REPRESENTATIONS BY QUATERNARY QUADRATIC FORMS WITH COEFFICIENTS 1, 3, 5 OR 15

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Abstract

We determine explicit formulas for the number of representations of a positive integer n by quaternary quadratic forms with coefficients 1, 3, 5 or 15. We use the theory of modular forms.

1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers and complex numbers, respectively. For $n \in \mathbb{N}$ we set $\sigma(n) = \sum_{1 \le d \mid n} d$. If $n \notin \mathbb{N}$ we set $\sigma(n) = 0$. For $a, b, c, d \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we define

$$N(a, b, c, d; n) := \text{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = ax^2 + by^2 + cz^2 + dt^2\}.$$

It is a classical result of Jacobi [8], [2], [16, Theorem 9.5] that

$$N(1, 1, 1, 1; n) = 8\sigma(n) - 32\sigma(n/4).$$

Jacobi's result $N(1, 1, 1, 1; n)$ was generalized to $N(a, b, c, d; n)$ for various coefficients $a, b, c, d \in \{1, p, q, pq\}$, where p and q are different primes. See, for example, [1] for $p = 2$ and $q = 3$, and [5] for $p = 2$ and $q = 7$. In this paper we determine explicit formulas for $N(a, b, c, d; n)$ for $a, b, c, d \in \{1, p, q, pq\}$ for $p = 3$ and $q = 5$.

For $q \in \mathbb{C}$ with $|q| < 1$, Ramanujan's theta function $\varphi(q)$ is defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

We have

$$\sum_{n=0}^{\infty} N(a, b, c, d; n) q^n = \varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d). \tag{1.1}$$

The Dedekind eta function $\eta(z)$ is the holomorphic function defined on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}). \tag{1.2}$$

Throughout the remainder of the paper we take $q = q(z) := e^{2\pi iz}$ with $z \in \mathbb{H}$. Hence we express the Dedekind eta function (1.2) as

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{1.3}$$

It is well known [6, p. 11] that $\varphi(q)$ can be expressed as

$$\varphi(q) = \frac{\eta^5(2z)}{\eta^2(z)\eta^2(4z)}. \tag{1.4}$$

Let N be a positive integer. A product of the form

$$f(z) = \prod_{1 \le \delta | N} \eta^{r_\delta}(\delta z), \tag{1.5}$$

where $r_\delta \in \mathbb{Z}$, not all zero, is called an eta quotient. When all of the exponents r_δ are nonnegative, $f(z)$ is said to be an eta product. We define the modular subgroup $\Gamma_0(N)$ by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ c \equiv 0 \pmod{N} \right\}.$$

Let $m \in \mathbb{Z}$. For each $t \in \{-12, -5, -4, -3, 1, 5, 12, 60\}$ we define a character χ_t by

$$\chi_t(m) = \left(\frac{t}{m}\right),\ m \in \mathbb{Z}. \tag{1.6}$$

Note that χ_1 is the trivial character. Let χ_{t_1} and χ_{t_2} be Dirichlet characters. For $n \in \mathbb{N}$ we define the generalized sum of divisors functions $\sigma_{(\chi_{t_1}, \chi_{t_2})}(n)$ by

$$\sigma_{(\chi_{t_1}, \chi_{t_2})}(n) := \sum_{1 \le m | n} \chi_{t_1}(m)\chi_{t_2}(n/m)m. \tag{1.7}$$

If $n \notin \mathbb{N}$ we set $\sigma_{(\chi_{t_1}, \chi_{t_2})}(n) = 0$. If $\chi_{t_1} = \chi_{t_2} = \chi_1$ then $\sigma_{(\chi_{t_1}, \chi_{t_2})}(n)$ coincides with the sum of divisors function $\sigma(n)$. For each

$$\begin{aligned}(t_1, t_2) = &(-20, -3), (-3, -20), (-15, -4), (-4, -15), (-4, -3), (-3, -4), \\ &(1, 1), (1, 5), (5, 1), (1, 12), (12, 1), (1, 60), (60, 1)\end{aligned}$$

we define the Eisenstein series $E_{t_1,t_2}(z)$ by

$$E_{t_1,t_2}(z) := c_{t_1,t_2} + \sum_{n=1}^{\infty} \sigma_{(\chi_{t_1},\chi_{t_2})}(n)q^n, \tag{1.8}$$

where

$$\begin{cases} c_{1,1} = -\dfrac{1}{24},\ c_{5,1} = -\dfrac{1}{5},\ c_{12,1} = -1,\ c_{60,1} = -12, \\ c_{t_1,t_2} = 0 \text{ if } (t_1,t_2) \neq (1,1),(5,1),(12,1),(60,1). \end{cases}$$

For $t_1 = t_2 = 1$ we write

$$L(q) := E_{1,1}(z) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n)q^n. \tag{1.9}$$

It is well known that $L(q)$ is a quasi-modular form of weight 2 (see [9, p. 38]), not a modular form.

