NUMBERS WITH THREE CLOSE FACTORIZATIONS AND LATTICE POINTS ON HYPERBOLAS

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Abstract

In this paper, we study numbers n that can be factored in three different ways as $n = A_1B_1 = A_2B_2 = A_3B_3$ with $A_1 < A_2 < A_3$ and $B_1 > B_2 > B_3$ satisfying $A_3 - A_1, B_1 - B_3 \leq C$. Then one must have $A_3, B_1 \leq C(C-1)^2/4$ and the upper bound is best possible. With this, we obtain an optimal lower bound for the gap between three close lattice points on the hyperbola $xy = n$.

1. Introduction and Main Results

Suppose a positive integer n can be factored as $n = AB$. Is it possible to have another factorization $n = (A+a)(B-b)$ with integers $0 < a, b \leq C$ and C small? (Note: We consider $A + a = B$ and $B - b = A$ as a different factorization as order matters.) The answer is yes. For this to hold, we require $AB = (A+a)(B-b)$ or $ab = aB - bA$. Let $d = (a, b)$ and $a = da'$, $b = db'$. Then $da'b' = a'B - b'A$. This implies that $a'|A$ and $b'|B$ since a' and b' are relatively prime. Let $A = a'A'$ and $B = b'B'$. Then $d = B' - A'$. Therefore, any such number would have the form $n = (a'A')(b'(A'+d)) = (a'(A'+d))(b'A')$ where a', b', d satisfy $da', db' \leq C$ and A' can be arbitrarily large. What if we ask for two such extra factorizations, i.e., $n = AB = (A+a_1)(B-b_1) = (A+a_2)(B-b_2)$?

Suppose $AB = (A+a_1)(B-b_1) = (A+a_2)(B-b_2)$ with $1 \leq a_1 < a_2 \leq C$ and $1 \leq b_1 < b_2 \leq C$. Then $a_1B - b_1A = a_1b_1$. Dividing by b_1B, we have $a_1/b_1 - A/B = a_1/B$. Similarly $a_2/b_2 - A/B = a_2/B$. Subtracting these two equations, we have

$$\frac{a_2}{b_2} - \frac{a_1}{b_1} = \frac{a_2 - a_1}{B} > 0. \tag{1}$$

Hence,

$$\frac{1}{C(C-1)} \leq \frac{1}{b_1 b_2} \leq \frac{a_2 b_1 - a_1 b_2}{b_1 b_2} = \frac{a_2 - a_1}{B} \leq \frac{C-1}{B}.$$

This gives $B \leq C(C-1)^2$. Similarly, one also has $A \leq C(C-1)^2$.

Hence, given $C \geq 2$, if $n = AB = (A+a_1)(B-b_1) = (A+a_2)(B-b_2)$ with $1 \leq a_1 < a_2 \leq C$ and $1 \leq b_1 < b_2 \leq C$, then $A, B \leq C(C-1)^2$. So A and B cannot be arbitrary large in terms of C.

Now one may ask if the upper bound $C(C-1)^2$ is sharp, and we have the following.

Theorem 1. *Let $C \geq 4$. If $n = AB = (A+a_1)(B-b_1) = (A+a_2)(B-b_2)$ with $1 \leq a_1 < a_2 \leq C$ and $1 \leq b_1 < b_2 \leq C$, then $A, B \leq C(C-1)^2/4$.*

Moreover, the above upper bound can be attained if and only if $C = 2N+1$ and

$$\begin{aligned} n =& [(2N-1)(N-1)(N+1)] \cdot [(2N+1)N^2] \\ =& [(2N-1)(N-1)(N+1) + (N-1)] \cdot [(2N+1)N^2 - N] \\ =& [(2N-1)(N-1)(N+1) + (2N-1)] \cdot [(2N+1)N^2 - (2N+1)]. \end{aligned}$$

If one excludes this family of n's, then one has $A, B < C^2(C-4)/4 + C$ for $C \geq 4$.

In a different perspective, we can think of finding three close factorizations of n as finding three close lattice points on the hyperbola $xy = n$. In [3], Cilleruelo and Jiménez-Urroz proved the following theorem.

