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PLAYING END-NIM WITH A MULLER TWIST

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Abstract

We study the game of End-Nim enhanced with a Muller twist. The twist is that after completing his turn, a player dictates at which end of the line of heaps his opponent is to play. We characterize the outcome classes for all positions and then present some results concerning Sprague-Grundy values.

1. Introduction

The reader will undoubtedly be familiar with the fundamental combinatorial game of Nim. This is a game for two players and is played with several heaps of beans or counters. On his turn, a player selects a heap and removes one or more beans from that heap. Under normal play, the player who removes the last bean is the winner. In the game of End-Nim, a variation of Nim considered by Albert and Nowakowski in [1], the heaps are placed in a line and a player may remove beans only from the leftmost or rightmost heap.

The engaging game of *Quarto*®, invented by Blaise Muller, is played by two players using 16 different pieces and a board with 16 squares. The game begins with one of the players, Bob, selecting one of the pieces and giving it to his opponent, Alice, who then places it on any square on the board. Alice then selects one of the 15 remaining pieces and gives it to Bob who then places it on one of the remaining squares on the board. Bob completes his turn by giving one of the 14 remaining pieces to Alice, and so on.

We say that a combinatorial game has a *Muller Twist*, as defined by Smith and Stănică in [3], if, on the completion of his turn, a player places a constraint on his opponent's next move. In this paper, we study the game of End-Nim played with a Muller Twist. In our game, the Muller Twist is that after a player has removed beans from a heap, he then dictates to his opponent from which end he must play. To be precise, our game begins with some heaps of beans which are arranged in a line. Play begins with Bob announcing to Alice in which heap, the leftmost or rightmost, she may play. Alice removes one or more beans from the specified heap and then completes her turn by announcing to Bob in which heap, the leftmost or rightmost, he is to play. The game continues in this fashion until the last bean is removed and the player who does so is declared the winner.

We find it more convenient to consider the following equivalent formulation in which the players always remove beans from the leftmost heap. However, after doing so, instead of specifying "left" or "right" for his opponent's next move, a player will either leave the remaining heaps as they are or reverse the order of the heaps. More precisely, a position in our game, which we call Muller End-Nim, is a sequence of positive integers. On his turn, a player reduces, or removes entirely, the first integer in the sequence. The player will then elect to either reverse the order of the sequence or to simply leave it as it is before play passes to his opponent. For example, given the position 3 7 1, a player will subsequently present one of the following 5 positions to his opponent: 2 7 1, 1 7 2, 1 7 1, 7 1, or 1 7. Finally, we make the assumption that, to start the game, the initial sequence has been given to the first player, by either the second player or a referee, with the directive to play in the leftmost heap.

We assume that the reader is familiar with basic concepts of combinatorial game theory such as outcome classes and Sprague-Grundy value, as described in [2]. Positions in the game are denoted simply by positive integers concatenated into strings such as 4 13 7 22 or abc. Exponentiation is used to express repetition so, for example, 1^3 denotes 1 1 1. We frequently use the symbol ω to represent a string of heaps and ω^T denotes the string obtained from ω by reversing the order of the heaps. The Sprague-Grundy value of the position α is denoted $\mathcal{G}(\alpha)$ and for a set S of nonnegative integers, we use $\text{mex}(S)$ to denote the minimum excluded value.

2. Outcome Classes

In this section, we will characterize the $\mathcal{P}$-positions of Muller End-Nim. To this end, we see first of all that a position must have 1 as its first heap to be in $\mathcal{P}$.

Proposition 1. *If $\alpha = a_1 a_2 \dots a_n$ is a $\mathcal{P}$-position then $a_1 = 1$.*

Proof. Suppose to the contrary that $a_1 \geq 2$. Then $\gamma = 1a_2 \dots a_n$ is an option of α and is therefore an $\mathcal{N}$-position. Thus γ has an option $\theta \in \mathcal{P}$. But every option of γ is also an option of α so $\theta \in \mathcal{P}$ is also an option of $\alpha \in \mathcal{P}$ which is impossible. Therefore, we must have $a_1 = 1$. □

We now characterize the $\mathcal{P}$-positions.

Theorem 1. *The set of* $\mathcal{P}$*-positions is precisely* $S = A \cup B$ *where*

$$A = \{1^{2n} \mid n \geq 0\}$$

and

$$B = \{1^n\omega 1^m \mid n \geq 1, m \geq 0, n+m \text{ is odd, first and last heaps in } \omega \neq \emptyset \text{ are not } 1\}.$$

Proof. Let α be a nonempty position in S. We first show that every option of α lies outside of S.

If $\alpha \in A$ then $\alpha = 1^{2n}$ for some $n \geq 1$. The sole option of this position is 1^{2n-1} which is not in S.