Let k be an integer. We write $M_k(\Gamma_0(N),\chi)$ to denote the space of modular forms of weight k with multiplier system χ for $\Gamma_0(N)$, and $E_k(\Gamma_0(N),\chi)$ and $S_k(\Gamma_0(N),\chi)$ to denote the subspaces of Eisenstein forms and cusp forms of $M_k(\Gamma_0(N),\chi)$, respectively. It is known (see for example [14, p. 83]) that

$$M_k(\Gamma_0(N)) = E_k(\Gamma_0(N)) \oplus S_k(\Gamma_0(N)). \tag{1.10}$$

We deduce from [14, Sec. 6.1, p. 93] that

$$\dim E_2(\Gamma_0(60),\chi_1) = 11,\ \ \dim S_2(\Gamma_0(60),\chi_1) = 7. \tag{1.11}$$

We also deduce from [14, Sec. 6.3, p. 98] that

$$\dim E_2(\Gamma_0(60),\chi_5) = 12,\ \ \dim S_2(\Gamma_0(60),\chi_5) = 6, \tag{1.12}$$
$$\dim E_2(\Gamma_0(60),\chi_{12}) = 8,\ \ \dim S_2(\Gamma_0(60),\chi_{12}) = 8, \tag{1.13}$$
$$\dim E_2(\Gamma_0(60),\chi_{60}) = 8,\ \ \dim S_2(\Gamma_0(60),\chi_{60}) = 8. \tag{1.14}$$

There are twenty-six quaternary quadratic forms $ax^2 + by^2 + cz^2 + dt^2$ with $a,b,c,d \in \{1,3,5,15\}$, $\gcd(a,b,c,d) = 1$ and $a \leq b \leq c \leq d$. Formulas for $N(a,b,c,d;n)$ for $(a,b,c,d) = (1,1,1,1)$, $(1,1,1,3)$, $(1,1,3,3)$, $(1,3,3,3)$, $(1,1,1,5)$, $(1,1,5,5)$, $(1,5,5,5)$, $(1,1,1,15)$ appear in the literature; see for example [2, 3, 4, 15]. In this paper we treat the remaining eighteen forms. For convenience, in Table 1, we group these eighteen quaternary forms according to the modular spaces $M_2(\Gamma_0(60),\chi)$ to which $\varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d)$ belong.

$M_2(\Gamma_0(60),\chi_1)$	$M_2(\Gamma_0(60),\chi_5)$	$M_2(\Gamma_0(60),\chi_{12})$	$M_2(\Gamma_0(60),\chi_{60})$
$(1,1,15,15)$	$(1,1,3,15)$	$(1,1,5,15)$	$(1,1,3,5)$
$(1,3,5,15)$	$(1,3,3,5)$	$(1,3,5,5)$	$(1,3,3,15)$
$(3,3,5,5)$	$(1,5,15,15)$	$(1,3,15,15)$	$(1,5,5,15)$
	$(3,5,5,15)$	$(3,3,5,15)$	$(1,15,15,15)$
			$(3,3,3,5)$
			$(3,5,5,5)$
			$(3,5,15,15)$

Table 1

We note that the form $(1,1,3,5)$ is one of Ramanujan's universal quaternary quadratic forms given in [13].

2. Preliminary Results

We use the following lemma to determine if certain eta quotients are modular forms. See [7, p. 174], [10, Corollary 2.3, p. 37], [9, Theorem 5.7, p. 99] and [11].

Lemma 2.1. (Ligozat) *Let $N \in \mathbb{N}$ and $f(z) = \prod_{1\leq\delta|N} \eta^{r_\delta}(\delta z)$ be an eta quotient and $s = \prod_{1\leq\delta|N} \delta^{|r_\delta|}$. Suppose that $k = \frac{1}{2}\sum_{1\leq\delta|N} r_\delta$ is an integer. If $f(z)$ satisfies the conditions*

(i) $\sum_{1\leq\delta|N} \delta \cdot r_\delta \equiv 0 \pmod{24}$,

(ii) $\sum_{1\leq\delta|N} \frac{N}{\delta} \cdot r_\delta \equiv 0 \pmod{24}$,

(iii) $\sum_{1\leq\delta|N} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta} \geq 0$ *for each positive divisor d of N,*

then $f(z) \in M_k(\Gamma_0(N),\chi)$, where χ is given by $\chi(m) = \left(\frac{(-1)^k s}{m}\right)$.

(iii)$'$ *In addition to the above conditions, if the inequality in* (iii) *is strict for each positive divisor d of N, then $f(z) \in S_k(\Gamma_0(N),\chi)$.*

We note that the eta quotients given by (3.1)–(3.7), (4.1)–(4.6), (5.1)–(5.8) and (6.1)–(6.8) are constructed with MAPLE in such a way that they satisfy the conditions of Lemma 2.1 for $N = 60$ and $k = 2$.

The following theorem follows directly from (1.4) and Lemma 2.1.

Theorem 2.1. *Let $\chi_1, \chi_5, \chi_{12}$ and χ_{60} be as in* (1.6). *If (a, b, c, d) is in the first, second, third or fourth column of* Table 1, *then*

$$\begin{aligned}
\varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d) &\in M_2(\Gamma_0(60), \chi_1),\\
\varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d) &\in M_2(\Gamma_0(60), \chi_5),\\
\varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d) &\in M_2(\Gamma_0(60), \chi_{12}),\\
\varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d) &\in M_2(\Gamma_0(60), \chi_{60}),
\end{aligned}$$

respectively.