Theorem 2. *On the hyperbola $xy = N$, there are at most k lattice points (x_1, y_1), ..., (x_k, y_k) such that $N^\gamma \leq x_1 < ... < x_k$ and $x_k - x_1 \leq N^{E_k(\gamma)}$ where*

$$E_k(\gamma) := \frac{\lfloor k\gamma \rfloor (2k\gamma - \lfloor k\gamma \rfloor - 1)}{k(k-1)}.$$

Applying this with $k = 3$ and $\gamma = 1/2$, we have $E_3(\gamma) = 1/6$. This matches with the above theorems. For example, Theorem 1 shows that $N = A_1 B_1$ with $A_1, B_1 < C^3/4$ and hence $N < C^6$ or $C > N^{1/6}$.

Granville and Jiménez-Urroz [5] gave a lower bound for an arc of the hyperbola $xy = n$ containing k integer lattice points. Suppose (x_1, y_1), (x_2, y_2) and (x_3, y_3) are three integer lattice points on $xy = n$ with $x_1 < x_2 < x_3$. They showed that

$$x_3 - x_1 \geq 2^{2/3} \frac{x_1}{n^{1/3}}. \tag{2}$$

From Theorem 1, we have the following corollary.

Corollary 1. *Suppose* (x_1, y_1), (x_2, y_2) *and* (x_3, y_3) *are three integer lattice points on* $xy = n$ *with* $x_1 < x_2 < x_3$. *Then*

$$\max(x_3 - x_1, y_1 - y_3) \geq 2^{2/3} n^{1/6} + (3 - \sqrt[6]{192}).$$

Moreover, for any $\epsilon > 0$, *there exist infinitely many integers* n *such that the hyperbola* $xy = n$ *contains three integer lattice points* (x_1, y_1), (x_2, y_2) *and* (x_3, y_3) *with* $x_1 < x_2 < x_3$ *satisfying* $\max(x_3 - x_1, y_1 - y_3) < 2^{2/3} n^{1/6} + 1 + \epsilon$.

Note that $(3 - \sqrt[6]{192}) = 0.598126...$ The lower bound in the first half of Corollary 1 is best possible. The second half of Corollary 1 shows that the constant $2^{2/3}$ in (2) is sharp and vanquishes any hope to prove the lower bound $\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2} \gg n^{1/4}$ as claimed by Granville and Jiménez-Urroz in [5].

In [4], Cilleruelo and Jiménez-Urroz studied the case of four close lattice points. They defined

$$\epsilon_k(\gamma) = \liminf\{\epsilon | n^\gamma \ll a_1 < ... < a_k = a_1 + n^\epsilon\}$$

where a_i divides n for all $1 \leq i \leq k$, i.e., the minimum ϵ such that for infinitely many n there exist k lattice points (a_i, b_i), $a_i \asymp n^\gamma$, on an arc of length n^ϵ of the hyperbola $xy = n$, and proved that

$$\epsilon_4(1/2) = E_4(1/2) = 1/6.$$

This suggests that there should be a similar result (with a smaller constant) for four close factorizations. In a forthcoming paper, the author will deal with four and five close factorizations.

More generally, I. Ruzsa proposed the following conjecture.

Conjecture 1. *For all* $\epsilon > 0$, *there exists an integer* k *such that only for a finite number of values of* n *there can be more than* k *lattice points on* $xy = n$ *with* $n^{1/2} \leq x \leq n^{1/2} + n^{1/2-\epsilon}$.

The author made slight contributions in [1] and [2] to this conjecture for n's that are perfect squares and almost squares. This would be one ultimate direction of research in this topic.

We will use the following notation. The symbol $f(x) \ll g(x)$ means that $|f(x)| \leq Cg(x)$ for some constant $C > 0$. The symbol $f(x) \asymp g(x)$ means that $f(x) \ll g(x)$ and $g(x) \ll f(x)$. The symbol $\lfloor x \rfloor$ stands for the greatest integer that is less than or equal to x and the symbol $a|b$ means a divides b.

