RANGES OF UNITARY DIVISOR FUNCTIONS

Colin Defant[1]
Department of Mathematics, University of Florida, Gainesville, Florida[2]
Current email: cdefant@princeton.edu

Received: 6/22/17, Revised: 11/26/17, Accepted: 2/20/18, Published: 3/9/18

Abstract

For any real t, the unitary divisor function σ_t^* is the multiplicative arithmetic function defined by $\sigma_t^*(p^\alpha) = 1 + p^{\alpha t}$ for all primes p and positive integers α. Let $\overline{\sigma_t^*(\mathbb{N})}$ denote the topological closure of the range of σ_t^*. We calculate an explicit constant $\eta^* \approx 1.9742550$ and show that $\overline{\sigma_{-r}^*(\mathbb{N})}$ is connected if and only if $r \in (0, \eta^*]$. We end with some open problems.

1. Introduction

For each $c \in \mathbb{C}$, the divisor function σ_c is defined by $\sigma_c(n) = \sum_{d|n} d^c$. Divisor functions, especially σ_1, σ_0, and σ_{-1}, are among the most extensively-studied arithmetic functions [2, 10, 12]. For example, two very classical number-theoretic topics are the study of perfect numbers and the study of friendly numbers. A positive integer n is said to be *perfect* if $\sigma_{-1}(n) = 2$, and n is said to be *friendly* if there exists $m \neq n$ with $\sigma_{-1}(m) = \sigma_{-1}(n)$ [14]. Motivated by the very difficult problems related to perfect and friendly numbers, Laatsch [11] studied $\sigma_{-1}(\mathbb{N})$, the range of σ_{-1}. He showed that $\sigma_{-1}(\mathbb{N})$ is a dense subset of the interval $[1, \infty)$ and asked if $\sigma_{-1}(\mathbb{N})$ is in fact equal to the set $\mathbb{Q} \cap [1, \infty)$. Weiner [16] answered this question in the negative, showing that $(\mathbb{Q} \cap [1, \infty)) \setminus \sigma_{-1}(\mathbb{N})$ is also dense in $[1, \infty)$.

The author has studied ranges of divisor functions in a variety of contexts [4, 5, 6, 7, 8]. For example, it is shown in [4] that $\mathcal{N}(c) \to \infty$ as $\Re(c) \to -\infty$, where $\mathcal{N}(c)$ denotes the number of connected components of $\overline{\sigma_c(\mathbb{N})}$. Here, the overline denotes the topological closure. In [15], Sanna develops an algorithm that can be used to calculate $\overline{\sigma_{-r}(\mathbb{N})}$ when $r > 1$ is real and is known with sufficient precision. In addition, he proves that $\mathcal{N}(-r)$ is finite for such r. The author [5] has since extended this result, showing that $\mathcal{N}(c)$ is finite whenever $\Re(c) \leq 0$ and $c \neq 0$. Very recently, Zubrilina [17] has obtained asymptotic estimates for $\mathcal{N}(-r)$ when

[1]This work was supported by National Science Foundation grant no. 1262930.
[2]Current Address: Department of Mathematics, Princeton University

$r > 1$. She has also shown that there is no real number r such that $\mathcal{N}(r) = 4$.

In this paper, we study the close relatives of the divisor functions known as unitary divisor functions. A *unitary divisor* of an integer n is a divisor d of n such that $\gcd(d, n/d) = 1$. The unitary divisor function σ_c^* is defined by

$$\sigma_c^*(n) = \sum_{\substack{d|n \\ \gcd(d,n/d)=1}} d^c$$

(see, for example, [1], [3], or [9]). The function σ_c^* is multiplicative and satisfies $\sigma_c^*(p^\alpha) = 1 + p^{\alpha c}$ for all primes p and positive integers α.

If $t \in [-1, 0)$, then one may use the same argument that Laatsch employed in [11] in order to show that $\overline{\sigma_t^*(\mathbb{N})} = [1, \infty)$. In particular, $\overline{\sigma_t^*(\mathbb{N})}$ is connected if $t \in [-1, 0)$. On the other hand, $\overline{\sigma_t^*(\mathbb{N})}$ is a discrete disconnected set if $t \geq 0$ (indeed, in this case, $\sigma_t(\mathbb{N}) \cap [0, s]$ is finite for every $s > 0$). The purpose of this paper is to prove the following theorem. Let ζ denote the Riemann zeta function.

Theorem 1. *Let η^* be the unique number in the interval $(1, 2]$ that satisfies the equation*

$$\frac{2^{\eta^*} + 1}{2^{\eta^*}} \cdot \frac{(3^{\eta^*} + 1)^2}{3^{2\eta^*} + 1} = \frac{\zeta(\eta^*)}{\zeta(2\eta^*)}. \tag{1}$$

If $r \in \mathbb{R}$, then $\overline{\sigma_{-r}(\mathbb{N})}$ is connected if and only if $r \in (0, \eta^*]$.

