Representations by Quaternary Quadratic Forms with Coefficients 1, 3, 5 or 15

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Abstract
We determine explicit formulas for the number of representations of a positive integer \( n \) by quaternary quadratic forms with coefficients 1, 3, 5 or 15. We use the theory of modular forms.

1. Introduction

Let \( \mathbb{N}, \mathbb{N}_0, \mathbb{Z} \) and \( \mathbb{C} \) denote the sets of positive integers, nonnegative integers, integers and complex numbers, respectively. For \( n \in \mathbb{N} \) we set \( \sigma(n) = \sum_{1 \leq d | n} d \). If \( n \notin \mathbb{N} \) we set \( \sigma(n) = 0 \). For \( a, b, c, d \in \mathbb{N} \) and \( n \in \mathbb{N}_0 \) we define

\[
N(a, b, c, d; n) := \text{card}\{ (x, y, z, t) \in \mathbb{Z}^4 \mid n = ax^2 + by^2 + cz^2 + dt^2 \}.
\]

It is a classical result of Jacobi [8], [2], [16, Theorem 9.5] that

\[
N(1, 1, 1, 1; n) = 8\sigma(n) - 32\sigma(n/4).
\]

Jacobi’s result \( N(1, 1, 1, 1; n) \) was generalized to \( N(a, b, c, d; n) \) for various coefficients \( a, b, c, d \in \{1, p, q, pq\} \), where \( p \) and \( q \) are different primes. See, for example, [1] for \( p = 2 \) and \( q = 3 \), and [5] for \( p = 2 \) and \( q = 7 \). In this paper we determine explicit formulas for \( N(a, b, c, d; n) \) for \( a, b, c, d \in \{1, p, q, pq\} \) for \( p = 3 \) and \( q = 5 \).

For \( q \in \mathbb{C} \) with \( |q| < 1 \), Ramanujan’s theta function \( \varphi(q) \) is defined by

\[
\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.
\]

We have

\[
\sum_{n=0}^{\infty} N(a, b, c, d; n) q^n = \varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d).
\]
The Dedekind eta function \( \eta(z) \) is the holomorphic function defined on the upper half plane \( \mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \) by
\[
\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}).
\tag{1.2}
\]
Throughout the remainder of the paper we take \( q = q(z) := e^{2\pi iz} \) with \( z \in \mathbb{H} \). Hence we express the Dedekind eta function (1.2) as
\[
\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).
\tag{1.3}
\]
It is well known [6, p. 11] that \( \varphi(q) \) can be expressed as
\[
\varphi(q) = \frac{\eta^3(2z)}{\eta(z)\eta(4z)}. \tag{1.4}
\]
Let \( N \) be a positive integer. A product of the form
\[
f(z) = \prod_{1 \leq \delta | N} \eta^{r_{\delta}}(\delta z), \tag{1.5}
\]
where \( r_{\delta} \in \mathbb{Z} \), not all zero, is called an eta quotient. When all of the exponents \( r_{\delta} \) are nonnegative, \( f(z) \) is said to be an eta product. We define the modular subgroup \( \Gamma_0(N) \) by
\[
\Gamma_0(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mid a, b, c, d \in \mathbb{Z}, \ ad - bc = 1, \ c \equiv 0 \pmod{N} \right\}.
\]
Let \( m \in \mathbb{Z} \). For each \( t \in \{-12, -5, -4, -3, 1, 5, 12, 60\} \) we define a character \( \chi_t \) by
\[
\chi_t(m) = \left( \frac{t}{m} \right), \ m \in \mathbb{Z}.
\tag{1.6}
\]
Note that \( \chi_1 \) is the trivial character. Let \( \chi_{t_1} \) and \( \chi_{t_2} \) be Dirichlet characters. For \( n \in \mathbb{N} \) we define the generalized sum of divisors functions \( \sigma_{(\chi_{t_1}, \chi_{t_2})}(n) \) by
\[
\sigma_{(\chi_{t_1}, \chi_{t_2})}(n) := \sum_{1 \leq m | n} \chi_{t_1}(m)\chi_{t_2}(n/m)m. \tag{1.7}
\]
If \( n \notin \mathbb{N} \) we set \( \sigma_{(\chi_{t_1}, \chi_{t_2})}(n) = 0 \). If \( \chi_{t_1} = \chi_{t_2} = \chi_1 \) then \( \sigma_{(\chi_{t_1}, \chi_{t_2})}(n) \) coincides with the sum of divisors function \( \sigma(n) \). For each
\[
(t_1, t_2) = (-20, -3), (-3, -20), (-15, -4), (-4, -15), (-4, -3), (-3, -4),
(1, 1), (1, 5), (5, 1), (1, 12), (12, 1), (1, 60), (60, 1)
\]
we define the Eisenstein series $E_{t_1, t_2}(z)$ by

$$E_{t_1, t_2}(z) := c_{t_1, t_2} + \sum_{n=1}^{\infty} \sigma(\chi_{t_1}, \chi_{t_2})(n)q^n,$$

(1.8)

where

$$
\begin{cases}
    c_{1,1} = -\frac{1}{24}, & c_{5,1} = -\frac{1}{5}, & c_{12,1} = -1, & c_{60,1} = -12, \\
    c_{t_1, t_2} = 0 \text{ if } (t_1, t_2) \neq (1,1), (5,1), (12,1), (60,1).
\end{cases}
$$

For $t_1 = t_2 = 1$ we write

$$L(q) := E_{1,1}(z) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n)q^n.$$

(1.9)

It is well known that $L(q)$ is a quasi-modular form of weight 2 (see [9, p. 38]), not a modular form.

