: Construction for the proof of Theorem 2.2.

First of all, let us call a play of the game on G *regular* if the players only make the following types of moves.

1. Left moves to the vertices $l_1, L'_1, L_2, l_2, L'_2, \ldots, l_n, L_n, L_{n+1}$ in his first $3n$ steps, respectively, where $l_i = x_i$ or $l_i = \bar{x_i}$ for all i.

2. Right moves to vertices $R_1, l_1, R'_1, R_2, l_2, R'_2, \ldots, R_n, l_n, R'_n$ in his first $3n$ steps,

respectively, where $l_i = y_i$ or $l_i = \bar{y_i}$ for all i.

3. Left moves to V_0 is his $3n$th step, then starts moving on the path P_i for some i (we require at least two steps on it).

4. Right moves to V_0', then if Left's $(3n+2)$nd move was to V_i for some i, then he starts moving on the path P_i' (we require at least two steps on it).

Sometimes we will refer to these moves as *regular* moves. A move is called *irregular* if it is not regular. In the following lemmas we prove that both players have to move *regularly* (in the above sense) during the game, because if someone makes an irregular move, then the other player would have a winning strategy.

Lemma 2.5. *Suppose that one of the players makes an irregular move in the first* $6n$ *steps of the game. Then the player who makes the first irregular move loses.*

Proof. Let A be the player who makes the first irregular move. This means that after this move he either starts moving on a path $S_{C_i,l}$, or moves back to some vertex corresponding a literal l, and starts moving on a path $S_{C_i,l}$ in his next step. In this case the other player, B, can win as follows: after A's first irregular step he moves regularly until his $(3n+1)$st step, then he gets himself stuck in the path Q_1 or Q_1'. This is possible because the paths $S_{C_i,l}$ are long enough. By doing this B will collect at least N^3 points since he will collect the treasure on the vertex V_0 or V_0', while A can collect at most $3(n+1)N^2$ points altogether. Therefore B wins if N is large enough. $\square$

Lemma 2.6. *Suppose that both players play regularly in the first* $6n$ *steps of the game, but one of them makes an irregular move later. Then the player making the first irregular move loses.*

Proof. Suppose that the first $6n$ steps were regular. Then Left is forced to move to V_0 and then Right is forced to move to V_0'. Up to this point both players collected exactly $3nN^2 + N^3$ points. For this reason we will ignore these points in the remaining part of the proof, and we will only count with the points collected after this step.

In the next ($(6n+3)$rd) step Left will move to the vertex V_i for some i. Now, there are 3 possible scenarios.

`Case 1:` Right makes an irregular move in his next ($(6n+4)$th) step by moving to the vertex V_j for some $j \neq i$.

We claim that in this case Left wins by taking the path Q_i.

`Subcase 1/1:` Right starts moving on the path P_j'.

In this case, both players will make $i+1$ moves in which Left will collect $2k^2 + (2k^2+k+1-i)i$ points, and Right will collect $2k^2(i+1)$ points. Therefore Left wins.

`Subcase 1/2:` $j > i$ and Right starts moving on the path Q'_j.

In this case, both players will make $i+1$ moves again in which Left will collect $2k^2 + (2k^2 + k + 1 - i)i$ points, and Right will collect $2k^2 + (2k^2 + k + 1 - j)i$ points. Thus Left wins, since $j > i$.

`Subcase 1/3:` $j < i$ and Right starts moving on the path Q'_j.

In this case, Left will make $j+2$ moves and Right will make $j+1$ moves. In these steps Left will collect $2k^2 + (2k^2 + k + 1 - i)(j+1)$ points, and Right will collect $2k^2 + (2k^2 + k + 1 - j)j + 1$ points. Then

$$2k^2+(2k^2+k+1-i)(j+1) > 2k^2(j+2) = 2k^2j+2k^2 > 2k^2+(2k^2+(k+1-j))j+1,$$

since $j, k+1-j \leq k$. Therefore Left wins again.