Remark 1.1. In the process of proving Theorem 1, we will show that there is indeed a unique solution to the equation (1) in the interval $(1, 2]$.

In all that follows, we assume $r > 1$ and study $\sigma_{-r}^*(\mathbb{N})$. We first observe that $\sigma_{-r}^*(\mathbb{N}) \subseteq [1, \zeta(r)/\zeta(2r))$. This is because if $q_1^{\beta_1} \cdots q_v^{\beta_v}$ is the prime factorization of some positive integer, then

$$\sigma_{-r}^*(q_1^{\beta_1} \cdots q_v^{\beta_v}) = \prod_{i=1}^{v} \sigma_{-r}^*(q_i^{\beta_i}) = \prod_{i=1}^{v} \left(1 + q_i^{-\beta_i r}\right) \leq \prod_{i=1}^{v} \left(1 + q_i^{-r}\right) < \prod_{p} \left(1 + p^{-r}\right)$$

$$= \prod_{p} \left(\frac{1 - p^{-2r}}{1 - p^{-r}}\right) = \frac{\zeta(r)}{\zeta(2r)}.$$

It is straightforward to show that 1 and $\zeta(r)$ are elements of $\overline{\sigma_{-r}^*(\mathbb{N})}$. Therefore, Theorem 1 tells us that $\overline{\sigma_{-r}^*(\mathbb{N})} = [1, \zeta(r)/\zeta(2r)]$ if and only if $r \in (0, \eta^*]$.

2. Proofs

In what follows, let p_i denote the i^{th} prime number. Let $\nu_p(x)$ denote the exponent of the prime p appearing in the prime factorization of the integer x.

To start, we need the following technical yet simple lemma.

Lemma 1. *If $s, m \in \mathbb{N}$ and $s \leq m$, then* $\dfrac{p_s^{2r}+1}{p_s^{2r}+p_s^r} \leq \dfrac{p_m^{2r}+1}{p_m^{2r}+p_m^r}$ *for all $r > 1$.*

Proof. Fix some $r > 1$, and write $h(x) = \dfrac{x^{2r}+1}{x^{2r}+x^r}$. Then

$$h'(x) = \frac{r}{x(x^r+1)^2}\left(x^r - 2 - \frac{1}{x^r}\right).$$

We see that $h(x)$ is increasing when $x \geq 3$. Hence, in order to complete the proof, it suffices to show that $h(2) \leq h(3)$. Let $f(s) = 2^s 3^{2s} + 2^{2s} + 2^s - (2^{2s}3^s + 3^{2s} + 3^s)$. For $s \geq 1$, we have

$$f''(s) = 18^s \log^2(18) + 4^s \log^2(4) + 2^s \log^2(2) - 12^s \log^2(12) - 9^s \log^2(9) - 3^s \log^2(3)$$
$$> 18^s \log^2(18) - 12^s \log^2(12) - 9^s \log^2(9) > 18^s \log^2(18) - 2(12^s \log^2(12)).$$

It is easy to verify that $18^s \log^2(18) - 2(12^s \log^2(12))$ is increasing in s for $s \geq 1$, so we obtain

$$f''(s) > 18 \log^2(18) - 2(12 \log^2(12)) > 0.$$

A simple calculation shows that $f'(1) > 0$, so it follows that $f'(s) > 0$ for all $s \geq 1$. Since $f(1) = 0$ and $r > 1$, we have $f(r) > 0$. Equivalently, $2^{2r}3^r + 3^{2r} + 3^r < 2^r 3^{2r} + 2^{2r} + 2^r$. It follows that $(2^{2r}+1)(3^{2r}+3^r) < (2^{2r}+2^r)(3^{2r}+1)$. This shows that $\dfrac{2^{2r}+1}{2^{2r}+2^r} < \dfrac{3^{2r}+1}{3^{2r}+3^r}$, which completes the proof. □

The following theorem replaces the question of whether or not $\overline{\sigma^*_{-r}(\mathbb{N})}$ is connected with a question concerning infinitely many inequalities. The advantage in doing this is that we will further reduce this problem to the consideration of a finite list of inequalities in Theorem 3. Recall from the introduction that $\overline{\sigma^*_{-r}(\mathbb{N})}$ is connected if and only if it is equal to the interval $[1, \zeta(r)/\zeta(2r)]$.