2. Proof of Theorem 1

Proof. Without loss of generality, we can assume that A, $A + a_1$, $A + a_2$ are three consecutive divisors of n. From (1), we have

$$\frac{a_2 b_1 - b_2 a_1}{b_1 b_2} = \frac{a_2 - a_1}{B}.$$

Let $D := (a_2 - a_1)b_1 - (b_2 - b_1)a_1 \geq 1$. Then

$$B = \frac{b_2 b_1 (a_2 - a_1)}{D} = \frac{b_2(a_1(b_2 - b_1) + D)}{D} = \frac{1}{D} a_1 b_2 (b_2 - b_1) + b_2.$$

Case 1: $a_2 = b_2$. From $AB = (A + a_2)(B - b_2)$, we have $a_2 = B - A > 0$. Now $A < A + a_1 < A + a_2 = B$ and $A + a_2 = B > B - b_1 > B - b_2 = A$. Since we assume that A, $A + a_1$, $A + a_2$ are three consecutive divisors of n, we must have $A + a_1 = B - b_1 =: M$ say. So

$$n = M^2 = (M + (a_2 - a_1))(M - (b_2 - b_1))$$

which implies

$$(a_2 - a_1)(b_2 - b_1) = ((a_2 - a_1) - (b_2 - b_1))M.$$

As the left-hand side is positive, the right hand side must be at least M. Therefore,

$$(C - 1)^2 \geq (a_2 - a_1)(b_2 - b_1) = ((a_2 - a_1) - (b_2 - b_1))M \geq M$$

which implies $A, B \leq (C - 1)^2 + (C - 1) \leq C(C - 1)^2/4$ when $C \geq 5$.

Now if $C = 4$, then $1 \leq a_1 < a_2 \leq 4$ and $1 \leq b_1 < b_2 \leq 4$. So

$$n = M^2 = (M + 3)(M - 2) \text{ or } (M + 3)(M - 1) \text{ or } (M + 2)(M - 1)$$

which implies $M = 6$ or $M = 2$. One can check directly for $n = 36$ or $n = 4$ that the statement of Theorem 1 holds true.

Case 2: $a_2 = b_2 + d$ for $d > 0$. We have

$$n = AB = (A + a_2)(B - (a_2 - d)) \tag{3}$$

which implies $a_2 B - (a_2 - d)A = a_2(a_2 - d)$. So $a_2 | dA$ and we write $dA = a_2 A'$. Substituting this into (3), we have $A'B = (A' + d)(B - (a_2 - d))$. After some algebra, $dB - (a_2 - d)A' = d(a_2 - d)$ which implies $a_2 - d | dB$. We write $dB = (a_2 - d)B'$.

Hence $A'dB = (A'+d)(dB - d(a_2-d))$ which gives $A'B' = (A'+d)(B'-d)$ and $B' = A'+d$ after some algebra. Therefore, the three factorizations are

$$n = \Big[\frac{a_2A'}{d}\Big]\Big[\frac{(a_2-d)(A'+d)}{d}\Big] = \Big[\frac{a_2A'}{d}+h\Big]\Big[\frac{(a_2-d)(A'+d)}{d}-k\Big]$$
$$= \Big[\frac{a_2(A'+d)}{d}\Big]\Big[\frac{(a_2-d)A'}{d}\Big]$$

for some $0 < h < a_2$ and $0 < k < a_2 - d$. From the second factorization of n above, we get

$$h\frac{(a_2-d)(A'+d)}{d} - k\frac{a_2A'}{d} - hk = 0.$$

After some algebra, we arrive at

$$h((a_2-d)-k) = A'\Big(\frac{a_2(k-h)}{d}+h\Big). \tag{4}$$

As the left hand side is positive, we must have $h \le k$ or $k < h < a_2k/(a_2-d)$.

For $h \le k$, we have $A' \le (a_2-d)-k$ which implies

$$A = \frac{a_2A'}{d} \le a_2(a_2-d-1) \le \frac{C(C-1)^2}{4}$$

for $C \ge 4$. Similarly,

$$B = \frac{(a_2-d)(A'+d)}{d} \le \frac{C(C-1)^2}{4}$$

for $C \ge 4$.