3. Modular Space $M_2(\Gamma_0(60))$

We define the eta products $A_r(q)$ and the integers $a_r(n)$ for $r \in \{1, 2, 3, 4, 5, 6, 7\}$ by

$$A_1(q) := \eta(z)\eta(3z)\eta(5z)\eta(15z), \tag{3.1}$$

$$A_2(q) := \eta(2z)\eta(6z)\eta(10z)\eta(30z), \tag{3.2}$$

$$A_3(q) := \eta(4z)\eta(12z)\eta(20z)\eta(60z), \tag{3.3}$$

$$A_4(q) := \eta(3z)\eta(5z)\eta(6z)\eta(10z), \tag{3.4}$$

$$A_5(q) := \eta(6z)\eta(10z)\eta(12z)\eta(20z), \tag{3.5}$$

$$A_6(q) := \eta^2(2z)\eta^2(10z), \tag{3.6}$$

$$A_7(q) := \eta^2(6z)\eta^2(30z), \tag{3.7}$$

$$A_r(q) = \sum_{n=1}^{\infty} a_r(n)q^n. \tag{3.8}$$

Note that

$$A_3(q) = A_2(q^2) = A_1(q^4), \quad A_5(q) = A_4(q^2), \quad A_7(q) = A_6(q^3).$$

For $1 < t \mid 60$, we define

$$L_t(q) := L(q) - tL(q^t), \tag{3.9}$$

which is a modular form in $M_2(\Gamma_0(t))$, see [14, Theorem 5.8, p. 88].

Theorem 3.1. *A basis for $M_2(\Gamma_0(60))$ is given by*

$$\{L_t(q) \mid t = 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\} \cup \{A_r(q)\}_{(1 \le r \le 7)}.$$

Proof. By taking both χ and ψ as the trivial character in [14, Theorem 5.9, p. 88] and appealing to (1.11), we have that $\{L_t(q) \mid t = 2,3,4,5,6,10,12,15,20,30,60\}$ is a basis for $E_2(\Gamma_0(60))$. By Lemma 2.1, $A_r(q) \in S_2(\Gamma_0(60))$ for each $r \in \{1,2,3,4,5,6,7\}$. The set $\{A_r(q)\}_{(1\leq r\leq 7)}$ can be shown to be linearly independent. Thus it follows from (1.11) that the set $\{A_r(q)\}_{(1\leq r\leq 7)}$ is a basis for $S_2(\Gamma_0(60))$. The assertion now follows from (1.10). □

To shorten the lengths of the identities in Theorems 3.2 and 3.3, we set

$$R(q) := L(q) - 2L(q^2) + 4L(q^4), \tag{3.10}$$

which is not a modular form.

Theorem 3.2.

$$\begin{aligned}
\varphi^2(q)\varphi^2(q^{15}) =& \frac{2}{3}R(q) - 2R(q^3) + \frac{10}{3}R(q^5) - 10R(q^{15}) \\
&+ \frac{2}{3}\big(A_1(q) - 2A_2(q) + 4A_3(q) + 4A_4(q) + 8A_5(q)\big), \\
\varphi(q)\varphi(q^3)\varphi(q^5)\varphi(q^{15}) =& \frac{1}{2}\big(R(q) + 3R(q^3) - 5R(q^5) - 15R(q^{15})\big) \\
&+ \frac{3}{2}A_1(q) + A_2(q) + 6A_3(q), \\
\varphi^2(q^3)\varphi^2(q^5) =& \frac{2}{3}R(q) - 2R(q^3) + \frac{10}{3}R(q^5) - 10R(q^{15}) \\
&- \frac{2}{3}\big(5A_1(q) + 14A_2(q) + 20A_3(q) - 4A_4(q) - 8A_5(q)\big).
\end{aligned}$$

Proof. We prove only the first identity as the other ones can be proven similarly. By (1.4) and Theorem 2.1 we have $\varphi^2(q)\varphi^2(q^{15}) \in M_2(\Gamma_0(60))$. By Theorem 3.1, $\varphi^2(q)\varphi^2(q^{15})$ must be a linear combination of $L_t(q)$ ($t = 2,3,4,5,6,10,12,15,20,30,60$) and $A_r(q)$ ($r \in \{1,2,3,4,5,6,7\}$), namely

$$\varphi^2(q)\varphi^2(q^{15}) = \sum_{2\leq d|60} x_d L_d(q) + \sum_{i=1}^{7} y_i A_i(q) \tag{3.11}$$

for some scalars x_d and y_i in $\mathbb{C}$ for $2 \leq d \mid 60$ and $1 \leq i \leq 7$. The Sturm bound for the modular space $M_2(\Gamma_0(60))$ is 24 (see [9, Theorem 3.13]). Equating the coefficients of q^n for $0 \leq n \leq 24$ on both sides of (3.11), we find a system of linear equations, with the unknowns x_i ($i \in \{2,3,4,5,6,10,12,15,20,30,60\}$) and y_j ($j \in \{1,2,3,4,5,6,7\}$). Using MAPLE [12] we solve the system and find that

$$\begin{aligned}
\varphi^2(q)\varphi^2(q^{15}) \quad = \quad & \frac{2}{3}L_2(q) + \frac{2}{3}L_3(q) - \frac{2}{3}L_4(q) - \frac{2}{3}L_5(q) - \frac{2}{3}L_6(q) + \frac{2}{3}L_{10}(q) \\
&+\frac{2}{3}L_{12}(q) + \frac{2}{3}L_{15}(q) - \frac{2}{3}L_{20}(q) - \frac{2}{3}L_{30}(q) + \frac{2}{3}L_{60}(q) \quad (3.12)
\end{aligned}$$

$$+\frac{2}{3}A_1(q) - \frac{4}{3}A_2(q) + \frac{8}{3}A_3(q) + \frac{8}{3}A_4(q) + \frac{16}{3}A_5(q).$$