If $\alpha \in B$ then $\alpha = 1^n\omega 1^m$ where $n \geq 1$, $m \geq 0$, $n+m$ is odd, and ω is a nonempty string whose first and last heaps are not 1. This position has two options, namely $1^{n-1}\omega 1^m$ and $1^m\omega^T 1^{n-1}$, neither of which is in S since $(n-1)+m$ is even.

Second, we now show that any position $\beta \notin S$ has an option that lies in S.

In the first case, suppose that β begins with a 1. If β consists entirely of 1s then $\beta = 1^{2r+1}$ for some $r \geq 0$. Now the only option of β is 1^{2r} which is in S. On the other hand, if β has some non 1s then we may write $\beta = 1^r\gamma 1^s$ where $r \geq 1$, $s \geq 0$, and γ is a nonempty string that begins and ends with a non 1. Moreover, since $\beta \notin S$, $r+s$ must be even and thus at least two. Now if $r \geq 2$ then the option $1^{r-1}\gamma 1^s$ is in S; otherwise, if $r = 1$, then the option $1^s\gamma^T 1^{r-1}$ is in S since s would have to be at least 1.

In the second case, suppose that $\beta \notin S$ does not begin with a 1. In this instance, we may write $\beta = x\psi$ where $x \geq 2$ and ψ is a string, possibly empty. We consider the three possibilities for ψ.

(a) If ψ is empty then $\beta = x$ and from this position we may move to the empty position which is in S.

(b) If ψ is not empty and consists entirely of 1s then $\beta = x1^r$ where $r \geq 1$. If r is odd then the option 1^{r+1} is in S; otherwise r is even and we remove the first heap to obtain the option 1^r which is in S.

(c) If ψ is not empty and contains some non 1s then we may write $\beta = x1^r\omega 1^s$ where $r, s \geq 0$ and ω is a nonempty string that begins and ends with a non 1. If $r+s$ is even then reducing the first heap to 1 gives the option $1^{r+1}\omega 1^s$ which is in S. On the other hand, suppose that $r+s$ is odd and consider two cases. If $r \geq 1$ then removing the first heap leaves $1^r\omega 1^s$ which is in S. Otherwise, $r = 0$ so $\beta = x\omega 1^s$ where s is odd. In this case, we remove the first heap and reverse the order of the remaining heaps to obtain the option $1^s\omega^T$ which is in S.

The proof is now complete. □

3. Sprague-Grundy Values

In this section, we consider the Sprague-Grundy values of Muller End-Nim positions. After first establishing some basic results, we then determine the value of every position having 2 or 3 heaps. Finally, we establish some results for positions having more than 3 heaps.

3.1. Some Basic Results

We begin by determining the value of a position that consists only of a single heap.

Theorem 2. *For all $a \geq 1$, $\mathcal{G}(a) = a$.*

Proof. The proof is by induction on a. Certainly $\mathcal{G}(1) = 1$ since the only option from 1 is the empty position which has value zero.

Suppose that the result holds for all heaps of size less than $a \geq 2$. The options of the position a are $a-1, a-2, \ldots, 1$ and the empty position which have values $a-1, a-2, \ldots, 1$ and 0 respectively. Thus $\mathcal{G}(a) = \text{mex}\{a-1, a-2, \ldots, 1, 0\} = a$. □

Our next result gives the value of a position in which all the heaps are of size 1.

Proposition 2. *If n is a positive integer then*

$$\mathcal{G}(1^n) = \begin{cases} 0 & n \text{ is even} \\ 1 & n \text{ is odd.} \end{cases}$$

Proof. The proof is by induction on n. The result holds in the case $n = 1$ since $\mathcal{G}(1) = 1$ by Theorem 2.

Now consider the position 1^n where $n \geq 2$. The only option of this position is 1^{n-1}. If n is even then the value of 1^{n-1} is 1 by the induction hypothesis so $\mathcal{G}(1^n) = \text{mex}\{1\} = 0$. Similarly, if n is odd then the value of 1^{n-1} is 0 so $\mathcal{G}(1^n) = 1$. □

We now obtain upper and lower bounds for the value of a position that depend upon the size of the first heap.

Lemma 1. *Let G be a position having H as an option. If every option of H is also an option of G then $\mathcal{G}(G) > \mathcal{G}(H)$.*

Proof. Let $\mathcal{G}(H) = m$. Then H has options of values $0, 1, \ldots, m-1$. Now, by hypothesis, G also has options of values $0, 1, \ldots, m-1$ and also an option of value m, namely H. Thus, G has options of values $0, 1, \ldots, m$ and so $\mathcal{G}(G) \geq m+1$. □

Proposition 3. *If a is the first heap in a position α then $\mathcal{G}(\alpha) \geq a-1$.*

Proof. We write $\alpha = a\omega$ where $a \geq 1$ and ω is a string, possibly empty. The proof is by induction on a.

