STEINHAUS TRIANGLES GENERATED BY VECTORS OF THE CANONICAL BASIS

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Abstract

We give a method to calculate the weight of binary Steinhaus triangles generated by the vectors of the canonical basis of the $\mathbb{F}_2$-vector space $\mathbb{F}_2^n$.

1. Introduction

In 1958 H. Steinhaus [17, Chapter VII] posed the following problem:

> "The figure given below consists of 14 plus signs and 14 minus signs. They are arranged in such a way that under each pair of equal signs there appears a positive sign and under opposite signs a minus sign.
>
> $$\begin{array}{ccccccccccccc} + & & + & & - & & + & & - & & + & & + \\ & + & & - & & - & & - & & - & & + & \\ & & - & & + & & + & & + & & - & & \\ & & & - & & + & & + & & - & & & \\ & & & & - & & + & & - & & & & \\ & & & & & - & & - & & & & & \\ & & & & & & + & & & & & & \end{array}$$
>
> If the first row had n signs, then in an analogous figure there would be $n(n+1)/2$ signs; our example corresponds to the case $n = 7$. As $n(n+1)/2$ is an even number for $n = 3, 4, 7, 8, 11, 12, \ldots$, etc., we can ask whether it is possible to construct a figure analogous to the above one and beginning with n signs in the highest row."

The figure above is called a *Steinhaus triangle*. We can replace the signs $+$ and $-$ by 1 and 0, respectively, and reformulate the rule of signs by the sum in the field $\mathbb{F}_2$ of order 2. The number of ones in a sequence is called its *weight*, and the number of ones in the whole triangle is called the *weight* of the triangle. More formally, given a sequence $\mathbf{x} = (x_0, \ldots, x_{n-1}) \in \mathbb{F}_2^n$, its *derivative* is the sequence $\partial\mathbf{x} = (x_0 + x_1, x_1 + x_2, \ldots, x_{n-2} + x_{n-1}) \in \mathbb{F}_2^{n-1}$. Recursively, for $r \geq 2$, we define $\partial^r\mathbf{x} = \partial\partial^{r-1}\mathbf{x}$; we also define $\partial^0\mathbf{x} = \mathbf{x}$. The *Steinhaus triangle* generated by $\mathbf{x}$ is the sequence $T(\mathbf{x}) = (\mathbf{x}, \partial\mathbf{x}, \partial^2\mathbf{x}, \ldots, \partial^{n-1}\mathbf{x})$. The length n of the initial sequence is the *size* of the triangle. The r-th derivative $\partial^r\mathbf{x}$ of $\mathbf{x}$ is the r-th row the triangle (thus, we number rows from 0 to $n-1$). We number the coordinates of $\partial^r\mathbf{x}$ from 0 to $n-1-r$. The *weight* of the sequence $\mathbf{x} = (x_0, \ldots, x_{n-1}) \in \mathbb{F}_2^n$ is $|\mathbf{x}| = \#\{i : x_i = 1\}$, and the *weight* of $T(\mathbf{x})$ is

$$|T(\mathbf{x})| = \sum_{i=0}^{n-1} |\partial^i\mathbf{x}|.$$

The problem posed by Steinhaus is the following one: given $n \equiv 0, 3 \pmod 4$, determine if there exist sequences $\mathbf{x} \in \mathbb{F}_2^n$ such that $|T(\mathbf{x})| = n(n+1)/4$. H. Harborth [14] solved the problem by constructing examples of such sequences. The problem has also been solved for sequences $\mathbf{x}$ with some additional conditions (S. Elihaou and D. Hachez [10, 11], S. Eliahou, J. M. Marín and M. P. Revuelta [12]), and also generalized for sequences in the cyclic group $\mathbb{Z}_m$ (J. C. Molluzzo [16], J. Chappelon [6, 7, 8], J. Chappelon and S. Eliahou [9]). Steinhaus triangles appear in the context of cellular automata; see A. Barbé [1, 2, 3] and J. Chappelon [7, 8]. In this context, A. Barbé [3] has studied some properties related to symmetries. Steinhaus triangles with rotational and dihedral symmetry are characterized by J. M. Brunat and M. Maureso in [4].