Substituting (3.9) into (3.12) we obtain

$$\begin{aligned}\varphi^2(q)\varphi^2(q^{15}) =& \frac{2}{3}L(q) - \frac{4}{3}L(q^2) - 2L(q^3) + \frac{8}{3}L(q^4) + \frac{10}{3}L(q^5) + 4L(q^6)\\ &- \frac{20}{3}L(q^{10}) - 8L(q^{12}) - 10L(q^{15}) + \frac{40}{3}L(q^{20}) + 20L(q^{30})\\ &- 40L(q^{60}) + \frac{2}{3}\big(A_1(q) - 2A_2(q) + 4A_3(q) + 4A_4(q) + 8A_5(q)\big).\end{aligned}$$

After rearranging the terms in the above equation we have

$$\begin{aligned}\varphi^2(q)\varphi^2(q^{15}) =& \frac{2}{3}\big(L(q) - 2L(q^2) + 4L(q^4)\big) - 2\big(L(q^3) - 2L(q^6) + 4L(q^{12})\big)\\ &+ \frac{10}{3}\big(L(q^5) - 2L(q^{10}) + 4L(q^{20})\big)\\ &- 10\big(L(q^{15}) - 2L(q^{30}) + 4L(q^{60})\big)\\ &+ \frac{2}{3}\big(A_1(q) - 2A_2(q) + 4A_3(q) + 4A_4(q) + 8A_5(q)\big).\end{aligned} \tag{3.13}$$

The assertion now follows from (3.10) and (3.13). □

We now give explicit formulas for $N(1,1,15,15;n)$, $N(1,3,5,15;n)$ and $N(3,3,5,5;n)$. For $n \in \mathbb{N}$ we set

$$r(n) := \sigma(n) - 2\sigma(n/2) + 4\sigma(n/4). \tag{3.14}$$

Theorem 3.3. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned}N(1,1,15,15;n) =& \frac{2}{3}r(n) - 2r(n/3) + \frac{10}{3}r(n/5) - 10r(n/15)\\ &+ \frac{2}{3}\big(a_1(n) - 2a_2(n) + 4a_3(n) + 4a_4(n) + 8a_5(n)\big),\\ N(1,3,5,15;n) =& \frac{1}{2}\big(r(n) + 3r(n/3) - 5r(n/5) - 15r(n/15)\big)\\ &+ \frac{3}{2}a_1(n) + a_2(n) + 6a_3(n),\\ N(3,3,5,5;n) =& \frac{2}{3}r(n) - 2r(n/3) + \frac{10}{3}r(n/5) - 10r(n/15)\\ &+ \frac{2}{3}\big(-5a_1(n) - 14a_2(n) - 20a_3(n) + 4a_4(n) + 8a_5(n)\big).\end{aligned}$$

Proof. Appealing to (1.1), (1.9), (3.1)–(3.8), and equating the coefficients of q^n on both sides of the equations in Theorem 3.2, we deduce the asserted results. □

4. Modular Space $M_2(\Gamma_0(60), \chi_5)$

Let $n \in \mathbb{N}$. We define the eta quotients $B_r(q)$ and the integers $b_r(n)$ for $r \in \{1,2,3,4,5,6\}$ by

$$B_1(q) = \frac{\eta^4(2z)\eta(3z)\eta(15z)}{\eta^2(z)}, \tag{4.1}$$

$$B_2(q) = \frac{\eta(z)\eta(5z)\eta^4(6z)}{\eta^2(3z)}, \tag{4.2}$$

$$B_3(q) = \frac{\eta(3z)\eta^4(10z)\eta(15z)}{\eta^2(5z)}, \tag{4.3}$$

$$B_4(q) = \frac{\eta(4z)\eta^4(6z)\eta(20z)}{\eta^2(12z)}, \tag{4.4}$$

$$B_5(q) = \frac{\eta(z)\eta(5z)\eta^4(30z)}{\eta^2(15z)}, \tag{4.5}$$

$$B_6(q) = \frac{\eta(4z)\eta(20z)\eta^4(30z)}{\eta^2(60z)}, \tag{4.6}$$

$$B_r(q) := \sum_{n=1}^{\infty} b_r(n)q^n. \tag{4.7}$$

Theorem 4.1. *Let χ_5 be as in* (1.6)*. A basis for $M_2(\Gamma_0(60), \chi_5)$ is given by*

$$\{E_{1,5}(tz), E_{5,1}(tz) \mid t = 1,2,3,4,6,12\} \cup \{B_r(q) \mid r = 1,2,3,4,5,6\}.$$