Let $k$ be an integer. We write $M_k(\Gamma_0(N), \chi)$ to denote the space of modular forms of weight $k$ with multiplier system $\chi$ for $\Gamma_0(N)$, and $E_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$ to denote the subspaces of Eisenstein forms and cusp forms of $M_k(\Gamma_0(N), \chi)$, respectively. It is known (see for example [14, p. 83]) that

$$M_k(\Gamma_0(N)) = E_k(\Gamma_0(N)) \oplus S_k(\Gamma_0(N)).$$

(1.10)

We deduce from [14, Sec. 6.1, p. 93] that

$$\dim E_2(\Gamma_0(60), \chi_1) = 11, \quad \dim S_2(\Gamma_0(60), \chi_1) = 7.$$

(1.11)

We also deduce from [14, Sec. 6.3, p. 98] that

$$\dim E_2(\Gamma_0(60), \chi_5) = 12, \quad \dim S_2(\Gamma_0(60), \chi_5) = 6,$$  

(1.12)

$$\dim E_2(\Gamma_0(60), \chi_{12}) = 8, \quad \dim S_2(\Gamma_0(60), \chi_{12}) = 8,$$  

(1.13)

$$\dim E_2(\Gamma_0(60), \chi_{60}) = 8, \quad \dim S_2(\Gamma_0(60), \chi_{60}) = 8.$$  

(1.14)

There are twenty-six quaternary quadratic forms $ax^2 + by^2 + cz^2 + dt^2$ with $a,b,c,d \in \{1,3,5,15\}$, $\gcd(a,b,c,d) = 1$ and $a \leq b \leq c \leq d$. Formulas for $N(a,b,c,d;n)$ for $(a,b,c,d) = (1,1,1,1), (1,1,1,3), (1,1,3,3), (1,3,3,3), (1,1,1,5), (1,1,5,5), (1,5,5,5), (1,1,1,15)$ appear in the literature; see for example [2, 3, 4, 15]. In this paper we treat the remaining eighteen forms. For convenience, in Table 1, we group these eighteen quaternary forms according to the modular spaces $M_2(\Gamma_0(60), \chi)$ to which $\varphi(q^n)\varphi(q^h)\varphi(q^l)\varphi(q^d)$ belong.
\[
\begin{array}{|c|c|c|c|}
\hline
 & M_2(\Gamma_0(60), \chi_1) & M_2(\Gamma_0(60), \chi_5) & M_2(\Gamma_0(60), \chi_{12}) \\
(1, 1, 15, 15) & (1, 1, 3, 15) & (1, 1, 5, 15) & (1, 1, 3, 5) \\
(1, 3, 5, 15) & (1, 3, 3, 5) & (1, 3, 5, 5) & (1, 5, 5, 15) \\
(3, 3, 5, 5) & (1, 5, 15, 15) & (3, 3, 5, 15) & (1, 15, 15, 15) \\
(3, 5, 15, 15) & (3, 5, 5, 15) & (3, 5, 5, 5) & (3, 5, 15, 15) \\
\hline
\end{array}
\]

Table 1

We note that the form \((1, 1, 3, 5)\) is one of Ramanujan’s universal quaternary quadratic forms given in [13].

2. Preliminary Results

We use the following lemma to determine if certain eta quotients are modular forms. See [7, p. 174], [10, Corollary 2.3, p. 37], [9, Theorem 5.7, p. 99] and [11].

**Lemma 2.1.** (Ligozat) Let \(N \in \mathbb{N}\) and \(f(z) = \prod_{1 \leq \delta | N} \eta^{s_{\delta}}(\delta z)\) be an eta quotient and \(s = \prod_{1 \leq \delta | N} \delta^{s_{\delta}}\). Suppose that \(k = \frac{1}{2} \sum_{1 \leq \delta | N} s_{\delta}\) is an integer. If \(f(z)\) satisfies the conditions

(i) \(\sum_{1 \leq \delta | N} \delta \cdot s_{\delta} \equiv 0 \pmod{24},\)

(ii) \(\sum_{1 \leq \delta | N} \frac{N}{\delta} \cdot s_{\delta} \equiv 0 \pmod{24},\)

(iii) \(\sum_{1 \leq \delta | N} \frac{\gcd(d, \delta)^2 \cdot s_{\delta}}{\delta} \geq 0\) for each positive divisor \(d\) of \(N,\)

then \(f(z) \in M_k(\Gamma_0(N), \chi),\) where \(\chi\) is given by \(\chi(m) = \left(\frac{-1}{m}\right)^{k_s} = \left(\frac{-1}{m}\right)^{k_s/m}\).