In the base case $a = 1$, the result holds since trivially $\mathcal{G}(1\omega) \geq 0$.

In the case $a \geq 2$, we note that $\alpha = a\omega$ has $\beta = (a-1)\omega$ as an option, and that every option of β is also an option of α. By Lemma 1, $\mathcal{G}(\alpha) > \mathcal{G}(\beta)$ and, by induction, $\mathcal{G}(\beta) \geq a - 2$ so $\mathcal{G}(\alpha) \geq a - 1$. □

Positions that achieve this lower bound for value are plentiful. Examples include 4 5, 2 7 4, 6 8 3 2 7, and 3 5 1 7 2 3.

Proposition 4. *If a is the first heap in a position α then $\mathcal{G}(\alpha) \leq 2a$.*

Proof. An option of α is obtained by removing at least one and at most a beans from the first heap and then possibly reversing the order of the remaining heaps. Thus there are at most $2a$ distinct options so the value of the position cannot exceed $2a$. □

There exist positions that achieve this upper bound for value. Examples include 3 5 3, 5 4 8 5, and 7 7 7 1 2 6.

The value of a position in which the first heap is of size one cannot exceed 2 by Proposition 4. Our next result shows that in fact the value of such a position is no more than 1.

Proposition 5. *The value of a position in which the first heap has size* 1 *is either* 0 *or* 1*.*

Proof. We consider positions of the form 1ω where ω is any (possibly empty) string. The proof is by induction on the number of heaps in 1ω.

For the base case, the position 1ω consists only of the single heap 1. Since $\mathcal{G}(1) = 1$, the result holds in this case.

Now consider the position 1ω where ω is not empty. This position admits only two options, namely ω and ω^T, so $\mathcal{G}(1\omega) \leq 2$. To complete the proof, we show that $\mathcal{G}(1\omega) = 2$ is not possible.

Note that $\mathcal{G}(1\omega) = 2$ if and only if the two options have the values 0 and 1. Without loss of generality, suppose that $\mathcal{G}(\omega) = 0$ and $\mathcal{G}(\omega^T) = 1$.

Then $\omega \in \mathcal{P}$ so by Theorem 1 either $\omega = 1^{2p}$ for some $p \geq 0$, or $\omega = 1^n z 1^m$ where $n \geq 1$, $m \geq 0$, $n + m$ is odd, and z is a nonempty string that begins and ends with non 1s. In the first instance, we have $\omega^T = 1^{2p} = \omega$ so $\mathcal{G}(\omega^T)$ is also zero which is a contradiction.

In the second instance, we have $\omega^T = 1^m z^T 1^n$. If $m \geq 1$ then ω^T is also a $\mathcal{P}$-position which contradicts $\mathcal{G}(\omega^T) = 1$. Thus we must have $m = 0$ so $\omega = 1^n z$ where n is odd and $\omega^T = z^T 1^n$.

Finally we show that ω^T has an option of value 1 which will contradict $\mathcal{G}(\omega^T) = 1$. We consider two cases.

Suppose in the first case that z consists of only a single heap so we may write $\omega^T = b1^n$ where $b \geq 2$ and n is odd. By clearing the first heap, we may move to the option 1^n which has value 1 by Proposition 2.

In the second case, suppose that z has more than one heap, that is, $z = z_1 z_2 \cdots z_r$ where $z_1, z_2, \ldots, z_r$ are positive integers and both z_1 and z_r are at least 2. Now $\omega^T = z_r z_{r-1} \cdots z_1 1^n$ and by reducing the first heap to one, or by removing it all together, we may move to a position α which begins with a 1 and is not a $\mathcal{P}$-position. Thus $\mathcal{G}(\alpha) \neq 0$ and as α has fewer heaps than 1ω, by induction, $\mathcal{G}(\alpha)$ must be 1. The proof is now complete. □

3.2. Two Heaps

In this section, we consider positions that are composed of exactly two heaps. The following table presents the values of the positions ab for $1 \leq a, b \leq 9$.

a \ b	1	2	3	4	5	6	7	8	9
1	0	0	0	0	0	0	0	0	0
2	2	1	1	1	1	1	1	1	1
3	3	3	2	2	2	2	2	2	2
4	4	4	4	3	3	3	3	3	3
5	5	5	5	5	4	4	4	4	4
6	6	6	6	6	6	5	5	5	5
7	7	7	7	7	7	7	6	6	6
8	8	8	8	8	8	8	8	7	7
9	9	9	9	9	9	9	9	9	8

We now establish the pattern suggested by the above table.

Theorem 3. *Let a and b be positive integers. Then*

$$\mathcal{G}(ab) = \begin{cases} a-1 & \text{if } a \leq b \\ a & \text{if } a > b\,. \end{cases}$$

Proof. The proof is by induction on the value of $a+b$. In the base case $a+b=2$ with $a=b=1$. The value of 11 is 0 by Proposition 2 and the result holds in this case.