Theorem 2. *If $r > 1$, then $\overline{\sigma^*_{-r}(\mathbb{N})} = [1, \zeta(r)/\zeta(2r))$ if and only if*

$$\frac{p_m^{2r}+p_m^r}{p_m^{2r}+1} \leq \prod_{i=m+1}^{\infty}\left(1 + \frac{1}{p_i^r}\right)$$

for all positive integers m.

Proof. First, suppose that $\dfrac{p_m^{2r}+p_m^r}{p_m^{2r}+1} \leq \displaystyle\prod_{i=m+1}^{\infty}\left(1 + \frac{1}{p_i^r}\right)$ for all positive integers m. We will show that the range of $\log \sigma^*_{-r}$ is dense in $[0, \log(\zeta(r)/\zeta(2r)))$, which will then imply that the range of σ^*_{-r} is dense in $[1, \zeta(r)/\zeta(2r))$. Fix some

$$x \in (0, \log(\zeta(r)/\zeta(2r))).$$

We will construct a sequence $(C_i)_{i=1}^{\infty}$ of elements of the range of $\log \sigma^*_{-r}$ that converges to x. First, let $C_0 = 0$. For each positive integer n, if $C_{n-1} < x$, let $C_n = C_{n-1} + \log\left(1 + p_n^{-\alpha_n r}\right)$, where α_n is the smallest positive integer that satisfies $C_{n-1} + \log\left(1 + p_n^{-\alpha_n r}\right) \le x$. If $C_{n-1} = x$, simply set $C_n = C_{n-1} = x$. For each $n \in \mathbb{N}$, $C_n \in \log \sigma^*_{-r}(\mathbb{N})$. Indeed, if $C_n \neq C_{n-1}$, then

$$C_n = \sum_{i=1}^{n} \log\left(1 + p_i^{-\alpha_i r}\right) = \log\left(\prod_{i=1}^{n}\left(1 + p_i^{-\alpha_i r}\right)\right) = \log \sigma^*_{-r}\left(\prod_{i=1}^{n} p_i^{\alpha_i}\right).$$

If, however, $C_n = C_{n-1} = x$, then we may let l be the smallest positive integer such that $C_l = x$ and show, in the same manner as above, that

$$C_n = C_l = \log \sigma^*_{-r}\left(\prod_{i=1}^{l} p_i^{\alpha_i}\right).$$

Let us write $\gamma = \lim_{n\to\infty} C_n$. Note that γ exists and that $\gamma \le x$ because the sequence $(C_i)_{i=1}^{\infty}$ is nondecreasing and bounded above by x. If we can show that $\gamma = x$, then we will be done. Therefore, let us assume instead that $\gamma < x$.

We have $C_n = C_{n-1} + \log(1 + p_n^{-\alpha_n r})$ for all positive integers n. Write $D_n = \log(1 + p_n^{-r}) - \log(1 + p_n^{-\alpha_n r})$ and $E_n = \sum_{i=1}^{n} D_i$. As

$$x + \lim_{n\to\infty} E_n > \gamma + \lim_{n\to\infty} E_n = \lim_{n\to\infty}(C_n + E_n) = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\log\left(1 + p_i^{-\alpha_i r}\right) + \sum_{i=1}^{n} D_i\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\log\left(1 + p_i^{-r}\right) = \log\left(\zeta(r)/\zeta(2r)\right),$$

we have $\lim_{n\to\infty} E_n > \log\left(\zeta(r)/\zeta(2r)\right) - x$. Therefore, we may let m be the smallest positive integer such that $E_m > \log\left(\zeta(r)/\zeta(2r)\right) - x$. If $\alpha_m = 1$ and $m > 1$, then $D_m = 0$. This forces $E_{m-1} = E_m > \log\left(\zeta(r)/\zeta(2r)\right) - x$, contradicting the minimality of m. If $\alpha_m = 1$ and $m = 1$, then $0 = E_m > \log\left(\zeta(r)/\zeta(2r)\right) - x$, which is also a contradiction since we originally chose $x < \log(\zeta(r)/\zeta(2r))$. Consequently, $\alpha_m > 1$. Due to the way we defined C_m and α_m, we have

$$C_{m-1} + \log\left(1 + p_n^{-(\alpha_m - 1)r}\right) > x.$$

Hence,

$$\log\left(1 + p_n^{-(\alpha_m - 1)r}\right) - \log\left(1 + p_n^{-\alpha_m r}\right) > x - C_m.$$