Proof. Let $r \in \{1,2,3,4,5,6\}$. By Lemma 2.1, we have $B_r(q) \in S_2(\Gamma_0(60), \chi_5)$. The set $\{B_r(q)\}_{(1 \le r \le 6)}$ can be shown to be linearly independent. Then appealing to (1.12), we deduce that $\{B_r(q)\}_{(1 \le r \le 6)}$ is a basis for $S_2(\Gamma_0(60), \chi_5)$. By taking $\epsilon = \chi_5$ and $\chi, \psi \in \{\chi_1, \chi_5\}$ in [14, Theorem 5.9, p. 88] and appealing to (1.12) we have that $\{E_{1,5}(tz), E_{5,1}(tz) \mid t = 1,2,3,4,6,12\}$ is a basis for $E_2(\Gamma_0(60), \chi_5)$. The assertion now follows from (1.10). □

To shorten the lengths of the identities in Theorem 4.2, we set

$$T_1(q) := E_{1,5}(z) + 6E_{1,5}(3z) + 4E_{1,5}(4z) + 24E_{1,5}(12z), \tag{4.8}$$

$$T_2(q) := 2E_{1,5}(z) - 3E_{1,5}(3z) + 8E_{1,5}(4z) - 12E_{1,5}(12z), \tag{4.9}$$

$$T_3(q) := 2E_{5,1}(z) + 3E_{5,1}(3z) + 8E_{5,1}(4z) + 12E_{5,1}(12z), \tag{4.10}$$

$$T_4(q) := E_{5,1}(z) - 6E_{5,1}(3z) + 4E_{5,1}(4z) - 24E_{5,1}(12z), \tag{4.11}$$

and for $n \in \mathbb{N}$ we define

$$t_1(n) := \sigma_{(\chi_1,\chi_5)}(n) + 6\sigma_{(\chi_1,\chi_5)}(n/3) + 4\sigma_{(\chi_1,\chi_5)}(n/4) + 24\sigma_{(\chi_1,\chi_5)}(n/12), \tag{4.12}$$

$$t_2(n) := 2\sigma_{(\chi_1,\chi_5)}(n) - 3\sigma_{(\chi_1,\chi_5)}(n/3) + 8\sigma_{(\chi_1,\chi_5)}(n/4) - 12\sigma_{(\chi_1,\chi_5)}(n/12), \quad (4.13)$$

$$t_3(n) := 2\sigma_{(\chi_5,\chi_1)}(n) + 3\sigma_{(\chi_5,\chi_1)}(n/3) + 8\sigma_{(\chi_5,\chi_1)}(n/4) + 12\sigma_{(\chi_5,\chi_1)}(n/12), \quad (4.14)$$

$$t_4(n) := \sigma_{(\chi_5,\chi_1)}(n) - 6\sigma_{(\chi_5,\chi_1)}(n/3) + 4\sigma_{(\chi_5,\chi_1)}(n/4) - 24\sigma_{(\chi_5,\chi_1)}(n/12). \quad (4.15)$$

Theorem 4.2. *Let χ_1 be the trivial character and χ_5 be as in* (1.6)*. Then*

$$\begin{aligned} N(1,1,3,15;n) =& t_2(n) - \frac{1}{5}t_3(n) + \frac{14}{5}b_1(n) + 2b_2(n) \\ & - 2b_3(n) + \frac{8}{5}b_4(n) - 6b_5(n) - 4b_6(n), \\ N(1,3,3,5;n) =& t_1(n) + \frac{1}{5}t_4(n) - \frac{2}{5}b_1(n) \\ & + 2b_2(n) + 2b_3(n) - \frac{4}{5}b_4(n) - 2b_5(n), \\ N(1,5,15,15;n) =& \frac{1}{5}\big(t_1(n) + t_4(n)\big) + \frac{2}{5}b_1(n) \\ & + \frac{2}{5}b_2(n) - \frac{2}{5}b_3(n) - 2b_5(n) + \frac{4}{5}b_6(n), \\ N(3,5,5,15;n) =& \frac{1}{5}\big(t_2(n) - t_3(n)\big) + \frac{2}{5}b_1(n) - \frac{6}{5}b_2(n) \\ & - \frac{14}{5}b_3(n) - \frac{4}{5}b_4(n) + 2b_5(n) + \frac{8}{5}b_6(n). \end{aligned}$$

Proof. We prove only the first identity as the other ones can be proven similarly. By Theorem 2.1, $\varphi^2(q)\varphi(q^3)\varphi(q^{15}) \in M_2(\Gamma_0(60),\chi_5)$. By Theorem 3.1, $\varphi^2(q)\varphi(q^3)\varphi(q^{15})$ must be a linear combination of $\{E_{1,5}(tz), E_{5,1}(tz) \mid t = 1,2,3,4,6,12\}$ and $\{B_r(q) \mid r = 1,2,3,4,5,6\}$, namely

$$\varphi^2(q)\varphi(q^3)\varphi(q^{15}) = \sum_{1\le d|12} x_d E_{1,5}(dz) + \sum_{1\le d|12} y_d E_{5,1}(dz) + \sum_{i=1}^{6} z_i B_i(q) \quad (4.16)$$

for some scalars x_d, y_d and z_i in $\mathbb{C}$ for $1 \le d \mid 12$ and $1 \le i \le 6$. By [14, Corollary 9.20], the Sturm bound for the modular space $M_2(\Gamma_0(60),\chi_5)$ is 24. Equating the coefficients of q^n for $0 \le n \le 24$ on both sides of (4.16) and appealing to (4.9) and (4.10) we obtain

$$\begin{aligned} \varphi^2(q)\varphi(q^3)\varphi(q^{15}) =& T_2(q) - \frac{1}{5}T_3(q) + \frac{14}{5}B_1(q) + 2B_2(q) \\ & - 2B_3(q) + \frac{8}{5}B_4(q) - 6B_5(q) - 4B_6(q). \end{aligned} \quad (4.17)$$