(iii)’ In addition to the above conditions, if the inequality in (iii) is strict for each positive divisor \(d\) of \(N,\) then \(f(z) \in S_k(\Gamma_0(N), \chi).\)

We note that the eta quotients given by (3.1)–(3.7), (4.1)–(4.6), (5.1)–(5.8) and (6.1)–(6.8) are constructed with MAPLE in such a way that they satisfy the conditions of Lemma 2.1 for \(N = 60\) and \(k = 2.\)

The following theorem follows directly from (1.4) and Lemma 2.1.
Theorem 2.1. Let $\chi_1, \chi_5, \chi_{12}$ and $\chi_{60}$ be as in (1.6). If $(a, b, c, d)$ is in the first, second, third or fourth column of Table 1, then

$$
\varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d) \in M_2(\Gamma_0(60), \chi_1),
\varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d) \in M_2(\Gamma_0(60), \chi_5),
\varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d) \in M_2(\Gamma_0(60), \chi_{12}),
\varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d) \in M_2(\Gamma_0(60), \chi_{60}),
$$

respectively.

3. Modular Space $M_2(\Gamma_0(60))$

We define the eta products $A_r(q)$ and the integers $a_r(n)$ for $r \in \{1, 2, 3, 4, 5, 6, 7\}$ by

$$
A_1(q) := \eta(z)\eta(3z)\eta(5z)\eta(15z),
A_2(q) := \eta(2z)\eta(6z)\eta(10z)\eta(30z),
A_3(q) := \eta(4z)\eta(12z)\eta(20z)\eta(60z),
A_4(q) := \eta(3z)\eta(5z)\eta(6z)\eta(10z),
A_5(q) := \eta(6z)\eta(10z)\eta(12z)\eta(20z),
A_6(q) := \eta^2(2z)\eta^2(10z),
A_7(q) := \eta^2(6z)\eta^2(30z),
$$

$$
A_r(q) = \sum_{n=1}^{\infty} a_r(n)q^n. \tag{3.8}
$$

Note that

$$
A_3(q) = A_2(q^2) = A_1(q^4), \quad A_5(q) = A_4(q^2), \quad A_7(q) = A_6(q^3).
$$

For $1 < t | 60$, we define

$$
L_t(q) := L(q) - tL(q^4), \tag{3.9}
$$

which is a modular form in $M_2(\Gamma_0(t))$, see [14, Theorem 5.8, p. 88].

Theorem 3.1. A basis for $M_2(\Gamma_0(60))$ is given by

$$
\{L_t(q) \mid t = 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\} \cup \{A_r(q)\}_{1 \leq r \leq 7}.
$$
Proof. By taking both \( \chi \) and \( \psi \) as the trivial character in [14, Theorem 5.9, p. 88] and appealing to (1.11), we have that \( \{ L_t(q) \mid t = 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60 \} \) is a basis for \( E_2(\Gamma_0(60)) \). By Lemma 2.1, \( A_r(q) \in S_2(\Gamma_0(60)) \) for each \( r \in \{1, 2, 3, 4, 5, 6, 7\} \). The set \( \{ A_r(q) \}_{1 \leq r \leq 7} \) can be shown to be linearly independent. Thus it follows from (1.11) that the set \( \{ A_r(q) \}_{1 \leq r \leq 7} \) is a basis for \( S_2(\Gamma_0(60)) \). The assertion now follows from (1.10).

To shorten the lengths of the identities in Theorems 3.2 and 3.3, we set
\[
R(q) := L(q) - 2L(q^2) + 4L(q^4),
\]
which is not a modular form.

**Theorem 3.2.**
\[
\varphi^2(q)\varphi^2(q^{15}) = \frac{2}{3}R(q) - 2R(q^3) + \frac{10}{3}R(q^5) - 10R(q^{15}) + \frac{2}{3}(A_1(q) - 2A_2(q) + 4A_3(q) + 4A_4(q) + 8A_5(q)),
\]
\[
\varphi(q)\varphi(q^3)\varphi(q^5) = \frac{1}{2}(R(q) + 3R(q^3) - 5R(q^5) - 15R(q^{15})) + \frac{3}{2}A_1(q) + A_2(q) + 6A_3(q),
\]
\[
\varphi^2(q^3)\varphi^2(q^5) = \frac{2}{3}R(q) - 2R(q^3) + \frac{10}{3}R(q^5) - 10R(q^{15}) - \frac{2}{3}(5A_1(q) + 14A_2(q) + 20A_3(q) - 4A_4(q) - 8A_5(q)).
\]