We now suppose that $a+b \geq 3$ and consider two cases.

First, suppose that $a \leq b$. If $a=1$ then the sole option of $1b$ is b which has value b by Theorem 2. Thus $\mathcal{G}(1b) = \text{mex}\{b\} = 0$ since $b \geq 1$ and so the result holds when $a=1$. If $a \geq 2$ then the options of ab are comprised of $S = \{kb \mid 1 \leq k \leq a-1\}$, $T = \{bj \mid 1 \leq j \leq a-1\}$, and b. By induction, the options in S have the values $0, 1, \ldots, a-2$, and those in T all have value b. Moreover, $\mathcal{G}(b) = b$ so $\mathcal{G}(ab) = \text{mex}\{0, 1, \ldots, a-2, b\} = a-1$ since $b \geq a$.

Conversely, suppose that $a > b$. The options of ab are comprised of $A = \{ib \mid b+1 \leq i \leq a-1\}$, $B = \{jb \mid 1 \leq j \leq b\}$, $C = \{bk \mid b \leq k \leq a-1\}$, $D = \{bl \mid 1 \leq l \leq b-1\}$, and b. By induction, the values of the positions in A are $a-1, a-2, \ldots, b+1$, and those in B have values $b-1, b-2, \ldots, 0$. Each position in C has value $b-1$, each position in D has value b, and $\mathcal{G}(b) = b$. Therefore, $\mathcal{G}(ab) = \text{mex}\{0, 1, \ldots, a-1\} = a$. □

3.3. Three Heaps

In this section, we consider positions that are composed of exactly three heaps. As an example, the following table presents the values of the positions $4ab$ for $1 \leq a, b, \leq 9$.

a \ b	1	2	3	4	5	6	7	8	9
1	4	5	6	7	4	4	4	4	4
2	5	5	6	7	4	4	4	4	4
3	5	6	6	7	4	4	4	4	4
4	5	6	7	4	4	4	4	4	4
5	4	6	7	8	3	3	3	3	3
6	4	5	7	8	3	3	3	3	3
7	4	5	6	8	3	3	3	3	3
8	4	5	6	7	3	3	3	3	3
9	4	5	6	7	3	3	3	3	3

The bottom right corner of the table suggests that the value of a position in which the first heap is smaller than the other two is simply one less than the size of the first heap. This is indeed the case and we may now prove it.

Proposition 6. *Let m be a positive integer. If a and b are positive integers with $a > m$ and $b > m$ then $\mathcal{G}(mab) = m - 1$.*

Proof. The proof is by induction on m.

For the base case $m = 1$, we note that, since $a > 1$ and $b > 1$, the position $1ab \in \mathcal{P}$ by Theorem 1. Therefore, $\mathcal{G}(1ab) = 0$ and the result holds in the case $m = 1$.

Now consider the position mab where $m \geq 2$. The options of this position are composed of $S = \{iab \mid 1 \leq i \leq m-1\}$, $T = \{baj \mid 1 \leq j \leq m-1\}$, ab, and ba.

The values of the positions in S are $0, 1, \ldots, m-2$ by induction. Moreover, by Proposition 3, the value of every position in T is at least $b-1$. Using the same result, we see that the values of ab and ba are at least $a-1$ and $b-1$, respectively.

Finally, we note that $a-1$ and $b-1$ are both at least m since $a > m$ and $b > m$. Thus the position mab has options having values $0, 1, \ldots, m-2$ but it has no option of value $m-1$ so we conclude that $\mathcal{G}(mab) = m - 1$. □

From Proposition 3, $m-1$ is a lower bound for the value of the position mab. Our previous result, together with the next one, characterize all three heap positions that attain this bound.

Proposition 7. *Let m, a, and b be a positive integers. If $\mathcal{G}(mab) = m-1$ then $a > m$ and $b > m$.*

Proof. Suppose that $\mathcal{G}(mab) = m-1$. Then, by Lemma 1 and Proposition 3, we have $\mathcal{G}(kab) = k-1$ for all $1 \leq k \leq m$. Since ab is an option of all these positions, $\mathcal{G}(ab)$ must be at least m, and for the same reason, $\mathcal{G}(ba) \geq m$.

Now if $a \leq b$ then, by Theorem 3, $\mathcal{G}(ab) = a-1$. Thus $a-1 \geq m$ so $a \geq m+1$ and $b \geq a \geq m+1$.

On the other hand, if $a > b$ then $\mathcal{G}(ba) = b-1$ and so $b-1 \geq m$ which implies that $b \geq m+1$ and $a > m+1$.

In either case, $a > m$ and $b > m$ as desired. □

The following corollary follows immediately from Proposition 7 and Proposition 3.