For $k < h < a_2k/(a_2-d)$, we consider the following two subcases:

Subcase 1: $d = 2$. Then $h = k+1$. Hence $A'(h - a_2/2) = h(a_2-1-h)$. When a_2 is odd and $h = (a_2+1)/2$, $A' = (a_2-3)(a_2+1)/2$. Otherwise $h - a_2/2$ is at least one and $A' \le (a_2-1)^2/4$. In any case, we have

$$A = \frac{a_2A'}{2} \le \frac{C(C-1)^2}{4}.$$

Similarly,

$$B = \frac{(a_2-2)(A'+2)}{2} \le \frac{C(C-1)^2}{4}$$

for $C \ge 3$.

Subcase 2: $d = 3$. Then $h = k+1$ or $h = k+2$. Hence,

$$A'\Big(h - \frac{a_2}{3}\Big) = h(a_2-2-h) \text{ or } A'\Big(h - \frac{2a_2}{3}\Big) = h(a_2-1-h).$$

When $h = (a_2+1)/3$ is an integer, $A' = (a_2+1)(2a_2-7)/3$ and

$$A = \frac{a_2 A'}{3} = \frac{a_2(a_2+1)(2a_2-7)}{9} \le \frac{C(C-1)^2}{4}.$$

Similarly,

$$B = \frac{(a_2-3)(A'+3)}{3} \le \frac{C(C-1)^2}{4}$$

for $C \ge 3$.

When $h = (2a_2+1)/3$ is an integer, $A' = (2a_2+1)(a_2-4)/3$ and

$$A = \frac{a_2 A'}{3} = \frac{a_2(a_2-4)(2a_2+1)}{9} \le \frac{C(C-1)^2}{4}.$$

Similarly,

$$B = \frac{(a_2-3)(A'+3)}{3} \le \frac{C(C-1)^2}{4}$$

for $C \ge 3$.

For the other situations, we have $h - a_2/3 \ge 2/3$ or $h - 2a_2/3 \ge 2/3$. Then

$$A' \le \frac{3h(a_2-1-h)}{2} \le \frac{3(C-1)^2}{8} \text{ and } A = \frac{a_2 A'}{3} \le \frac{C(C-1)^2}{4}.$$

Similarly,

$$B = \frac{(a_2-3)(A'+3)}{3} \le \frac{C(C-1)^2}{4}$$

for $C \ge 2$.

Subcase 3: $d \ge 4$. Let $a_2 = (1+\phi)a_1 \le C$ and $b_2 = (1+\theta)b_1 \le C-d$. Thus,

$$b_1 \le \frac{C-d}{1+\theta},\ a_1 \le \frac{C}{1+\theta},\ \text{and } b_2 - b_1 = \theta b_1 \le \frac{\theta(C-d)}{1+\theta}.$$

Note that $D = (\phi-\theta)a_1 b_1 \ge 1$. Therefore,

$$B = \frac{1}{D}a_1 b_2(b_2-b_1) + b_2 \le \frac{C}{1+\theta}(C-d)\frac{\theta(C-d)}{1+\theta} + C = \frac{\theta}{(1+\theta)^2}C(C-d)^2 + C.$$

It remains to note that $\theta/(1+\theta)^2$ has maximum value $1/4$ when $\theta = 1$. This gives

$$B \le C(C-4)^2/4 + C \le C(C-1)^2/4$$

for $C \ge 3$. Similarly,

$$A = \frac{a_1 a_2(b_2-b_1)}{D} \le \frac{C}{1+\theta}C\frac{\theta(C-d)}{1+\theta} = \frac{\theta}{(1+\theta)^2}C^2(C-d)$$

which gives

$$A \le C^2(C-4)/4 \le C(C-1)^2/4$$

for $C \ge 3$.

Case 3: $a_2 + d = b_2$ for $d > 0$. This follows by symmetry from case 2.