Proof. We prove only the first identity as the other ones can be proven similarly. By (1.4) and Theorem 2.1 we have \( \varphi^2(q)\varphi^2(q^{15}) \in M_2(\Gamma_0(60)) \). By Theorem 3.1, \( \varphi^2(q)\varphi^2(q^{15}) \) must be a linear combination of \( L_t(q) \) (\( t = 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60 \)) and \( A_r(q) \) (\( r \in \{1, 2, 3, 4, 5, 6, 7\} \)), namely
\[
\varphi^2(q)\varphi^2(q^{15}) = \sum_{2 \leq d \mid 60} x_d L_d(q) + \sum_{i=1}^{7} y_i A_i(q)
\]
for some scalars \( x_d \) and \( y_i \) in \( \mathbb{C} \) for \( 2 \leq d \mid 60 \) and \( 1 \leq i \leq 7 \). The Sturm bound for the modular space \( M_2(\Gamma_0(60)) \) is 24 (see [9, Theorem 3.13]). Equating the coefficients of \( q^n \) for \( 0 \leq n \leq 24 \) on both sides of (3.11), we find a system of linear equations, with the unknowns \( x_i \) (\( i \in \{2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\} \)) and \( y_j \) (\( j \in \{1, 2, 3, 4, 5, 6, 7\} \)). Using MAPLE [12] we solve the system and find that
\[
\varphi^2(q)\varphi^2(q^{15}) = \frac{2}{3}L_2(q) + \frac{2}{3}L_3(q) - \frac{2}{3}L_4(q) - \frac{2}{3}L_5(q) - \frac{2}{3}L_6(q) + \frac{2}{3}L_{10}(q) + \frac{2}{3}L_{12}(q) + \frac{2}{3}L_{15}(q) - \frac{2}{3}L_{20}(q) - \frac{2}{3}L_{30}(q) + \frac{2}{3}L_{60}(q)
\]
\[ + \frac{2}{3} A_1(q) - \frac{4}{3} A_2(q) + \frac{8}{3} A_3(q) + \frac{8}{3} A_4(q) + \frac{16}{3} A_5(q). \]

Substituting (3.9) into (3.12) we obtain
\[ \varphi^2(q) \varphi^2(q^{15}) = \frac{2}{3} L(q) - \frac{4}{3} L(q^2) - 2L(q^3) + \frac{8}{3} L(q^4) + \frac{10}{3} L(q^5) + 4L(q^6) \]
\[ - \frac{20}{3} L(q^{10}) - 8L(q^{12}) - 10L(q^{15}) + \frac{40}{3} L(q^{20}) + 20L(q^{30}) \]
\[ - 40L(q^{60}) + \frac{2}{3} (A_1(q) - 2A_2(q) + 4A_3(q) + 4A_4(q) + 8A_5(q)). \]

After rearranging the terms in the above equation we have
\[ \varphi^2(q) \varphi^2(q^{15}) = \frac{2}{3} (L(q) - 2L(q^2) + 4L(q^4)) - 2(L(q^3) - 2L(q^6) + 4L(q^{12})) \]
\[ + \frac{10}{3} (L(q^5) - 2L(q^{10}) + 4L(q^{20})) \]
\[ - 10(L(q^{15}) - 2L(q^{30}) + 4L(q^{60})) \]
\[ + \frac{2}{3} (A_1(q) - 2A_2(q) + 4A_3(q) + 4A_4(q) + 8A_5(q)). \quad (3.13) \]

The assertion now follows from (3.10) and (3.13). \[ \square \]

We now give explicit formulas for \( N(1, 1, 15; n) \), \( N(1, 3, 5, 15; n) \) and \( N(3, 3, 5, 5; n) \). For \( n \in \mathbb{N} \) we set
\[ r(n) := \sigma(n) - 2\sigma(n/2) + 4\sigma(n/4). \quad (3.14) \]

**Theorem 3.3.** Let \( n \in \mathbb{N} \). Then
\[ N(1, 1, 15; n) = \frac{2}{3} r(n) - 2r(n/3) + \frac{10}{3} r(n/5) - 10r(n/15) \]
\[ + \frac{2}{3} (a_1(n) - 2a_2(n) + 4a_3(n) + 4a_4(n) + 8a_5(n)), \]
\[ N(1, 3, 5, 15; n) = \frac{1}{2} (r(n) + 3r(n/3) - 5r(n/5) - 15r(n/15)) \]
\[ + \frac{3}{2} a_1(n) + a_2(n) + 6a_3(n), \]
\[ N(3, 3, 5, 5; n) = \frac{2}{3} r(n) - 2r(n/3) + \frac{10}{3} r(n/5) - 10r(n/15) \]
\[ + \frac{2}{3} (-5a_1(n) - 14a_2(n) - 20a_3(n) + 4a_4(n) + 8a_5(n)). \]

**Proof.** Appealing to (1.1), (1.9), (3.1)–(3.8), and equating the coefficients of \( q^n \) on both sides of the equations in Theorem 3.2, we deduce the asserted results. \[ \square \]
4. Modular Space $M_2(\Gamma_0(60), \chi_5)$

Let $n \in \mathbb{N}$. We define the eta quotients $B_r(q)$ and the integers $b_r(n)$ for $r \in \{1, 2, 3, 4, 5, 6\}$ by

$$B_1(q) = \frac{\eta^4(2z)\eta(3z)\eta(15z)}{\eta^2(z)}, \quad (4.1)$$
$$B_2(q) = \frac{\eta(z)\eta(5z)\eta^4(6z)}{\eta^2(3z)}, \quad (4.2)$$
$$B_3(q) = \frac{\eta(3z)\eta^4(10z)\eta(15z)}{\eta^2(5z)}, \quad (4.3)$$
$$B_4(q) = \frac{\eta(4z)\eta^4(6z)\eta(20z)}{\eta^2(12z)}, \quad (4.4)$$
$$B_5(q) = \frac{\eta(z)\eta(5z)\eta^4(30z)}{\eta^2(15z)}, \quad (4.5)$$
$$B_6(q) = \frac{\eta(4z)\eta(20z)\eta^4(30z)}{\eta^2(60z)}, \quad (4.6)$$

$$B_r(q) := \sum_{n=1}^{\infty} b_r(n)q^n. \quad (4.7)$$

**Theorem 4.1.** Let $\chi_5$ be as in (1.6). A basis for $M_2(\Gamma_0(60), \chi_5)$ is given by

$$\{E_{1,5}(tz), E_{5,1}(tz) \mid t = 1, 2, 3, 4, 6, 12\} \cup \{B_r(q) \mid r = 1, 2, 3, 4, 5, 6\}.$$