Corollary 1. *Let m be a positive integer. If a and b are positive integers with $a \leq m$ or $b \leq m$ then $\mathcal{G}(mab) \geq m$.*

Our next result establishes the pattern suggested by the top right corner of the table at the beginning of this section.

Proposition 8. *Let m be a positive integer. If a and b are positive integers with $a \leq m$ and $b > m$ then $\mathcal{G}(mab) = m$.*

Proof. The proof is by induction on m.

In the base case $m = 1$, we consider a position of the form $11b$ where $b \geq 2$. By Theorem 1, this position is not in $\mathcal{P}$ so its value is nonzero. Therefore, by Proposition 5, $\mathcal{G}(11b) = 1$ and the result holds when $m = 1$.

We now consider a position mab where $a \leq m$, $b > m$, and $m \geq 2$. The options of this position are composed of $S = \{iab \mid a \leq i \leq m-1\}$, $T = \{jab \mid 1 \leq j \leq a-1\}$, $U = \{bak \mid 1 \leq k \leq m-1\}$, ab, and ba.

By induction, the values of the positions in S are $m-1, m-2, \ldots, a$. Moreover, the values of the positions in T are $a-2, a-3, \ldots, 0$ by Proposition 6. Furthermore, since $b > m$, every position in U has value at least b by Corollary 1.

Finally, since $a \leq m < b$, we have $\mathcal{G}(ab) = a-1$ and $\mathcal{G}(ba) = b$ by Theorem 3.

We see that the position mab has options of values $0, 1, \ldots, m-1$ but, since $b > m$, it does not have an option of value m. Thus $\mathcal{G}(mab) = m$. □

We now turn our attention to the value of positions of the form mab where $b \leq m$. The table of values of the positions $4ab$ is suggestive but the patterns are more complicated than those established earlier for the case $b > m$. The next four propositions describe the values of the positions in which $b \leq m$.

Proposition 9. *Let m be a positive integer. If a and b are positive integers with $a \geq m + b$ and $b \leq m$ then $\mathcal{G}(mab) = m + b - 1$.*

Proof. The proof is by induction on m.

For the base case $m = 1$, we consider a position of the form $1a1$ where $a \geq 2$. By Theorem 1, this position is not in $\mathcal{P}$ so, by Proposition 5, its value is $1 = 1 + 1 - 1$ and the result holds in this case.

Now consider a position $\alpha = mab$ where $m \geq 2$, $a \geq m + b$, and $b \leq m$. The options of α are composed of $S = \{iab \mid b \leq i \leq m - 1\}$, $T = \{jab \mid 1 \leq j \leq b - 1\}$, $U = \{bak \mid b + 1 \leq k \leq m - 1\}$, $V = \{bal \mid 1 \leq l \leq b\}$, as well as ab and ba.

By induction, the values of the positions in S and V are $m + b - 2, m + b - 3, \ldots, 2b - 1$ and $2b - 1, 2b - 2, \ldots, b$, respectively. Moreover, by Proposition 6, the values of the positions in T and U are $0, 1, \ldots, b - 2$, and $b - 1$, respectively. Since $a > b$, the values of ab and ba are a and $b - 1$, respectively, by Theorem 3.

Thus $\mathcal{G}(mab) = \text{mex}\{0, 1, \ldots, m + b - 2, a\} = m + b - 1$ as desired. □

Proposition 10. *Let m be a positive integer. If a and b are positive integers with $b + 1 \leq a \leq m + b - 1$ and $b \leq m$ then $\mathcal{G}(mab) = m + b$.*

The proof is by induction on m and makes use of Proposition 6 and Proposition 9. We omit the details of the proof but instead illustrate with an example.

Consider the position 13 10 6.

By induction, the options $12\ 10\ 6, 11\ 10\ 6, \ldots, 6\ 10\ 6$ have the values $18, 17, \ldots, 12$ and, by Proposition 6, $5\ 10\ 6, 4\ 10\ 6, \ldots, 1\ 10\ 6$ have the values $4, 3, \ldots, 0$.

Each of the options $6\ 10\ 12, 6\ 10\ 11, \ldots, 6\ 10\ 7$ has value 5 by Proposition 6, 6 10 6 and 6 10 5 have the values 12 and 11 respectively by induction, and $6\ 10\ 4, 6\ 10\ 3, 6\ 10\ 2, 6\ 10\ 1$ have the values $9, 8, 7$, and 6 by Proposition 9.

Finally, the options 10 6 and 6 10 have the values 10 and 5 respectively, by Theorem 3 and we conclude that $\mathcal{G}(13\ 10\ 6) = \text{mex}\{0, 1, \ldots, 18\} = 19$.

Proposition 11. *For any positive integer m, the value of the position mmm is m.*

Proof. The proof is by induction on m.