`Case 2:` Right moves to V'_i in the $(6n+4)$th step, but in the next step Left makes an irregular move, i.e., starts moving on the path Q_i.

We claim that in this case Right wins if he starts moving on the corresponding path Q'_i. Indeed, in this case both players will make $i+1$ moves, in which Left will collect $2k^2 + (2k^2 + k - i)i$, and Right will collect $2k^2 + (2k^2 + k - i)i + 1$ points. Hence Right wins.

`Case 3:` Right moves to V'_i in the $(6n+4)$th step, and in the next step Left starts moving on the path P_i, but after that in the $(6n+6)$th step Right makes an irregular move, i.e., starts moving on the path Q'_i.

In this case Left will make $i+2$ moves and Right will make $i+1$ moves. By taking these steps, Left will collect $2k^2(i+2)$ points, while Right will collect $2k^2 + 2(k^2+k-i)i+1$ points. We have already seen (in Subcase 1/3) that the inequality $2k^2(i+2) > 2k^2 + 2(k^2+k-i)i+1$ holds. This implies that Left wins in this case as well.

By definition, if the first $6n+6$ steps are regular, then it must be a regular play of the game. $\square$

From now on we will always assume that the players play regularly. By Lemmas 2.5. and 2.6. we can assume this. In this case the first $6n+6$ steps of the game can be considered as if they were setting values of the Boolean variables $x_1, y_1, \ldots, x_n, y_n$ alternately (by assigning the true literal with their moves), then starting to move on the paths P_i and P'_i (of the same index). We claim that after setting the values of the variables $x_1, y_1, \ldots, x_n, y_n$, Left can win if and only if the formula $\Phi(x_1, y_1, \ldots, x_n, y_n)$ is false. If the players play regularly, then they will collect the same amount of treasure in the first $6n+6$ steps, hence we can ignore these points as in the proof of Lemma 2.6, and only count the points collected afterwards.

First, assume that Φ is false. This implies that C_i must be false for some $1 \leq i \leq k$. Then Left must take the path P_i. In this case Left and Right will move alternately until Right arrives to C_i. In his next step, Right has to take a path

$S_{C_i,l}$ for some l. Since C_i is false, the value of this l must be true, which means that somebody has already moved to l earlier. In this case Left will make $k+N$ steps, and Right will make $k+N-1$ steps, and they will collect $2(k+N)k^2$ and $2k^3$ points, respectively. Hence Left wins.

Now assume that Φ is true. Then the path Left takes does not matter, as Right can win with the following strategy. Suppose that Right has already arrived to C_i for some $1 \le i \le k$. This must be true, since Φ is true. This implies that C_i contains a literal l, which is true. We claim that Right wins, if he takes the path $S_{C_i,\bar{l}}$. Indeed, in this case Right will eventually arrive to the vertex $\bar{l}$ and collect at least N^2 points, while Left could only collect at most $(k+N+1)k^2$ points, which is less than N^2 if N is large enough.

Finally, the reduction works as follows. It can be seen from our previous arguments that Left has a winning strategy if and only if he can choose the value of x_1 such that for any choice of y_1 he can choose the value of x_2 such that etc., and finally he can choose the value of x_n such that for any choice of y_n the formula $\Phi(x_1, y_1, \ldots, x_n, y_n)$ will be false. This holds if and only if the formula Ψ is false. Therefore, by this construction we reduced the QBF problem to Problem 2.3, which finishes the proof of the theorem. $\square$

One can notice that in our proof of Theorem 2.2 the constructed graph G is "almost disconnected", and therefore a good question is what happens if we consider the game Pirates and Treasure only for graphs in which the initial positions of the players are in different components. By slightly modifying the proof of Theorem 2.2, we can show that Problem 2.3 is **PSPACE**-complete in this case as well.

Theorem 2.7. *The following problem is* **PSPACE**-*complete.*

Problem 2.8. *Input: A graph $G = (V, E)$, the initial positions of the players Left and Right (denoted by L and R, respectively) which are in different components of G, and a $W : V \setminus \{L, R\} \to \mathbb{R}_0^+$ weight function.*

Question: Does Left have a winning strategy in the Pirates and Treasure game corresponding to the graph G and the weight function W?