An easy induction that involves only the binomial number recurrence shows that the entry of a Steinhaus triangle $T(\mathbf{x})$ on row r and column c is

$$T(\mathbf{x})(r, c) = \sum_{i=0}^{r} \binom{r}{i} x_{c+i}, \tag{1}$$

so it depends linearly on the entries of $\mathbf{x}$. Thus, the set $S(n)$ of Steinhaus triangles of size n is an $\mathbb{F}_2$-vector space of dimension n that can be identified with a vector subspace of $\mathbb{F}_2^{n(n+1)/2}$, that is, with a (binary) linear code of length $n(n+1)/2$ and dimension n. Then, the weight distribution of Steinhaus triangles, or equivalently, how many triangles exist of every possible weight, is a particular case of the general problem of weight distribution in linear codes. This is, in general, a difficult problem, and it seems that it is also difficult for the particular case of Steinhaus triangles.

Obviously, in $S(n)$ there exists only one Steinhaus triangle of weight 0, which is the one generated by the sequence $\mathbf{0} = (0, \ldots, 0)$. It is easy to see that the next possible weight is n and that it is generated by the sequences $\mathbf{1} = (1, \ldots, 1)$, $\mathbf{e}_0 = (1, 0, \ldots, 0)$, and $\mathbf{e}_{n-1} = (0, \ldots, 0, 1)$. H. Harborth, in his paper [14] solving the original problem, observed that the maximum weight is $\lceil n(n+1) \rceil$ and gave the sequences that generates triangles of this maximum weight. G. J. Chang [5] determined the triangles having the first four non-zero weights and the last two weights. Also, there is an asymptotic result by F. M. Malysev and E. V. Kutyreva [15], who estimated the number of Steinhaus triangles (which they call Boolean Pascal triangles, and take the initial sequence in the bottom of the triangle) of sufficiently large size n containing a given number $\omega \leq kn$ $(k > 0)$ of ones. Not much else is known about the weight distribution of Steinhaus triangles.

Here, we study the weight of Steinhaus triangles generated by the vectors of the canonical basis. The main tool is Lucas's Theorem, so let us recall it. Given a prime number p, Lucas's Theorem gives a way to calculate binomial numbers in $\mathbb{F}_p$, the field of order p.

Theorem 1 (Lucas's Theorem). *Let p be a prime number,*

$$r = \alpha_t p^t + \alpha_{t-1} p^{t-1} + \cdots + \alpha_1 p + \alpha_0, \text{ and}$$
$$s = \beta_t p^t + \beta_{t-1} p^{t-1} + \cdots + \beta_1 p + \beta_0,$$

with $\alpha_i, \beta_i \in \{0, 1, \ldots, p-1\}$ for $i \in \{0, \ldots, t\}$. Then,

$$\binom{r}{s} = \binom{\alpha_t}{\beta_t} \binom{\alpha_{t-1}}{\beta_{t-1}} \cdots \binom{\alpha_0}{\beta_0} \quad \text{in } \mathbb{F}_p.$$

See [13] for a nice proof by N. J. Fine.

2. Canonical Basis

For $n \geq 1$ and $0 \leq k \leq n-1$, the k-th vector of the canonical basis of $\mathbb{F}_2^n$ is

$$\mathbf{e}_k^{(n)} = (\underbrace{0, \ldots, 0}_{k \text{ terms}}, 1, \underbrace{0, \ldots, 0}_{n-1-k \text{ terms}}).$$

In order to simplify notation, we define $T(k, n) = T(\mathbf{e}_k^{(n)})$ and $w(k, n) = |T(k, n)|$. We give a way to calculate $w(k, n)$ that depends on $t = \lfloor \log_2 k \rfloor + 1$ and on the quotient and the remainder of dividing n by 2^t.

We apply equation (1) to the vectors of the canonical basis of $\mathbb{F}_2^n$. The coordinates of $\mathbf{e}_k^{(n)}$ are $x_k = 1$ and $x_{c+i} = 0$ for $i \neq k - c$. Therefore,

$$T(k, n)(r, c) = \binom{r}{k - c} = \binom{r}{r - k + c}.$$

In particular, $T(k,n)(r,c) = 0$ for $c > k$, that is, in any row of $T(k,n)$, all entries in positions c with $c > k$ are 0.