**Proof.** Let $r \in \{1, 2, 3, 4, 5, 6\}$. By Lemma 2.1, we have $B_r(q) \in S_2(\Gamma_0(60), \chi_5)$. The set $\{B_r(q)\}_{1 \leq r \leq 6}$ can be shown to be linearly independent. Then appealing to (1.12), we deduce that $\{B_r(q)\}_{1 \leq r \leq 6}$ is a basis for $S_2(\Gamma_0(60), \chi_5)$. By taking $\epsilon = \chi_5$ and $\chi, \psi \in \{\chi_1, \chi_5\}$ in [14, Theorem 5.9, p. 88] and appealing to (1.12) we have that $\{E_{1,5}(tz), E_{5,1}(tz) \mid t = 1, 2, 3, 4, 6, 12\}$ is a basis for $E_2(\Gamma_0(60), \chi_5)$. The assertion now follows from (1.10). \hfill \Box

To shorten the lengths of the identities in Theorem 4.2, we set

$$T_1(q) := E_{1,5}(z) + 6E_{1,5}(3z) + 4E_{1,5}(4z) + 24E_{1,5}(12z), \quad (4.8)$$
$$T_2(q) := 2E_{1,5}(z) - 3E_{1,5}(3z) + 8E_{1,5}(4z) - 12E_{1,5}(12z), \quad (4.9)$$
$$T_3(q) := 2E_{5,1}(z) + 3E_{5,1}(3z) + 8E_{5,1}(4z) + 12E_{5,1}(12z), \quad (4.10)$$
$$T_4(q) := E_{5,1}(z) - 6E_{5,1}(3z) + 4E_{5,1}(4z) - 24E_{5,1}(12z), \quad (4.11)$$

and for $n \in \mathbb{N}$ we define

$$t_1(n) := \sigma_{(\chi_1, \chi_5)}(n) + 6\sigma_{(\chi_1, \chi_5)}(n/3) + 4\sigma_{(\chi_1, \chi_5)}(n/4) + 24\sigma_{(\chi_1, \chi_5)}(n/12), \quad (4.12)$$
\[ t_2(n) := 2\sigma_{(x_1,x_5)}(n) - 3\sigma_{(x_1,x_5)}(n/3) + 8\sigma_{(x_1,x_5)}(n/4) - 12\sigma_{(x_1,x_5)}(n/12), \quad (4.13) \]
\[ t_3(n) := 2\sigma_{(x_5,x_1)}(n) + 3\sigma_{(x_5,x_1)}(n/3) + 8\sigma_{(x_5,x_1)}(n/4) + 12\sigma_{(x_5,x_1)}(n/12), \quad (4.14) \]
\[ t_4(n) := \sigma_{(x_5,x_1)}(n) - 6\sigma_{(x_5,x_1)}(n/3) + 4\sigma_{(x_5,x_1)}(n/4) - 24\sigma_{(x_5,x_1)}(n/12). \quad (4.15) \]

**Theorem 4.2.** Let \( \chi_1 \) be the trivial character and \( \chi_5 \) be as in (1.6). Then

\[
N(1,1,3,15;n) = t_2(n) - \frac{1}{5} t_3(n) + \frac{14}{5} b_1(n) + 2b_2(n)
- 2b_3(n) + \frac{8}{5} b_4(n) - 6b_5(n) - 4b_6(n),
\]
\[
N(1,3,3,5;n) = t_1(n) + \frac{1}{5} t_4(n) - \frac{2}{5} b_1(n)
+ 2b_2(n) + 2b_3(n) - 4\frac{4}{5} b_4(n) - 2b_5(n),
\]
\[
N(1,5,15,15;n) = \frac{1}{5} (t_1(n) + t_4(n)) + \frac{2}{5} b_1(n)
+ \frac{2}{5} b_2(n) - \frac{2}{5} b_3(n) - 2b_5(n) + 4\frac{4}{5} b_6(n),
\]
\[
N(3,5,15,15;n) = \frac{1}{5} (t_2(n) - t_3(n)) + \frac{2}{5} b_1(n) - \frac{6}{5} b_2(n)
- \frac{14}{5} b_3(n) - 4\frac{4}{5} b_4(n) + 2b_5(n) + 8\frac{8}{5} b_6(n).
\]