Proof. We modify the construction in the proof of Theorem 2.2 in a way that the vertices L and R will be in different components of G.

Let us consider a formula Φ and the corresponding construction G as in the proof of Theorem 2.2. We make the following modifications.

1. We delete the edges (L_i, x_i), $(L_i, \bar{x}_i)$, (R'_{i-1}, R_i) and put the graph in Figure 2 in place of them. We denote the new neighbors of R'_{i-1} by x'_i and $\bar{x}'_i$ (x'_i is the left neighbor in the figure). The numbers on the vertices in Figure 2 denote their weight. We choose the value of ε in such a way that the sum of the weights of the newly added vertices is less than 1.

2. The endpoint of the paths S_{C_i,x_i} and $S_{C_i,\bar{x}_i}$ are $\bar{x}'_i$ and x'_i, respectively (instead of $\bar{x}_i$ and x_i).

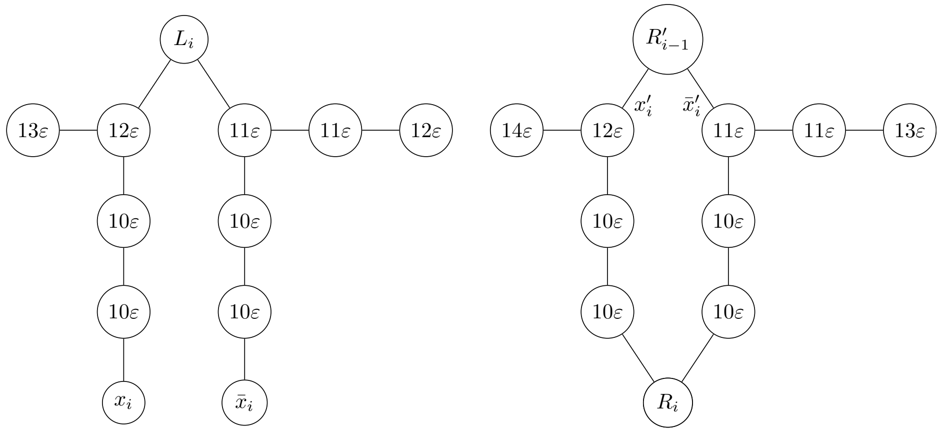

Figure 2: Construction for the proof of Theorem 2.7.

Let G denote the graph obtained this way. This graph can be computed again from the formula Ψ in polynomial time. We claim that Left has a winning strategy in the Pirates and Treasure game on the graph G if and only if the formula Ψ is false. We define the concept of *regular* play of a game again: let us call a play of the game on G *regular* if both players make only the following types of moves.

1. The players make a regular move on the edges of G which existed before the modification, according to the definition in the proof of Theorem 2.2.

2. Left walks from L_i to l in his $(6(i-1)+1)$st, $(6(i-1)+2)$nd, $(6(i-1)+3)$rd and $(6(i-1)+4)$th steps, Right moves l' in his $(6(i-1)+1)$st and walks to R_i in his $(6(i-1)+2)$nd, $(6(i-1)+3)$rd and $(6(i-1)+4)$th steps, where $l = x_i$ or $l = \bar{x}_i$. (After Left's $(6(i-1)+1)$st move it turns out whether he walks towards x_i or $\bar{x}_i$).

Now, we prove again that both players have to move *regularly* during the game.

Lemma 2.9. *Suppose that one of the players makes an irregular move in the first $12n$ steps of the game. Then the player who makes the first irregular move loses.*

Proof. Let A be the player who makes the first irregular move. Then we distinguish 4 cases according to the first irregular move.

`Case 1:` Player A starts moving on a path $S_{C_i,l}$ (maybe after a few other moves).

As in the proof of Theorem 2.2, the other player B wins by moving regularly until his $(3n+1)$st step, then getting himself stuck in the path Q_1 or Q'_1. This is possible, because the paths $S_{C_i,l}$ are long enough. Now, the same argument proves that B wins.