From Proposition 2, $\mathcal{G}(111) = 1$ so the result holds in the base case $m = 1$.

We now consider the position mmm where $m \geq 2$. The options of this position are composed of $S = \{kmm \mid 1 \leq k \leq m - 1\}$, $T = \{mmj \mid 1 \leq j \leq m - 1\}$, and mm.

By Proposition 6, the values of the positions in S are $\{0, 1, \ldots, m-1\}$. Moreover, the values of the positions in T are $\{m + 1, m + 2, \ldots, 2m - 1\}$ by Proposition 10, and by Theorem 3, the value of mm is $m - 1$.

Therefore, $\mathcal{G}(mmm) = \text{mex}\{0, 1, \ldots, m - 1, m + 1, m + 2, \ldots, 2m - 1\} = m$. □

Proposition 12. *Let m be a positive integer. If a and b are positive integers such that $\{a \leq b-1$ and $b \leq m\}$ or $\{a = b$ and $b \leq m-1\}$ then $\mathcal{G}(mab) = m + b - 1$.*

The proof is by induction on m and requires Proposition 6, Proposition 8, and Proposition 10. The details of the proof are omitted; the following example is perhaps more illustrative.

Consider the position 15 3 7 which satisfies the hypotheses.

Options 14 3 7 down to 7 3 7 have values 20 down to 13 by induction. Options 6 3 7 down to 3 3 7 have values 6 down to 3 by Proposition 8, and options 2 3 7 and 1 3 7 have values 1 and 0 respectively, by Proposition 6.

Each of the options 7 3 14 down to 7 3 8 has value 7 by Proposition 8. Options 7 3 7 down to 7 3 3 have values 13 down to 9 by induction, and options 7 3 2 and 7 3 1 have values 9 and 8 respectively by Proposition 10.

Finally, by Theorem 3, $\mathcal{G}(3\ 7) = 2$ and $\mathcal{G}(7\ 3) = 7$ and we see that $\mathcal{G}(15\ 3\ 7) = \text{mex}\{0, 1, \ldots, 20\} = 21$.

To summarize, we have determined the value of every position that has exactly 3 heaps and now restate our results in the following two theorems.

Theorem 4. *Let m be a positive integer. If a and b are positive integers with $b > m$ then*

$$\mathcal{G}(mab) = \begin{cases} m & \text{if } a \leq m \\ m-1 & \text{if } a > m. \end{cases}$$

Theorem 5. *Let m be a positive integer. If a and b are positive integers with $b \leq m$ then*

$$\mathcal{G}(mab) = \begin{cases} m+b-1 & \text{if } a \leq b-1 \\ m+b-1 & \text{if } a = b < m \text{ and } b < m \\ m & \text{if } a = b = m \\ m+b & \text{if } b+1 \leq a \leq m+b-1 \\ m+b-1 & \text{if } a \geq m+b. \end{cases}$$

Earlier in this section, we characterized those positions mab which attained the lower bound of $m-1$ for value. To conclude, we now present those positions which achieve the upper bound of $2m$.

Corollary 2. *Let m, a, and b be positive integers. Then $\mathcal{G}(mab) = 2m$ if and only if $m \geq 2$, $m+1 \leq a \leq 2m-1$, and $b = m$.*

3.4. Several Heaps

We now turn our attention to positions with more than 3 heaps. We are not able to describe the value of all such positions, but can do so in some particular cases.

Our first result generalizes Proposition 6 to the case of more than 3 heaps.

Proposition 13. *Let m be a positive integer and ω a string, possibly empty. If a and b are positive integers with $a > m$ and $b > m$ then $\mathcal{G}(ma\omega b) = m - 1$.*

Proof. The proof is by induction on m.

For the base case $m = 1$, we note that, since $a > 1$ and $b > 1$, the position $1a\omega b \in \mathcal{P}$ by Theorem 1. Therefore $\mathcal{G}(1a\omega b) = 0$ and the result holds in the case $m = 1$.

Now consider the position $ma\omega b$ where $m \geq 2$. The options of this position are $S = \{ia\omega b \mid 1 \leq i \leq m-1\}$, $T = \{b\omega^T aj \mid 1 \leq j \leq m-1\}$, $a\omega b$, and $b\omega^T a$.

The values of the positions in S are $m-2, m-3, \ldots, 0$ by induction. Moreover, by Proposition 3, the value of every position in T is at least $b-1$, and the values of $a\omega b$ and $b\omega^T a$ are at least $a-1$ and $b-1$, respectively.

Finally, since $a > m$ and $b > m$, we have that $a-1$ and $b-1$ are both at least m. Thus the position $ma\omega b$ has options having values $0, 1, \ldots, m-2$ but it has no option of value $m-1$ so we conclude that $\mathcal{G}(ma\omega b) = m-1$. □

The next result is a generalization of Proposition 8 to several heaps.