**Proof.** We prove only the first identity as the other ones can be proven similarly. By Theorem 2.1, \( \varphi^2(q)\varphi(q^3)\varphi(q^{15}) \in M_2(\Gamma_0(60), \chi_5) \). By Theorem 3.1, \( \varphi^2(q)\varphi(q^3)\varphi(q^{15}) \) must be a linear combination of \( \{E_{1,5}(tz), E_{5,1}(tz) \mid t = 1, 2, 3, 4, 6, 12 \} \) and \( \{B_r(q) \mid r = 1, 2, 3, 4, 5, 6 \} \), namely

\[
\varphi^2(q)\varphi(q^3)\varphi(q^{15}) = \sum_{1 \leq d \mid 12} x_d E_{1,5}(dz) + \sum_{1 \leq d \mid 12} y_d E_{5,1}(dz) + \sum_{i=1}^{6} z_i B_i(q) \quad (4.16)
\]

for some scalars \( x_d, y_d \) and \( z_i \) in \( \mathbb{C} \) for \( 1 \leq d \mid 12 \) and \( 1 \leq i \leq 6 \). By [14, Corollary 9.20], the Sturm bound for the modular space \( M_2(\Gamma_0(60), \chi_5) \) is 24. Equating the coefficients of \( q^n \) for \( 0 \leq n \leq 24 \) on both sides of (4.16) and appealing to (4.9) and (4.10) we obtain

\[
\varphi^2(q)\varphi(q^3)\varphi(q^{15}) = T_2(q) - \frac{1}{5} T_3(q) + \frac{14}{5} B_1(q) + 2B_2(q)
- 2B_3(q) + \frac{8}{5} B_4(q) - 6B_5(q) - 4B_6(q). \quad (4.17)
\]

The assertion now follows from (1.1), (4.7), (4.13), (4.14) and (4.17).
5. Modular Space $M_2(\Gamma_0(60), \chi_{12})$

Let $n \in \mathbb{N}$. We define the eta quotients $C_r(q)$ and the integers $c_r(n)$ for $r \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ by

$$C_1(q) = \frac{\eta(2z)\eta(3z)\eta(4z)\eta^2(5z)\eta(12z)}{\eta(2z)\eta(3z)},$$

$$C_2(q) = \frac{\eta(z)\eta(4z)\eta(6z)\eta(12z)\eta^2(15z)}{\eta(2z)\eta(3z)},$$

$$C_3(q) = \frac{\eta^2(2z)\eta(5z)\eta^2(6z)\eta^3(20z)}{\eta(z)\eta^2(10z)\eta(12z)},$$

$$C_4(q) = \frac{\eta^5(2z)\eta(15z)\eta(20z)}{\eta^2(z)\eta(4z)\eta(30z)},$$

$$C_5(q) = \frac{\eta(3z)\eta^2(10z)\eta^3(12z)\eta^2(30z)}{\eta^2(6z)\eta(15z)\eta(20z)},$$

$$C_6(q) = \frac{\eta^5(2z)\eta(5z)\eta^2(60z)}{\eta(z)\eta^2(4z)\eta(30z)},$$

$$C_7(q) = \frac{\eta^3(3z)\eta^2(10z)\eta(12z)\eta^2(30z)}{\eta(5z)\eta^2(12z)\eta(60z)},$$

$$C_8(q) = \frac{\eta^2(4z)\eta(5z)\eta(10z)\eta(15z)\eta(60z)}{\eta(20z)\eta(30z)},$$

$$C_r(q) := \sum_{n=1}^{\infty} c_r(n)q^n.$$  \hfill (5.9)

**Theorem 5.1.** A basis for $M_2(\Gamma_0(60), \chi_{12})$ is given by

$$\{E_{1,12}(tz), E_{12,1}(tz), E_{-4,-3}(tz), E_{-3,-4}(tz) \mid t = 1, 5\} \cup \{C_r(q)\}_{1 \leq r \leq 8}.$$

**Proof.** By taking $\epsilon = \chi_{12}$ and $\chi, \psi \in \{\chi_{-4}, \chi_{-3}, \chi_1, \chi_{12}\}$ in [14, Theorem 5.9, p. 88] and appealing to (1.13), we deduce that $\{E_{1,12}(tz), E_{12,1}(tz), E_{-4,-3}(tz), E_{-3,-4}(tz) \mid t = 1, 5\}$ is a basis for $E_2(\Gamma_0(60), \chi_{12})$. Let $r \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. By Lemma 2.1, $C_r(q) \in S_2(\Gamma_0(60), \chi_{12})$. The set $\{C_r(q)\}_{1 \leq r \leq 8}$ can be shown to be linearly independent. Thus appealing to (1.13) we deduce that $\{C_r(q)\}_{1 \leq r \leq 8}$ is a basis for $S_2(\Gamma_0(60), \chi_{12})$. We complete the proof by appealing to (1.10). \hfill \square