`Case 2:` Right chooses the "wrong" path.

If the players moved regularly up to their $6(i-1)$th step, then they collected the same amount of points, thus we can ignore these points (as we did it in the earlier proofs). Here we distinguish two subcases.

`Subcase 2/1:` Left moves to the left neighbor of L_i, but Right chooses the right one ($\bar{x}'_i$).

In this case Left moves to the 13ε vertex in his next step, then Right has one more move in which he collects at most 11ε points. Then Left collects 25ε and Right collects at most 22ε points, thus Left wins.

`Subcase 2/2:` Left moves to the right neighbor of L_i, but Right chooses the left one (x'_i).

In the following two steps Left moves to the vertices worth 11ε and 12ε. Then he collects 34ε points. If Right moves to the vertex worth 14ε, then he will collect 26ε points; and if he moves to the vertex worth 10ε, then he will collect 30ε points. Either way, Right loses.

`Case 3:` Left starts moving on the "short" path in his $(6(i-1)+2)$nd step.

In this case Right wins by taking the short path, as well, because in both cases Right will collect ε points more than Left.

`Case 4:` Right starts moving on the "short" path in his $(6(i-1)+2)$nd step.

We distinguish two subcases again.

`Subcase 4/1:` Both players have chosen the left path in their $(6(i-1)+1)$st steps.

Then Left will collect 32ε points and Right will collect 26ε points, hence Left wins.

`Subcase 4/2:` Both players have chosen the right path in their $(6(i-1)+1)$st steps.

Then Left will collect $30\varepsilon + N^2 > 30\varepsilon + 1$ points and Right will collect 35ε, hence Left wins again. □

Lemma 2.10. *Suppose that both players play regularly in the first $12n$ steps of the game, but one of them makes an irregular move later. Then the player who makes the first irregular move loses.*

Proof. This proof is the same as that of Lemma 2.6. □

Completion of Proof of Theorem 2.7. We can notice that if the players move regularly in the above sense, then for all $1 \le i \le n$ during the game they visit the vertex x_i' if and only if they visit the vertex x_i, and they visit the vertex $\bar{x}_i'$ if and only if they visit vertex $\bar{x}_i$. Using this, the same argument as in the proof of Theorem 2.2 shows that the formula Ψ is false if and only if Left has a winning strategy. This gives us a polynomial time reduction from the QBF problem to Problem 2.3, which proves the theorem. □

3. About Connectivity and Convexity at a Fixed Graph

In this section we examine what happens if we fix the graph and also fix the initial positions of the ships, but we do not fix the weight function on the vertices of the graph. The main question is: how do the outcome and the strategies of the game depend on our particular choice of the weights?

We know that in Pirates and Treasure either Left or Right has a winning strategy or they both have a non-losing strategy.

Definition 3.1. Let $G = (V, E)$ a graph, $V = \{L, R, v_1, \ldots, v_n\}$, where L and R denote the initial position of player Left and Right, respectively. Then let

$\mathcal{L}_G := \{(x_1, x_2, \ldots, x_n) | \forall i (x_i \ge 0)$, and Left has a winning strategy if the weight on vertex v_i is x_i for all $i\}$;

$\mathcal{R}_G := \{(x_1, x_2, \ldots, x_n) | \forall i (x_i \ge 0)$, and Right has a winning strategy if the weight on vertex v_i is x_i for all $i\}$;

$\mathcal{T}_G := \{(x_1, x_2, \ldots, x_n) | \forall i (x_i \ge 0)$, $\exists i (x_i > 0)$, and both Left and Right have a non-losing strategy if the weight on vertex v_i is x_i for all $i\}$.

Sometimes we omit the subscripts if the graph G is clear from the context.