The assertion now follows from (1.1), (4.7), (4.13), (4.14) and (4.17). □

5. Modular Space $M_2(\Gamma_0(60), \chi_{12})$

Let $n \in \mathbb{N}$. We define the eta quotients $C_r(q)$ and the integers $c_r(n)$ for $r \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ by

$$C_1(q) = \frac{\eta(2z)\eta(3z)\eta(4z)\eta^2(5z)\eta(12z)}{\eta(z)\eta(6z)}, \tag{5.1}$$

$$C_2(q) = \frac{\eta(z)\eta(4z)\eta(6z)\eta(12z)\eta^2(15z)}{\eta(2z)\eta(3z)}, \tag{5.2}$$

$$C_3(q) = \frac{\eta^2(2z)\eta(5z)\eta^2(6z)\eta^3(20z)}{\eta(z)\eta^2(10z)\eta(12z)}, \tag{5.3}$$

$$C_4(q) = \frac{\eta^5(2z)\eta^2(15z)\eta(20z)}{\eta^2(z)\eta(4z)\eta(30z)}, \tag{5.4}$$

$$C_5(q) = \frac{\eta(3z)\eta^2(10z)\eta^3(12z)\eta^2(30z)}{\eta^2(6z)\eta(15z)\eta(20z)}, \tag{5.5}$$

$$C_6(q) = \frac{\eta^5(2z)\eta(5z)\eta^2(60z)}{\eta(z)\eta^2(4z)\eta(30z)}, \tag{5.6}$$

$$C_7(q) = \frac{\eta^3(3z)\eta^2(10z)\eta(12z)\eta^2(30z)}{\eta(5z)\eta^2(6z)\eta(60z)}, \tag{5.7}$$

$$C_8(q) = \frac{\eta^2(4z)\eta(5z)\eta(10z)\eta(15z)\eta(60z)}{\eta(20z)\eta(30z)}, \tag{5.8}$$

$$C_r(q) := \sum_{n=1}^{\infty} c_r(n) q^n. \tag{5.9}$$

Theorem 5.1. *A basis for $M_2(\Gamma_0(60), \chi_{12})$ is given by*

$$\{E_{1,12}(tz), E_{12,1}(tz), E_{-4,-3}(tz), E_{-3,-4}(tz) \mid t = 1, 5\} \cup \{C_r(q)\}_{(1 \le r \le 8)}.$$

Proof. By taking $\epsilon = \chi_{12}$ and $\chi, \psi \in \{\chi_{-4}, \chi_{-3}, \chi_1, \chi_{12}\}$ in [14, Theorem 5.9, p. 88] and appealing to (1.13), we deduce that $\{E_{1,12}(tz), E_{12,1}(tz), E_{-4,-3}(tz), E_{-3,-4}(tz) \mid t = 1, 5\}$ is a basis for $E_2(\Gamma_0(60), \chi_{12})$. Let $r \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. By Lemma 2.1, $C_r(q) \in S_2(\Gamma_0(60), \chi_{12})$. The set $\{C_r(q)\}_{(1 \le r \le 8)}$ can be shown to be linearly independent. Thus appealing to (1.13) we deduce that $\{C_r(q)\}_{(1 \le r \le 8)}$ is a basis for $S_2(\Gamma_0(60), \chi_{12})$. We complete the proof by appealing to (1.10). □

To shorten the lengths of the identities in Theorem 5.2, we set

$$u_1(n) := 6\sigma_{\chi_1,\chi_{12}}(n) - \sigma_{(\chi_{12},\chi_1)}(n) - 2\sigma_{(\chi_{-3},\chi_{-4})}(n) + 3\sigma_{(\chi_{-4},\chi_{-3})}(n),$$
$$u_2(n) := 6\sigma_{\chi_1,\chi_{12}}(n) + \sigma_{(\chi_{12},\chi_1)}(n) + 2\sigma_{(\chi_{-3},\chi_{-4})}(n) + 3\sigma_{(\chi_{-4},\chi_{-3})}(n),$$
$$u_3(n) := 2\sigma_{\chi_1,\chi_{12}}(n) - \sigma_{(\chi_{12},\chi_1)}(n) + 2\sigma_{(\chi_{-3},\chi_{-4})}(n) - \sigma_{(\chi_{-4},\chi_{-3})}(n),$$
$$u_4(n) := 2\sigma_{\chi_1,\chi_{12}}(n) + \sigma_{(\chi_{12},\chi_1)}(n) - 2\sigma_{(\chi_{-3},\chi_{-4})}(n) - \sigma_{(\chi_{-4},\chi_{-3})}(n).$$