Proposition 14. *Let m be a positive integer and ω a nonempty string. If a and b are positive integers with $a \leq m$ and $b \geq m+2$ then*

$$\mathcal{G}(ma\omega b) = \begin{cases} m & \text{if } a < m \\ m \text{ or } m-1 & \text{if } a = m. \end{cases}$$

Proof. The proof is by induction on m. In the base case $m = 1$, the result holds since we must also have $a = 1$ and the value of the position is either 0 or 1 by Proposition 5.

Consider now a position $ma\omega b$ where $m \geq 2$, $a \leq m$, and $b \geq m+2$.

Suppose that $a = m$. The options of the position $mm\omega b$ are comprised of $A = \{km\omega b \mid 1 \leq k \leq m-1\}$, $B = \{b\omega^T mj \mid 1 \leq j \leq m-1\}$, $m\omega b$, and $b\omega^T m$. By Proposition 13, the values of the positions in A are $0, 1, \ldots, m-2$. Moreover, by Proposition 3, the value of every position in B and the position $b\omega^T m$ is at least $b-1 \geq m+1$. It remains to consider the value of $m\omega b$ which, again by Proposition 3, is at least $m-1$. Finally, we see that $\mathcal{G}(mm\omega b) = \text{mex}\{0, 1, \ldots, m-2, \mathcal{G}(m\omega b)\}$ equals m if $\mathcal{G}(m\omega b) = m-1$ but $m-1$ if $\mathcal{G}(m\omega b) \geq m$.

On the other hand, suppose that $a < m$. The options of $ma\omega b$ are comprised of $S = \{ia\omega b \mid a \leq i \leq m-1\}$, $T = \{ja\omega b \mid 1 \leq j \leq a-1\}$, $U = \{b\omega^T ak \mid 1 \leq k \leq m-1\}$, $a\omega b$, and $b\omega^T a$. By Proposition 13, the values of the positions in T are $0, 1, \ldots, a-2$ and, by Proposition 3, every position in U, as well as $b\omega^T a$, has value at least $b-1 \geq m+1$. Moreover, by induction, the positions in S of the form $ia\omega b$ with $a+1 \leq i \leq m-1$ have the values $a+1, a+2, \ldots, m-1$.

It remains to consider the values of $aa\omega b$ and $a\omega b$. By induction, the value of $aa\omega b$ is either a or $a-1$. Let ω_1 denote the first heap in the string ω. If $\omega_1 > a$ then $\mathcal{G}(a\omega b) = a-1$ by Proposition 13. On the other hand, if $\omega_1 \leq a$ then, by induction, $\mathcal{G}(a\omega b) = a$ or $a-1$. (In the event that ω_1 is the only heap in the string ω then we may make the same conclusions about $\mathcal{G}(a\omega b)$ by using Theorem 4.) To summarize, each of the positions $aa\omega b$ and $a\omega b$ has the value a or $a-1$. However, $aa\omega b$ and $a\omega b$ cannot have the same value since the latter is an option of the former. Therefore, we must have $\{\mathcal{G}(aa\omega b), \mathcal{G}(a\omega b)\} = \{a-1, a\}$.

To conclude then, in the case $a < m$, the position $ma\omega b$ has options of values $0, 1, \ldots, m-1$ but no option of value m. Thus $\mathcal{G}(ma\omega b) = m$. □

We note that in the case $a = m$ in the above result both possibilities may occur. For example, $\mathcal{G}(2\ 2\ 1\ 4) = 1$ and $\mathcal{G}(2\ 2\ 3\ 4) = 2$.

Finally in this section, we consider positions in which the last heap is of size 1 and show that the value of such a position is closely related to the size of the first heap.

Lemma 2. *Let ω be a string, possibly empty. For any $m \geq 2$,*

$$\mathcal{G}(m\omega 1) = \begin{cases} m+1 & \text{if } 2 \leq \mathcal{G}(\omega 1) \leq m \\ m & \text{otherwise.} \end{cases}$$

Proof. The proof is by induction on m.

For the base case $m = 2$, we consider the position $2\omega 1$. From this position, we will move to one of the positions $1\omega 1$, $1\omega^T 1$, $\omega 1$, or $1\omega^T$. By Proposition 5, each of the positions $1\omega 1$, $1\omega^T 1$, and $1\omega^T$ has the value 0 or 1. Moreover, the positions $1\omega 1$ and $1\omega^T$ cannot have the same value because the latter is an option of the former. Thus $\{\mathcal{G}(1\omega 1), \mathcal{G}(1\omega^T)\} = \{0, 1\}$ so the value of $2\omega 1 = \text{mex}\{0, 1, \mathcal{G}(\omega 1)\}$ which is 3 if $\mathcal{G}(\omega 1) = 2$ and is 2 otherwise.