Because of our previous remark, these definitions are meaningful and give a partition of $\mathbb{R}^n_{\ge 0} \setminus 0$. We would like to investigate how certain properties of these sets $\mathcal{L}_G, \mathcal{R}_G, \mathcal{T}_G$ depend on the graph G. It is obvious that these sets are cone-like subsets of $\mathbb{R}^n_{\ge 0} \setminus 0$, i.e., if $\mathbf{x} \in \mathcal{L}$ (similarly for $\mathcal{R}$ and $\mathcal{T}$), then $\lambda \mathbf{x} \in \mathcal{L}$ for all $\lambda > 0$.

Proposition 3.2. *For every graph G the sets $\mathcal{L}_G, \mathcal{R}_G$ are open and $\mathcal{T}_G, \mathcal{L}_G \cup \mathcal{T}_G, \mathcal{R}_G \cup \mathcal{T}_G$ are closed subsets of $\mathbb{R}^n_{\ge 0} \setminus 0$.*

The proof of Proposition 3.2 is not so difficult but rather technical, hence we omit this proof in this paper.

After these essential topological properties of $\mathcal{L}, \mathcal{R}$ and $\mathcal{T}$, one could also be interested in the topological nature (e.g., connectivity properties) of $\mathcal{L}, \mathcal{R}$ and $\mathcal{T}$. To prove connectivity for $\mathcal{L} \cup \mathcal{T}$, we present a nice observation. Let e_i denote the vector whose ith coordinate is 1, and all other coordinates are 0.

Lemma 3.3. *For every $\mathbf{x} \in \mathcal{L}$ there exists an i such that the vertex v_i is a neighbor of L in G, and the vectors $\mathbf{x}$ and e_i are in the same connectivity component of $\mathcal{L}$.*

Proof. By the definition of $\mathcal{L}$ Left has a winning strategy for the weighting corresponding to $\mathbf{x}$. Let v_i be a possible first move of Left in his winning strategy. Then, by increasing the value of the ith coordinate of $\mathbf{x}$, it remains in $\mathcal{L}$. Let $\lambda > 0$ such that the ith coordinate of $\mathbf{x}' = \mathbf{x} + \lambda e_i$ is larger than the sum of all other coordinates of $\mathbf{x}'$. Then, by decreasing the value of all but the ith coordinate of $\mathbf{x}'$ to 0 it still remains in $\mathcal{L}$. We have found a continuous curve within $\mathcal{L}$ connecting the points $\mathbf{x}$ and $\lambda' e_i$ for some $\lambda' > 0$, therefore $\mathbf{x}$ and e_i (recall that $\mathcal{L}$ is cone-like) are in the same connectivity component of $\mathcal{L}$. □

Proposition 3.4. *$\mathcal{L} \cup \mathcal{T}$ is (path) connected.*

Proof. Using Lemma 3.3 we only have to prove that if v_i and v_j are neighbors of L in G, then e_i and e_j are in the same connectivity component of $\mathcal{L} \cup \mathcal{T}$. For this, it is clearly enough to prove that $\lambda e_i + \mu e_j \in \mathcal{L} \cup \mathcal{T}$ for all $\lambda, \mu \geq 0$ (at least one of them is non-zero). If $\lambda \geq \mu$, then if Left moves first to v_i, he does not lose. If $\lambda \leq \mu$, then if Left moves first to v_j he does not lose. Hence Left can not lose with the weighting corresponding to the point $\lambda e_i + \mu e_j$. □

It is natural to ask whether the set $\mathcal{L}$ is always connected, as well. The answer is no, in fact, it is possible that $\mathcal{L}$ has as many connectivity components as many neighbors L has. Clearly (by Lemma 3.3) the latter is always an upper bound for the number of connectivity components of $\mathcal{L}$.

Example 3.1. Let $V(G) := \{L, R, v_1, v_2, \ldots, v_n\}$ and

$$E(G) := \{(L, v_1), (L, v_2), \ldots, (L, v_n)\} \cup \{(R, v_1), (R, v_2), \ldots, (R, v_n)\}.$$

Then the vectors e_i are in pairwise different connectivity components of $\mathcal{L}$, and hence by Lemma 3.3 the class $\mathcal{L}$ has exactly n connectivity components.