As $T(0,n)(r,0) = 1$ and $T(0,n)(r,c) = 0$ for $c > 0$, we have $w(0,n) = n$ for all $n \geq 1$. Thus, in the following, we assume $k \geq 1$.

Note that the triangles $T(k,n)$ and $T(n-1-k,n)$ are symmetric with respect to the vertical line passing through the bottom vertex of the triangle:

$$\begin{aligned} T(n-1-k,n)(r,n-1-r-c) =& \binom{r}{r-(n-1-k)+(n-1-r-c)} \\ =& \binom{r}{k-c} = T(k,n)(r,c). \end{aligned}$$

As we are interested in the values of $w(k,n) = |T(k,n)|$, and the triangles $T(k,n)$ and $T(n-1-k,n)$ have the same weight because of symmetry, we can assume $n \geq 2k+1 \geq 3$.

For $m \in \{1,\dots,n\}$, denote by $s(k,n,m)$ the sum of the weights of the rows $0,1,\dots,m-1$ of $T(n,k)$. Our main result is the following.

Theorem 2. *Let $k \geq 1$ and $n \geq 2k+1$ be integers. Let $t = \lfloor \log_2 k \rfloor + 1$, $q = \lfloor n/2^t \rfloor$, $r = n - 2^t q$, $\lambda = s(k, k+1+2^t, 2^t)$, and $\mu = w(k, r+2^t)$. Then,*

$$w(k,n) = (q-1)\lambda + \mu.$$

Proof. We have $2^{t-1} \leq k < 2^t$, hence $n \geq 2k+1 \geq 2^t+1$. We claim that row 2^t of $T(k,n)$ is just $\mathbf{e}_k^{(n-2^t)}$. Indeed, we have

$$T(k,n)(2^t,c) = \binom{2^t}{k-c}.$$

If $c < k$, and $k - c = \alpha_a 2^a + \cdots + \alpha_1 2 + \alpha_0$, we have $a < t$ and some $\alpha_i = 1$ because $k - c > 0$. By Lucas's Theorem,

$$\binom{2^t}{k-c} = \binom{1}{0}\binom{0}{0}\cdots\binom{0}{0}\binom{0}{\alpha_a}\cdots\binom{0}{\alpha_i}\cdots\binom{0}{\alpha_0} = 0$$

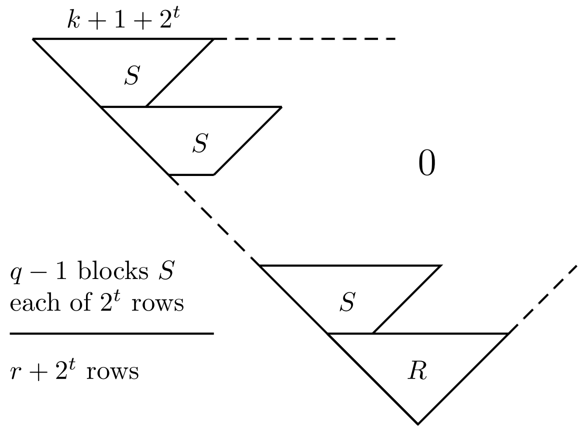
because $\binom{0}{\alpha_i} = \binom{0}{1} = 0$. Also, we have $T(k,n)(2^t,k) = \binom{2^t}{0} = 1$, and, for $c > k$, $T(k,n)(2^t,c) = \binom{2^t}{k-c} = 0$, so $\partial^{2^t}\mathbf{e}_k^{(n)} = \mathbf{e}_k^{(n-2^t)}$.