We now consider a position $m\omega 1$ where $m \geq 3$. The options of this position are composed of $A = \{r\omega 1 \mid 2 \leq r \leq m-1\}$, $B = \{1\omega^T s \mid 1 \leq s \leq m-1\}$, $C = \{1\omega 1, 1\omega^T\}$, and $\omega 1$. By Proposition 5, every position in B has value 0 or 1, and, as noted above, the values of the positions in C are precisely the set $\{0, 1\}$.

For sake of clarity, let v denote $\mathcal{G}(\omega 1)$, the value of the position $\omega 1$.

Suppose that $2 \leq v \leq m-1$ and consider the values of the positions in A. By induction, we see that the position $k\omega 1$ has value $k+1$ if $v \leq k \leq m-1$ and value k if $2 \leq k \leq v-1$. In other words, the values of the positions in A are precisely the set $\{2, 3, \ldots, m\} \setminus \{v\}$. However, the value of $\omega 1$ is v so $\mathcal{G}(m\omega 1) = \text{mex}\{0, 1, \ldots, m\} = m+1$.

On the other hand, if $v \leq 1$ or $v \geq m$ then, by induction, the values of the positions in A are $2, 3, \ldots, m-1$ and so $\mathcal{G}(m\omega 1) = \text{mex}\{0, 1, 2, \ldots, m-1, v\}$ which equals $m+1$ if $v = m$ and m otherwise. □

Theorem 6. *Let ω be a string, possibly empty, and a be any positive integer. For any $m \geq 2$,*

$$\mathcal{G}(ma\omega 1) = \begin{cases} m & \text{if } a = 1 \\ m+1 & \text{if } 2 \leq a \leq m-1 \\ m \text{ or } m+1 & \text{if } a = m \\ m & \text{if } a > m\ . \end{cases}$$

Proof. By Lemma 2, the value of $ma\omega 1$ is either m or $m+1$ so in the case of $a = m$ the result is immediate.

In the case $a = 1$, we have $\mathcal{G}(1\omega 1) \leq 1$ by Proposition 5 so it follows that $\mathcal{G}(m1\omega 1) = m$ by Lemma 2.

Now suppose that $a \geq 2$. From Lemma 2, the value of $a\omega 1$ is either a or $a+1$. In particular, if $a > m$ then $\mathcal{G}(a\omega 1) > m$ so $\mathcal{G}(ma\omega 1) = m$ by Lemma 2. Or, if $2 \leq a \leq m-1$ then $2 \leq \mathcal{G}(a\omega 1) \leq m$ and applying Lemma 2 again, we obtain $\mathcal{G}(ma\omega 1) = m+1$. □

We note that in positions where the first two heaps are of the same size m, the value can be either m or $m+1$. For example, $\mathcal{G}(2\ 2\ 1\ 1) = 3$ and $\mathcal{G}(2\ 2\ 2\ 1) = 2$.

4. Conclusion

Much more remains to be discovered about the values of positions in Muller End-Nim. We conclude by enumerating some possibilities for further research and make some conjectures suggested by numerical evidence.

- The most important and tantalizing question is to find a convenient way to determine the value of any position. It appears that, in general, the value of a position does not depend merely upon the initial and terminal heaps as in the case of Proposition 13. For example, $\mathcal{G}(6\ 8\ 4\ 3\ 4\ 5) = \mathcal{G}(6\ 8\ 6\ 3\ 4\ 5) = 11$ yet $\mathcal{G}(6\ 8\ 5\ 3\ 4\ 5) = 12$.

- Proposition 4 gives a simple upper bound on the value of a position, namely double the size of the first heap. It appears, however, that a more refined upper bound may exist that depends upon the sizes of the first and last heaps.

  Conjecture 1. *Let a and b be positive integers and ω any (possibly empty) string. Then $\mathcal{G}(a\omega b) \leq a + b + 1$.*

  The positions analyzed by our computer program were limited to heaps of size 9 or less. With this restriction, the position 3 4 1 2 2 was the smallest found to meet this upper bound.

- By Proposition 3, we know that the value of any position is at least one less than the size of the first heap. We make the following conjecture regarding a necessary condition on those positions that attain this lower bound for value.

  Conjecture 2. *Let a and b be positive integers and ω any (possibly empty) string. If $\mathcal{G}(a\omega b) = a - 1$ then $a - b \leq 1$.*

  Our computer program found that the position 5 4 2 4 was the smallest to meet the above bound.

- We make the following conjecture which extends the result of Proposition 2 to positions in which all the heaps are of the same size.

Conjecture 3. *If m and n are positive integers then*

$$\mathcal{G}(m^n) = \begin{cases} m-1 & n \text{ is even} \\ m & n \text{ is odd.} \end{cases}$$

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