Theorem 5.2. *Let χ_1 be the trivial character and χ_{12} be as in* (1.6)*. Then*

$$\begin{aligned}
N(1,1,5,15;n) =& \frac{15}{13}u_1(n/5) + \frac{2}{13}u_2(n) - \frac{16}{13}c_3(n) + \frac{28}{13}c_4(n) - \frac{20}{13}c_6(n),\\
N(1,3,5,5;n) =& \frac{3}{13}u_1(n) - \frac{10}{13}u_2(n/5) - \frac{56}{13}c_1(n) - \frac{168}{13}c_2(n) - \frac{56}{13}c_3(n)\\
&+ \frac{94}{13}c_4(n) - \frac{60}{13}c_5(n) - \frac{10}{13}c_6(n) - \frac{30}{13}c_7(n) + \frac{56}{13}c_8(n),\\
N(1,3,15,15;n) =& \frac{3}{13}u_3(n) - \frac{10}{13}u_4(n/5) + \frac{16}{13}c_1(n) - \frac{48}{13}c_2(n) - \frac{16}{13}c_3(n)\\
&+ \frac{8}{13}c_4(n) - \frac{40}{13}c_5(n) + \frac{8}{13}c_6(n) - \frac{4}{13}c_7(n) + \frac{32}{13}c_8(n),\\
N(3,3,5,15;n) =& \frac{15}{13}u_3(n/5) + \frac{2}{13}u_4(n) + \frac{8}{3}c_1(n) + \frac{64}{13}c_2(n) + \frac{64}{39}c_3(n)\\
&- \frac{146}{39}c_4(n) + \frac{36}{13}c_5(n) + \frac{22}{39}c_6(n) + \frac{14}{13}c_7(n) - \frac{128}{39}c_8(n).
\end{aligned}$$

Proof. The proof is similar to that of Theorem 4.2. □

6. Modular Space $M_2(\Gamma_0(60), \chi_{60})$

Let $n \in \mathbb{N}$. We define the eta quotients $D_r(q)$ and the integers $d_r(n)$ for $r \in \{1,2,3,4,5,6,7,8\}$ by

$$D_1(q) = \frac{\eta(3z)\eta^9(10z)\eta(12z)}{\eta^3(5z)\eta(6z)\eta^3(20z)}, \tag{6.1}$$

$$D_2(q) = \frac{\eta^{10}(2z)\eta(3z)\eta(5z)\eta(12z)\eta(20z)}{\eta^4(z)\eta^4(4z)\eta(6z)\eta(10z)}, \tag{6.2}$$

$$D_3(q) = \frac{\eta^5(2z)\eta^2(3z)\eta(20z)}{\eta^2(z)\eta(4z)\eta(6z)}, \tag{6.3}$$

$$D_4(q) = \frac{\eta(5z)\eta^9(6z)\eta(20z)}{\eta^3(3z)\eta(10z)\eta^3(12z)}, \tag{6.4}$$

$$D_5(q) = \frac{\eta(z)\eta(3z)\eta(4z)\eta(12z)\eta(30z)}{\eta(2z)}, \tag{6.5}$$

$$D_6(q) = \frac{\eta(3z)\eta(12z)\eta^3(20z)\eta^3(30z)}{\eta(6z)\eta(15z)\eta^2(60z)}, \tag{6.6}$$

$$D_7(q) = \frac{\eta(2z)\eta(5z)\eta(15z)\eta(20z)\eta(60z)}{\eta(30z)}, \tag{6.7}$$

$$D_8(q) = \frac{\eta^9(2z)\eta(15z)\eta(60z)}{\eta^3(z)\eta^3(4z)\eta(30z)}, \tag{6.8}$$

$$D_r(q) := \sum_{n=1}^{\infty} d_r(n)q^n. \tag{6.9}$$

Theorem 6.1. *A basis for $M_2(\Gamma_0(60),\chi_{60})$ is given by*

$$\{E_{1,60}(z), E_{60,1}(z), E_{5,12}(z), E_{12,5}(z), E_{-20,-3}(z), E_{-3,-20}(z), E_{-15,-4}(z), E_{-4,-15}(z)\} \cup \{D_r(q)\}_{(1\leq r\leq 8)}.$$

Proof. By taking $\epsilon=\chi_{60}$ and $\chi,\psi \in \{\chi_{-20},\chi_{-15},\chi_{-4},\chi_{-3},\chi_1,\chi_5,\chi_{12},\chi_{60}\}$ in [14, Theorem 5.9, p. 88] and appealing to (1.14), we deduce that

$$\{E_{1,60}(z), E_{60,1}(z), E_{5,12}(z), E_{12,5}(z), E_{-20,-3}(z), E_{-3,-20}(z), E_{-15,-4}(z), E_{-4,-15}(z)\}$$

is a basis for $E_2(\Gamma_0(60),\chi_{60})$. By Lemma 2.1, $D_r(q) \in S_2(\Gamma_0(60),\chi_{60})$ for each $r \in \{1,2,3,4,5,6,7,8\}$. The set $\{D_r(q)\}_{(1\leq r\leq 8)}$ can be shown to be linearly independent. Thus by (1.14), $\{D_r(q)\}_{(1\leq r\leq 8)}$ is a basis for $S_2(\Gamma_0(60),\chi_{60})$. The assertion now follows from (1.10). □