Therefore, if we delete the first 2^t rows of the triangle $T(k,n)$, we obtain the triangle $T(k,n-2^t)$. Then, $w(k,n) - w(k,n-2^t) = s(k,n,2^t)$. In any row, all entries in positions $c > k$ are 0. Then, $s(k,n,2^t) = s(k,k+1+2^t,2^t)$. Let q and r be the quotient and the remainder of dividing n by 2^t. Then, for a fixed k, the sequence

$$w(k,r+2^t),\ w(k,r+2\cdot 2^t),\ w(k,r+3\cdot 2^t),\ \dots,\ w(k,r+q\cdot 2^t) = w(k,n)$$

is an arithmetic progression with difference $\lambda = s(k,k+1+2^t,2^t)$. If $\mu = w(k,r+2^t)$, we have $w(k,n) = \lambda(q-1) + \mu$. $\square$

With the notation of the previous theorem, let S be the region formed by the 2^t first rows of $T(k, k+1+2^t)$, that is, the region used to calculate $s(k, k+1+2^t, 2^t)$, and let $R = T(k, r+2^t)$. Then, $T(k, n)$ has the following schema:



For instance, let us compute $w(6, 203)$. We have $k = 6$; $t = 3$; $2^t = 8$, $n = 203 = 8{\cdot}25{+}3$, that is $q = 25$ and $r = 3$. Let us calculate $\lambda = s(k, k{+}1{+}2^t, 2^t) = s(6, 15, 8)$ and $\mu = w(k, r+2^t) = w(6, 11)$:

$$\begin{array}{c}
0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\
0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\
0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \\
0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0 \\
0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0 \\
0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0 \\
1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0 \\
1\ 1\ 1\ 1\ 1\ 1\ 1\ 0
\end{array} \qquad (S)$$

$$\begin{array}{c}
0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0 \\
0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0 \\
0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0 \\
0\ 0\ 0\ 1\ 1\ 1\ 1\ 0 \\
0\ 0\ 1\ 0\ 0\ 0\ 1 \\
0\ 1\ 1\ 0\ 0\ 1 \\
1\ 0\ 1\ 0\ 1 \\
1\ 1\ 1\ 1 \\
0\ 0\ 0 \\
0\ 0 \\
0
\end{array} \qquad (R)$$

We see that $\lambda = s(6, 15, 8) = 26$ and $\mu = w(6, 11) = 21$. Then $w(6, 203) = \lambda(q-1) + \mu = 26 \cdot 24 + 21 = 645$.

Note that $\lambda = s(k, k+1+2^t, 2^t)$ depends only on k, and not on n. Moreover, $\mu = w(k, 2^t + r)$ depends on k and r, but r, as the remainder of a division by 2^t, is bounded by $2^t \leq 2k$.

The tables bellow give the values λ and μ for $k \in \{1, 2, 3, 4, 5, 6, 7\}$. As in Theorem 2, $t = \lfloor \log_2 k \rfloor + 1$ and r is the remainder of the division of n by 2^t.

k	**1**		**2**				**3**			
t	1		2				2			
λ	3		8				9			
r	0	1	0	1	2	3	0	1	2	3
μ	2	3	5	7	8	11	4	6	8	9

k	**4**								**5**							
t	3								3							
λ	22								24							
r	0	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7
μ	13	17	19	21	22	27	30	33	13	15	19	21	23	24	30	33

k	**6**								**7**							
t	3								3							
λ	26								27							
r	0	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7
μ	11	15	17	21	23	25	26	33	8	12	16	18	22	24	26	27

We can see the relation

$$w(k, n) - w(k, n - 2^t) = s(k, k+1+2^t, 2^t) \tag{2}$$

in another way. In fact, for a fixed k, the weights $w(k, n)$ satisfy the linear recurrence (2) of order 2^t with constant coefficients. The characteristic polynomial is $x^{2^t} - 1$, and its roots are the 2^t-th roots of unity,

$$\alpha^j = \cos(2\pi j/2^t) + i \sin(2\pi j/2^t), \quad j \in \{0, \ldots, 2^t - 1\}.$$

As the right hand side of the recurrence is a constant, the solution of the recurrence is of the form

$$w(k, n) = A_0(k) + A_1(k)n + B_1(k)\alpha^n + B_2(k)\alpha^{2n} + \cdots + B_{2^t-1}(k)\alpha^{(2^t-1)n} \tag{3}$$

for certain complex constants $A_0(k), A_1(k), B_1(k), \ldots, B_{2^t-1}(k)$. These constants can be determined by finding the initial conditions $w(k, 2k+1)$, $w(k, 2k+2)$, ..., $w(k, 2k+2^t+1)$ and solving the system of the 2^t+1 equations in the unknowns $A_0(k), A_1(k), B_1(k), \ldots, B_{2^t-1}(k)$ obtained by substituting in (3) $n = 2k + j$, for