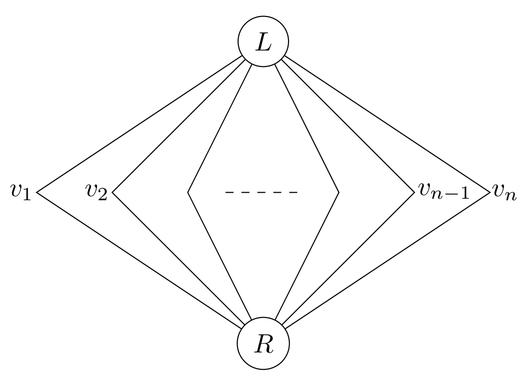

Figure 3: There could be n connectivity components.

Proof. Since $\mathcal{L}$ is an open subset of $\mathbb{R}^n_{\geq 0}$, the connectivity components and the path connectivity components of $\mathcal{L}$ are the same. Therefore it is enough to prove that if $i \neq j$, then e_i and e_j cannot be connected by a continuous curve within $\mathcal{L}$. For the contradiction, let us suppose that $\gamma : [0,1] \to \mathcal{L}$ is a continuous curve for which $\gamma(0) = e_i$ and $\gamma(1) = e_j$. Let $\alpha(x)$ be the ith coordinate of $\gamma(x)$, and let $\beta(x)$ be the maximum of the other coordinates of $\gamma(x)$. Then $\alpha(0) = \beta(1) = 1, \alpha(1) = \beta(0) = 0$, and the functions α and β are continuous on $[0,1]$, as well, thus by Bolzano's Theorem there exists an $x \in [0,1]$ such that $\alpha(x) = \beta(x)$. This means that the vector $\gamma(x)$ has at least two maximum coordinates, and it is easy to see that this implies $\gamma(x) \in \mathcal{T}$. This contradiction completes the proof. □

Considering the examples we discussed so far, it is easy to see that the sets $\mathcal{L} \cup \mathcal{T}$ and the components of $\mathcal{L}$ were always convex sets themselves, so one could ask whether it is true in general. The answer in negative again as the following example shows.

Example 3.2. Consider the graph G as in Figure 4. If $x < 1$ or $2 < x$, then Left wins, but if $1 < x < 2$, Right could win.

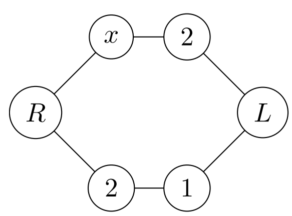

Figure 4: Non-convex example.

In the construction of Example 3.2, it can be seen that if we fix the weight of each vertex but x, then the sets $\mathcal{L}, \mathcal{R}, \mathcal{T}$ divide the half-line in $\mathbb{R}^n_{\geq 0} \setminus 0$ (depending on the choice of x) into more than 2 intervals. The following theorem generalizes this fact.

Theorem 3.5. *For every strictly monotone increasing* $0 < a_k < a_{k-1} \ldots < a_1$ *sequence of integers (note the unusual way of indexing!) there exists a graph* G *with a distinguished vertex* $z \in V(G)$ *and with fixed weights on the vertices distinct to* z*, such that Left has a winning strategy if and only if* $w(z) \in (a_{i+1}, a_i)$ *for some odd* i *(or* $w(z) < a_k$ *and* k *is odd), Right has a winning strategy if and only if* $w(z) \in (a_{i+1}, a_i)$ *for some even* i *(or* $w(z) < a_k$ *and* k *is even) and both have a non-losing strategy if and only if* $w(z)$ *is equal to some* a_i.

Proof. Consider an arbitrary but fixed strictly monotone increasing $0 < a_k < a_{k-1} \ldots < a_1$ sequence of integers. We construct a graph with the desired properties. Our graph will consist of two disjoint paths of length $3k$, a central vertex

c and the vertex z. The ith vertex of the first path is denoted by f_i and the jth vertex of the second path is denoted by s_j. The starting point of Left is f_1, and the starting point of Right is s_1 ($L = f_1$ and $R = s_1$). The central vertex is connected with every sixth vertex of the first path starting from the fourth one (the vertices f_{6i-2} where $1 \le i \le k/2$) and with every sixth vertex of the second path starting from the first one (the vertices s_{6j-5} where $1 \le j \le \lceil k/2 \rceil$). The central vertex is also connected with the vertex z, and there are no more edges in the graph.