To shorten the lengths of the identities in Theorem 5.2, we set

$$u_1(n) := 6\sigma_{\chi_1,\chi_{12}}(n) - \sigma_{(\chi_{12},\chi_1)}(n) - 2\sigma_{(\chi_{-3},\chi_{-4})}(n) + 3\sigma_{(\chi_{-4},\chi_{-3})}(n),$$

$$u_2(n) := 6\sigma_{\chi_1,\chi_{12}}(n) + \sigma_{(\chi_{12},\chi_1)}(n) + 2\sigma_{(\chi_{-3},\chi_{-4})}(n) + 3\sigma_{(\chi_{-4},\chi_{-3})}(n),$$

$$u_3(n) := 2\sigma_{\chi_1,\chi_{12}}(n) - \sigma_{(\chi_{12},\chi_1)}(n) + 2\sigma_{(\chi_{-3},\chi_{-4})}(n) - \sigma_{(\chi_{-4},\chi_{-3})}(n),$$

$$u_4(n) := 2\sigma_{\chi_1,\chi_{12}}(n) + \sigma_{(\chi_{12},\chi_1)}(n) - 2\sigma_{(\chi_{-3},\chi_{-4})}(n) - \sigma_{(\chi_{-4},\chi_{-3})}(n).$$
Theorem 5.2. Let $\chi_1$ be the trivial character and $\chi_{12}$ be as in (1.6). Then

\begin{align*}
N(1, 1, 5, 15; n) &= \frac{15}{13} u_1(n/5) + \frac{2}{13} u_2(n) - \frac{16}{13} c_3(n) + \frac{28}{13} c_4(n) - \frac{20}{13} c_6(n), \\
N(1, 3, 5, 5; n) &= \frac{3}{13} u_1(n) - \frac{10}{13} u_2(n/5) - \frac{56}{13} c_1(n) - \frac{168}{13} c_2(n) - \frac{56}{13} c_3(n) \\
&\quad + \frac{94}{13} c_4(n) - \frac{60}{13} c_5(n) - \frac{10}{13} c_6(n) - \frac{30}{13} c_7(n) + \frac{56}{13} c_8(n), \\
N(1, 3, 15, 15; n) &= \frac{3}{13} u_3(n) - \frac{10}{13} u_4(n/5) + \frac{16}{13} c_1(n) - \frac{48}{13} c_2(n) - \frac{16}{13} c_3(n) \\
&\quad + \frac{8}{13} c_4(n) - \frac{40}{13} c_5(n) + \frac{8}{13} c_6(n) - \frac{4}{13} c_7(n) + \frac{32}{13} c_8(n), \\
N(3, 3, 5, 15; n) &= \frac{15}{13} u_3(n/5) + \frac{2}{13} u_4(n) + \frac{8}{3} c_1(n) + \frac{64}{13} c_2(n) + \frac{64}{39} c_3(n) \\
&\quad - \frac{146}{39} c_4(n) + \frac{36}{13} c_5(n) + \frac{22}{39} c_6(n) + \frac{14}{13} c_7(n) - \frac{128}{39} c_8(n).
\end{align*}

Proof. The proof is similar to that of Theorem 4.2. 

6. Modular Space $M_2(\Gamma_0(60), \chi_{60})$

Let $n \in \mathbb{N}$. We define the eta quotients $D_r(q)$ and the integers $d_r(n)$ for $r \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ by

\begin{align*}
D_1(q) &= \frac{\eta(3z)\eta^3(10z)\eta(12z)}{\eta^3(5z)\eta(6z)\eta^3(20z)}, \\
D_2(q) &= \frac{\eta^{10}(2z)\eta(3z)\eta(5z)\eta(12z)\eta(20z)}{\eta^4(z)\eta^4(4z)\eta(6z)\eta(10z)}, \\
D_3(q) &= \frac{\eta^5(2z)\eta^2(3z)\eta(20z)}{\eta^2(z)\eta(4z)\eta(6z)}, \\
D_4(q) &= \frac{\eta^3(5z)\eta^3(6z)\eta(20z)}{\eta^3(3z)\eta(10z)\eta^3(12z)}, \\
D_5(q) &= \frac{\eta(z)\eta(3z)\eta(4z)\eta(12z)\eta(30z)}{\eta(2z)}, \\
D_6(q) &= \frac{\eta(3z)\eta(12z)\eta^3(20z)\eta^3(30z)}{\eta(6z)\eta(15z)\eta^2(60z)}, \\
D_7(q) &= \frac{\eta(2z)\eta(5z)\eta(15z)\eta(20z)\eta(60z)}{\eta(30z)}, \\
D_8(q) &= \frac{\eta^3(2z)\eta(15z)\eta(60z)}{\eta^3(z)\eta^3(4z)\eta(30z)}, \\
D_r(q) &= \sum_{n=1}^{\infty} d_r(n)q^n.
\end{align*}
Theorem 6.1. A basis for $M_2(\Gamma_0(60), \chi_{60})$ is given by

$$\{E_{1,60}(z), E_{60,1}(z), E_{5,12}(z), E_{12,5}(z), E_{-20,-3}(z), E_{-3,-20}(z),$$

$$E_{-15,-4}(z), E_{-4,-15}(z)\} \cup \{D_r(q)\}_{1 \leq r \leq 8}.$$

Proof. By taking $\epsilon = \chi_{60}$ and $\chi, \psi \in \{\chi_{-20}, \chi_{-15}, \chi_{-4}, \chi_{-3}, \chi_1, \chi_5, \chi_{12}, \chi_{60}\}$ in [14, Theorem 5.9, p. 88] and appealing to (1.14), we deduce that

$$\{E_{1,60}(z), E_{60,1}(z), E_{5,12}(z), E_{12,5}(z), E_{-20,-3}(z), E_{-3,-20}(z), E_{-15,-4}(z), E_{-4,-15}(z)\}$$

is a basis for $E_2(\Gamma_0(60), \chi_{60})$. By Lemma 2.1, $D_r(q) \in S_2(\Gamma_0(60), \chi_{60})$ for each $r \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. The set $\{D_r(q)\}_{1 \leq r \leq 8}$ can be shown to be linearly independent. Thus by (1.14), $\{D_r(q)\}_{1 \leq r \leq 8}$ is a basis for $S_2(\Gamma_0(60), \chi_{60})$. The assertion now follows from (1.10). \qed