$j = 1, \ldots, 2^t + 1$. We skip the (long) calculations, but the results of this process for $1 \le k \le 7$ are the following:

$$w(1,n) = \frac{1}{4}\left(-5 + 6n + (-1)^n\right)$$

$$w(2,n) = \frac{1}{4}\left(-13 + 8n - (-1)^n + 2\cos\frac{n\pi}{2}\right)$$

$$w(3,n) = \frac{1}{8}\left(-45 + 18n + 3(-1)^n + 2\cos\frac{n\pi}{2} + 6\sin\frac{n\pi}{2}\right)$$

$$\begin{aligned} w(4,n) = \frac{1}{8}\Big(&-71 + 22n - 3(-1)^n + 2(2+\sqrt{2})\cos\frac{n\pi}{4} \\ &+2\sin\frac{n\pi}{2} - 6\cos\frac{n\pi}{2} + 2(2-\sqrt{2})\cos\frac{3n\pi}{4}\Big) \end{aligned}$$

$$\begin{aligned} w(5,n) = \frac{1}{16}\Big(&-196 + 48n + (4+3\sqrt{2})\cos\frac{n\pi}{4} + (2+3\sqrt{2})\sin\frac{n\pi}{4} \\ &- 2\cos\frac{n\pi}{2} - 6\sin\frac{n\pi}{2} \\ &+ (4-3\sqrt{2})\cos\frac{3n\pi}{4} + (-2+3\sqrt{2})\sin\frac{3n\pi}{4} \\ &+ 8\cos n\pi \\ &+ (4-3\sqrt{2})\cos\frac{5n\pi}{4} + (2-3\sqrt{2})\sin\frac{5n\pi}{4} \\ &- 2\cos\frac{3n\pi}{2} + 6\sin\frac{3n\pi}{2} \\ &+(4+3\sqrt{2})\cos\frac{7n\pi}{4} - (2+3\sqrt{2})\sin\frac{7n\pi}{4}\Big) \end{aligned}$$

$$\begin{aligned} w(6,n) = \frac{1}{8}\Big(&-128 + 26n + (1+\sqrt{2})\cos\frac{n\pi}{4} + (4+2\sqrt{2})\sin\frac{n\pi}{4} \\ &+ 4\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2} \\ &+ (1-\sqrt{2})\cos\frac{3n\pi}{4} + (-4+2\sqrt{2})\sin\frac{3n\pi}{4} \\ &- 4\cos(n\pi) \\ &+ (1-\sqrt{2})\cos\frac{5n\pi}{4} + (4-2\sqrt{2})\sin\frac{5n\pi}{4} \\ &+ 4\cos\frac{3n\pi}{2} + \sin\frac{3n\pi}{2} \\ &+(1+\sqrt{2})\cos\frac{7n\pi}{4} + (-4-2\sqrt{2})\sin\frac{7n\pi}{4}\Big) \end{aligned}$$

$$
\begin{aligned}
w(7,n) =& \frac{1}{16}\Big(-315+54n+(-1-3\sqrt{2})\cos\frac{n\pi}{4}+(7+6\sqrt{2})\sin\frac{n\pi}{4} \\
&+3\cos\frac{n\pi}{2}+9\sin\frac{n\pi}{2} \\
&+(-1+3\sqrt{2})\cos\frac{3n\pi}{4}+(-7+6\sqrt{2})\sin\frac{3n\pi}{4} \\
&+9\cos(n\pi) \\
&+(-1+3\sqrt{2})\cos\frac{5n\pi}{4}+(7-6\sqrt{2})\sin\frac{5n\pi}{4} \\
&+3\cos\frac{3n\pi}{2}-9\sin\frac{3n\pi}{2} \\
&+(-1-3\sqrt{2})\cos\frac{7n\pi}{4}+(-7-6\sqrt{2})\sin\frac{7n\pi}{4}\Big).
\end{aligned}
$$

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