The weights of the vertices are the following: $w(z)$ will be varied, $w(f_{3i-1}) = w(s_{3i-1}) = a_i$ where $1 \le i \le k$, and all the other weights are 0. Figure 5 shows the construction for $k = 6$.

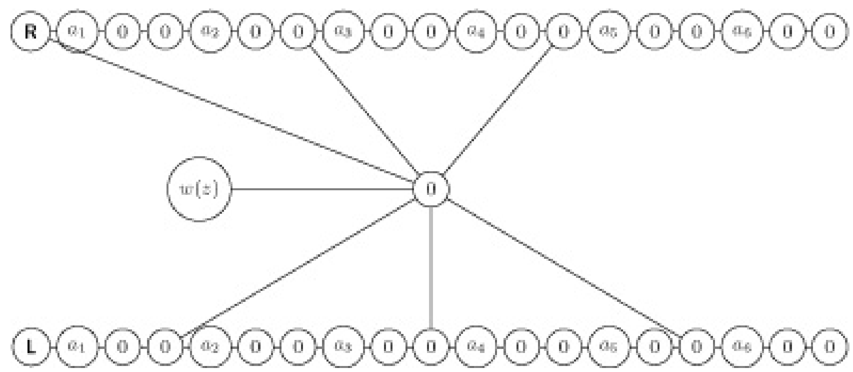

Figure 5: Varying $w(z)$ ruins convexity

We will show first that if a player goes to the c central vertex, and his next step is to go back into his own path, then he will lose. Because we do not specify which player moves like this, we will denote the vertices of his path by v_j instead of f_j or s_j and we will refer to this player as Player A, and the other player as Player B. Suppose that he stepped to c from v_{3i+1}, and he arrived at $v_{3i+1+6j}$ when moving away from c.

We have two different cases: the first case is when he collects the weights on the vertices between v_{3i+1} and $v_{3i+1+6j}$. In this case he takes $3i+6j+2$ steps, so Player B must take at least $3i + 6j + 2$ steps too (Player B might have $3i + 6j + 3$ if he is the Left). Thus Player A collects the first $(i + 2j)$ non-zero weights, but Player B collects the first $(i+2j+1)$ ones, so Player B wins. The second case is when Player A collects the weights on the vertices from $v_{3i+1+6j}$ to v_{3k+1}. In this case Player A takes $3i + 2 + (3k + 1 - (3i + 1 + 6j)) = 3k + 2 - 6j$ steps, so Player B must take at least $3k + 2 - 6j$ steps too. Player A collects $k - 2j$ non-zero vertices, while Player B collects also $k - 2j = \frac{(3k+2-6j)-2}{3}$ non-zero vertices. But Player B collects the largest treasures from the vertices, hence he wins.

Now, we would like to show that if a player goes to the c central vertex, and in his next step he goes to the other player's path, then he would lose. Again, we do not specify which player plays like this, we will denote the vertices of his path by v_j instead of f_j or s_j, the vertices of the other player's path by u_j, and we will refer to this player as Player A, and the other player as Player B. Suppose that he stepped to c from v_{3i+1}, and he arrived at $u_{3i+4+6j}$ when stepping away from c.

We have two different cases: the first case is when Player A leaves his spot toward Player B. In this case Player B collects one non-zero vertex more than Player A, and Player B collects the largest treasures while Player A does not, thus in this case Player B wins. The second case is when Player A leaves his spot in the opposite direction. In this case Player B collects at least as many non-zero vertex as Player A, and Player B collects the largest treasures while Player A does not, hence in this case Player B wins.