Theorem 6.2. Let $\chi_1$ be the trivial character and $\chi_{12}$ be as in (1.6). Then

\begin{align*}
N(1,1,3,5;n) &= \frac{1}{12} \left(30\sigma_{(1,1,3)}(n) - \sigma_{(3,60,1)}(n) - 6\sigma_{(1,5,12)}(n) \\
&\quad + 5\sigma_{(1,1,5)}(n) - 3\sigma_{(1,20,1)}(n) + 10\sigma_{(1,3,1)}(n) \\
&\quad - 2\sigma_{(1,1,15)}(n) + 15\sigma_{(1,4,15)}(n) \right) - d_8(n),
\end{align*}

\begin{align*}
N(1,3,3,15;n) &= \frac{1}{12} \left(10\sigma_{(1,3,1)}(n) - \sigma_{(3,60,3)}(n) + 2\sigma_{(3,5,12)}(n) - 5\sigma_{(3,12,3)}(n) \\
&\quad - \sigma_{(3,20,3)}(n) + 10\sigma_{(3,3,1)}(n) + 2\sigma_{(3,15,3)}(n) \right) - 5\sigma_{(1,4,15)}(n) + d_4(n),
\end{align*}

\begin{align*}
N(1,5,5,15;n) &= \frac{1}{12} \left(6\sigma_{(1,5,1)}(n) - \sigma_{(5,60,1)}(n) + 6\sigma_{(5,5,12)}(n) - \sigma_{(12,5,1)}(n) \\
&\quad + 3\sigma_{(5,20,5)}(n) - 2\sigma_{(5,3,1)}(n) - 2\sigma_{(5,15,5)}(n) \\
&\quad + 3\sigma_{(5,4,15)}(n) \right) + d_1(n),
\end{align*}

\begin{align*}
N(1,15,15,15;n) &= \frac{1}{12} \left(2\sigma_{(1,15,1)}(n) - \sigma_{(15,60,1)}(n) + 2\sigma_{(5,5,12)}(n) - \sigma_{(12,5,15)}(n) \\
&\quad - \sigma_{(15,20,5)}(n) + 2\sigma_{(5,3,1)}(n) + 2\sigma_{(15,15,5)}(n) \\
&\quad - \sigma_{(15,4,15)}(n) \right) - \frac{4}{9}d_1(n) + \frac{1}{3}d_4(n) + \frac{16}{9}d_6(n),
\end{align*}

\begin{align*}
N(3,3,3,5;n) &= \frac{1}{12} \left(10\sigma_{(3,3,1)}(n) - \sigma_{(60,1)}(n) - 2\sigma_{(3,5,12)}(n) + 5\sigma_{(12,5,5)}(n) \\
&\quad + \sigma_{(3,20,3)}(n) - 10\sigma_{(3,3,1)}(n) + 2\sigma_{(5,15,5)}(n) \\
&\quad - 5\sigma_{(3,4,15)}(n) \right) - \frac{25}{18}d_1(n) - \frac{10}{3}d_3(n) \\
&\quad - \frac{5}{6}d_4(n) + 2d_5(n) + \frac{50}{9}d_6(n) + \frac{5}{3}d_8(n).
\end{align*}
\[ N(3, 5, 5; n) = \frac{1}{12} \left( 6\sigma_{(x_1,x_60)}(n) - \sigma_{(x_60,x_1)}(n) - 6\sigma_{(x_{12},x_3)}(n) \\
- 3\sigma_{(x_{20},x_{-3})}(n) + 2\sigma_{(x_{-3},x_{-20})}(n) - 2\sigma_{(x_{-15},x_{-4})}(n) \\
+ 3\sigma_{(x_{-4},x_{-15})}(n) \right) + \frac{16}{3} d_7(n) - \frac{5}{3} d_6(n), \]

\[ N(3, 5, 15, 15; n) = \frac{1}{12} \left( 2\sigma_{(x_1,x_{60})}(n) - \sigma_{(x_{60},x_1)}(n) - 2\sigma_{(x_{12},x_3)}(n) \\
+ \sigma_{(x_{12},x_{15})}(n) + \sigma_{(x_{20},x_{-3})}(n) - 2\sigma_{(x_{-3},x_{-20})}(n) \\
+ 2\sigma_{(x_{-15},x_{-4})}(n) - \sigma_{(x_{-4},x_{-15})}(n) \right) - \frac{1}{6} d_4(n) - \frac{2}{5} d_5(n) \\
- \frac{1}{10} d_4(n) - \frac{2}{5} d_5(n) + \frac{2}{3} d_6(n) + \frac{1}{5} d_6(n). \]

**Proof.** The proof is similar to that of Theorem 4.2. \( \square \)

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