Combining the above cases, we see that the optimal situations come from Case (2), Subcase (1), and we have the first part of Theorem 1. Moreover, this upper bound can be attained exactly when $a_2 = 2N+1 = C$, $A = (2N-1)(N+1)(N-1)$ and $B = (2N+1)N^2$. Then

$$\begin{aligned} n =& [(2N-1)(N-1)(N+1)] \cdot [(2N+1)N^2] \\ =& [(2N-1)(N-1)(N+1) + (N-1)] \cdot [(2N+1)N^2 - N] \\ =& [(2N-1)(N-1)(N+1) + (2N-1)] \cdot [(2N+1)N^2 - (2N+1)]. \end{aligned}$$

Putting these n's aside, the next biggest upper bound comes from Case (2), Subcase (3), which gives the last part of Theorem 1. ☐

3. Proof of Corollary 1

Proof. Suppose (x_1, y_1), (x_2, y_2) and (x_3, y_3) are three integer lattice points on the hyperbola $xy = n$ with $x_1 < x_2 < x_3$. Then

$$n = AB = (A + a_1)(B - b_1) = (A + a_2)(B - b_2)$$

where $A = x_1$, $B = y_1$, $a_1 = x_2 - x_1$, $b_1 = y_1 - y_2$, $a_2 = x_3 - x_1$, $b_2 = y_1 - y_3$. Let $C = \max(a_2, b_2)$.

Suppose $C = 2$. Then $n = AB = (A+1)(B-1) = (A+2)(B-2)$ which is impossible.

Suppose $C = 3$. Then either (i) $n = AB = (A+1)(B-b_1) = (A+3)(B-b_2)$ or (ii) $n = AB = (A+2)(B-b_1) = (A+3)(B-b_2)$ for some $1 \le b_1 < b_2 \le 3$.

In case (i), we have $B = b_1 A + b_1$. This, together with $AB = (A+3)(B-b_2)$, gives $(b_2 - b_1)(A+3) = 2b_1 A$. Since $b_2 \le 3$, we must have $b_1 = 1$ and then $b_2 = 2$ and $A = 3$. Hence, $n = 12$ and one can check that

$$C \ge 2^{2/3} n^{1/6} + (3 - \sqrt[6]{192})$$

with equality holding when $C = 3$. Note that $3 - \sqrt[6]{192} = 0.598126...$

In case (ii), we have $B = b_1(A+2)/2$. This together with $AB = (A+3)(B-b_2)$ gives $3b_1(A+1) = 2b_2(A+3)$. One can check that none of the choices for $1 \le b_1 < b_2 \le 3$ is possible.

Thus, it remains to consider the case when $C \ge 4$. Theorem 1 tells us that

$$n = AB \le \left(\frac{C(C-1)^2}{4}\right)^2,$$

so $4\sqrt{n} \le C(C-1)^2$. Now $(C-2/3)^3 - C(C-1)^2 = C/3 - 8/27 > 0$ since $C \ge 1$. So $4\sqrt{n} \le C(C-1)^2 < (C-2/3)^3$ and

$$2^{2/3}n^{1/6} + \frac{2}{3} < C = \max(x_3 - x_1, y_1 - y_3),$$

which gives the first half of Corollary 1. The second half of Corollary 1 follows from the case when $C = 2N+1$ in Theorem 1. Recall

$$n = (N-1)N^2(N+1)(2N-1)(2N+1)$$

where

$$n = [(2N-1)(N-1)(N+1)] \cdot [(2N+1)N^2] =: x_1 \cdot y_1,$$
$$n = [(2N-1)(N-1)(N+1) + (N-1)] \cdot [(2N+1)N^2 - N] =: x_2 \cdot y_2,$$

and

$$n = [(2N-1)(N-1)(N+1) + (2N-1)] \cdot [(2N+1)N^2 - (2N+1)] =: x_3 \cdot y_3.$$

Then $\max(x_3 - x_1, y_1 - y_3) = 2N+1 = C$, and $16n = (C+1)C(C-1)^2(C-2)(C-3)$. For any t, we have

$$\begin{aligned} 16n - (C-t)^6 =& (C+1)C(C-1)^2(C-2)(C-3) - (C-t)^6 \\ =& 6(t-1)C^5 - (15t^2-10)C^4 + 20t^3C^3 - (15t^4+11)C^2 + (6t^5+6)C - t^6. \end{aligned}$$

If $t = 1 + \epsilon$, for sufficiently large C (hence sufficiently large N) the above quantity is greater than zero. Hence $16n > (C-t)^6$ which implies $2^{2/3}n^{1/6} + 1 + \epsilon > C = \max(x_3 - x_1, y_3 - y_1)$. This gives the second half of Corollary 1. □

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