We will now examine under which circumstances would it be feasible for a player to step on the vertex c, then on the vertex z. If a player does that, and he steps to c from the vertex v_{3j+1}, then he collects $w(z) + \sum\limits_{1 \le i \le j} a_i$ and the other player collects $\sum\limits_{1 \le i \le j+1} a_i$, so the first player will win if and only if $w(z) > a_{j+1}$, and the game ends in a draw if and only if $w(z) = a_{j+1}$.

Thus the player, who has the opportunity to collect the vertex z such that $w(z)$ is greater than the next a_j he can collect, and who has this opportunity earlier than the opposing player, wins. If neither player has such an opportunity, then both have a non-losing strategy, and this happens exactly when $w(z)$ is equal to some a_j. $\square$

It is natural to ask similar connectivity questions for $\mathcal{R}$ and $\mathcal{R} \cup \mathcal{T}$, too. In order to examine these sets, we define a graph G^L in the following way. For a graph $G = (V, E)$, where $V = \{L', R', v_2, v_3, \ldots, v_n\}$, let $V(G^L) := \{L, R, v_1, v_2, \ldots, v_n\}$, where $R = L', v_1 = R'$ and $E(G^L) := E(G) \cup \{(L, v_1)\}$. When we consider games on the graphs G and G^L, we will assume that Left's initial position is L' (and L, respectively), and Right's initial position is R' (and R, respectively).

Proposition 3.6.

1. *If Left has a winning (non-losing) strategy for a given weighting in the graph G, then Right has a winning (non-losing) strategy in the graph G^L for the same weighting extended in v_1 as 0.*

2. *If Right has a winning (non-losing) strategy for a given weighting in the graph G^L, then Left has a winning (non-losing) strategy in the graph G for the same weighting restricted to $V \setminus \{L', R'\}$.*

Proof (sketch).

1. If after the first step of Left Right plays according to a winning (non-losing) strategy of Left in G, he wins (does not lose).

2. If Left plays just like as a winning (non-losing) strategy of Right after the first step of Left in G^L, he wins (does not lose). □

Corollary 3.7. $\mathbf{x} \in \mathcal{L}_G \Leftrightarrow (0, \mathbf{x}) \in \mathcal{R}_{G^L}$,
$\mathbf{x} \in \mathcal{L}_G \cup \mathcal{T}_G \Leftrightarrow (0, \mathbf{x}) \in \mathcal{R}_{G^L} \cup \mathcal{T}_{G^L}$.
For any $c \in \mathbb{R}_{\geq 0}$ $(c, \mathbf{x}) \in \mathcal{R}_{G^L} \Rightarrow \mathbf{x} \in \mathcal{L}_G$,
$(c, \mathbf{x}) \in \mathcal{R}_{G^L} \cup \mathcal{T}_{G^L} \Rightarrow \mathbf{x} \in \mathcal{L}_G \cup \mathcal{T}_G$.

Proof. This is just a reformulation of Proposition 3.6. □

Using this corollary, it is easy to see that the number of connectivity components of $\mathcal{R}_{G^L}$ is at least the number of connectivity components of $\mathcal{L}_G$ (in fact they are equal). Then, if G is as in Example 3.1, we get that $\mathcal{R}_{G^L}$ could have any number of connectivity components. Example 3.2 shows that $\mathcal{L}_G$ does not need to be convex, hence (by Corollary 3.7) the same is true for $\mathcal{R}_{G^L}$.

We have seen that for every graph G the degree of L (the initial position of player Left) is an upper estimate for the number of connectivity components of $\mathcal{L}_G$. Although $\mathcal{L}_G \cup \mathcal{T}_G$ is always connected, one could ask whether the same applies for the sets $\mathcal{R}_G$ and $\mathcal{R}_G \cup \mathcal{T}_G$ in general.

Question 3.1. Is it true in general that the number of connectivity components of $\mathcal{R}_G$ (if there is any) is at most the degree of R (the initial position of player Right)?

Question 3.2. Is that true that $\mathcal{R}_G \cup \mathcal{T}_G$ is always connected (if it is nonempty)?

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