Theorem 6.2. *Let χ_1 be the trivial character and χ_{12} be as in* (1.6). *Then*

$$\begin{aligned}
N(1,1,3,5;n) =& \frac{1}{12}\Big(30\sigma_{(\chi_1,\chi_{60})}(n) - \sigma_{(\chi_{60},\chi_1)}(n) - 6\sigma_{(\chi_5,\chi_{12})}(n)\\
&+ 5\sigma_{(\chi_{12},\chi_5)}(n) - 3\sigma_{(\chi_{-20},\chi_{-3})}(n) + 10\sigma_{(\chi_{-3},\chi_{-20})}(n)\\
&- 2\sigma_{(\chi_{-15},\chi_{-4})}(n) + 15\sigma_{(\chi_{-4},\chi_{-15})}(n)\Big) - d_8(n),\\
N(1,3,3,15;n) =& \frac{1}{12}\Big(10\sigma_{(\chi_1,\chi_{60})}(n) - \sigma_{(\chi_{60},\chi_1)}(n) + 2\sigma_{(\chi_5,\chi_{12})}(n) - 5\sigma_{(\chi_{12},\chi_5)}(n)\\
&- \sigma_{(\chi_{-20},\chi_{-3})}(n) + 10\sigma_{(\chi_{-3},\chi_{-20})}(n) + 2\sigma_{(\chi_{-15},\chi_{-4})}(n)\Big)\\
&- 5\sigma_{(\chi_{-4},\chi_{-15})}(n) + d_4(n),\\
N(1,5,5,15;n) =& \frac{1}{12}\Big(6\sigma_{(\chi_1,\chi_{60}}(n) - \sigma_{(\chi_{60},\chi_1)}(n) + 6\sigma_{(\chi_5,\chi_{12})}(n) - \sigma_{(\chi_{12},\chi_5)}(n)\\
&+ 3\sigma_{(\chi_{-20},\chi_{-3})}(n) - 2\sigma_{(\chi_{-3},\chi_{-20})}(n) - 2\sigma_{(\chi_{-15},\chi_{-4})}(n)\\
&+ 3\sigma_{(\chi_{-4},\chi_{-15})}(n)\Big) + d_1(n),\\
N(1,15,15,15;n) =& \frac{1}{12}\Big(2\sigma_{(\chi_1,\chi_{60})}(n) - \sigma_{(\chi_{60},\chi_1)}(n) + 2\sigma_{(\chi_5,\chi_{12})}(n) - \sigma_{(\chi_{12},\chi_5)}(n)\\
&- \sigma_{(\chi_{-20},\chi_{-3})}(n) + 2\sigma_{(\chi_{-3},\chi_{-20})}(n) + 2\sigma_{(\chi_{-15},\chi_{-4})}(n)\\
&- \sigma_{(\chi_{-4},\chi_{-15})}(n)\Big) - \frac{4}{9}d_1(n) + \frac{1}{3}d_4(n) + \frac{16}{9}d_6(n),\\
N(3,3,3,5;n) =& \frac{1}{12}\Big(10\sigma_{(\chi_1,\chi_{60})}(n) - \sigma_{(\chi_{60},\chi_1)}(n) - 2\sigma_{(\chi_5,\chi_{12})}(n) + 5\sigma_{(\chi_{12},\chi_5)}(n)\\
&+ \sigma_{(\chi_{-20},\chi_{-3})}(n) - 10\sigma_{(\chi_{-3},\chi_{-20})}(n) + 2\sigma_{(\chi_{-15},\chi_{-4})}(n)\\
&- 5\sigma_{(\chi_{-4},\chi_{-15})}(n)\Big) - \frac{25}{18}d_1(n) - \frac{10}{3}d_3(n)\\
&- \frac{5}{6}d_4(n) + 2d_5(n) + \frac{50}{9}d_6(n) + \frac{5}{3}d_8(n),
\end{aligned}$$

$$N(3,5,5,5;n) = \frac{1}{12}\Big(6\sigma_{(\chi_1,\chi_{60})}(n) - \sigma_{(\chi_{60},\chi_1)}(n) - 6\sigma_{(\chi_5,\chi_{12})}(n) + \sigma_{(\chi_{12},\chi_5)}(n) - 3\sigma_{(\chi_{-20},\chi_{-3})}(n) + 2\sigma_{(\chi_{-3},\chi_{-20})}(n) - 2\sigma_{(\chi_{-15},\chi_{-4})}(n) + 3\sigma_{(\chi_{-4},\chi_{-15})}(n)\Big) + \frac{16}{3}d_7(n) - \frac{5}{3}d_8(n),$$

$$N(3,5,15,15;n) = \frac{1}{12}\Big(2\sigma_{(\chi_1,\chi_{60})}(n) - \sigma_{(\chi_{60},\chi_1)}(n) - 2\sigma_{(\chi_5,\chi_{12})}(n) + \sigma_{(\chi_{12},\chi_5)}(n) + \sigma_{(\chi_{-20},\chi_{-3})}(n) - 2\sigma_{(\chi_{-3},\chi_{-20})}(n) + 2\sigma_{(\chi_{-15},\chi_{-4})}(n) - \sigma_{(\chi_{-4},\chi_{-15})}(n)\Big) - \frac{1}{6}d_1(n) - \frac{2}{5}d_3(n) - \frac{1}{10}d_4(n) - \frac{2}{5}d_5(n) + \frac{2}{3}d_6(n) + \frac{1}{5}d_8(n).$$

Proof. The proof is similar to that of Theorem 4.2. □

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