Using our original assumption that $\dfrac{p_m^{2r} + p_m^r}{p_m^{2r} + 1} \le \prod_{i=m+1}^{\infty}\left(1 + \dfrac{1}{p_i^r}\right)$, we have

$$\log\left(\frac{p_m^{2r} + p_m^r}{p_m^{2r} + 1}\right) \le \sum_{i=m+1}^{\infty}\log\left(1 + \frac{1}{p_i^r}\right) = \log\left(\frac{\zeta(r)}{\zeta(2r)}\right) - E_m - C_m$$

$$< x - C_m < \log\left(1 + p_n^{-(\alpha_m - 1)r}\right) - \log\left(1 + p_n^{-\alpha_m r}\right) = \log\left(\frac{p_m^{\alpha_m r} + p_m^r}{p_m^{\alpha_m r} + 1}\right).$$

Thus,

$$\frac{p_m^{2r} + p_m^r}{p_m^{2r} + 1} < \frac{p_m^{\alpha_m r} + p_m^r}{p_m^{\alpha_m r} + 1}.$$

Rewriting this inequality, we get $p_m^{2r} + p_m^{(\alpha_m+1)r} < p_m^{3r} + p_m^{\alpha_m r}$. Dividing through by $p_m^{\alpha_m r}$ yields $p_m^{(2-\alpha_m)r} + p_m^r < 1 + p_m^{(3-\alpha_m)r}$, which is impossible since $\alpha_m \geq 2$. This contradiction proves that $\gamma = x$, so $\overline{\sigma^*_{-r}(\mathbb{N})} = [1, \zeta(r)/\zeta(2r)]$.

To prove the converse, suppose there exists some positive integer m such that

$$\frac{p_m^{2r} + p_m^r}{p_m^{2r} + 1} > \prod_{i=m+1}^{\infty} \left(1 + \frac{1}{p_i^r}\right).$$

We may write this inequality as

$$\frac{p_m^{2r} + 1}{p_m^{2r} + p_m^r} < \prod_{i=m+1}^{\infty} \left(1 + \frac{1}{p_i^r}\right)^{-1}. \tag{2}$$

Fix a positive integer N. If $\nu_{p_s}(N) = 1$ for all $s \in \{1, 2, \ldots, m\}$, then

$$\sigma^*_{-r}(N) \geq \prod_{s=1}^{m} \left(1 + \frac{1}{p_s^r}\right) = \frac{\zeta(r)}{\zeta(2r)} \prod_{i=m+1}^{\infty} \left(1 + \frac{1}{p_i^r}\right)^{-1}.$$

On the other hand, if $\nu_{p_s}(N) \neq 1$ for some $s \in \{1, 2, \ldots, m\}$, then $\sigma^*_{-r}\left(p_s^{\nu_{p_s}(N)}\right) \leq 1 + \dfrac{1}{p_s^{2r}}$. This implies that

$$\sigma^*_{-r}(N) \leq \left(1 + \frac{1}{p_s^{2r}}\right) \prod_{\substack{i=1 \\ i \neq s}}^{\infty} \left(1 + \frac{1}{p_i^r}\right) = \frac{\zeta(r)}{\zeta(2r)} \frac{1 + p_s^{-2r}}{1 + p_s^{-r}} = \frac{\zeta(r)}{\zeta(2r)} \frac{p_s^{2r} + 1}{p_s^{2r} + p_s^r}$$

in this case. Using Lemma 1, we have

$$\sigma^*_{-r}(N) \leq \frac{\zeta(r)}{\zeta(2r)} \frac{p_m^{2r} + 1}{p_m^{2r} + p_m^r}.$$

As N was arbitrary, we have shown that there is no element of the range of σ^*_{-r} in the interval

$$\left(\frac{\zeta(r)}{\zeta(2r)} \frac{p_m^{2r} + 1}{p_m^{2r} + p_m^r}, \frac{\zeta(r)}{\zeta(2r)} \prod_{i=m+1}^{\infty} \left(1 + \frac{1}{p_i^r}\right)^{-1}\right).$$

This interval is a gap in the range of σ^*_{-r} because of the inequality (2). $\square$

As mentioned above, we wish to reduce the task of checking the infinite collection of inequalities given in Theorem 2 to that of checking finitely many inequalities. We do so in Theorem 3, the proof of which requires the following lemma.

Lemma 2. *If* $j \in \mathbb{N} \setminus \{1,2,3,4,6,9\}$*, then* $\dfrac{p_{j+1}}{p_j} < \sqrt[3]{2}$.

Proof. In [13], it is shown that $\dfrac{p_{j+1}}{p_j} \le \dfrac{6}{5} < \sqrt[3]{2}$ for all $j \ge 10$. We easily verify the cases $j = 5, 7, 8$ by hand. □

Theorem 3. *If* $r \in (1,3]$*, then* $\overline{\sigma^*_{-r}(\mathbb{N})} = [1, \zeta(r)/\zeta(2r)]$ *if and only if*

$$\frac{p_m^{2r} + p_m^r}{p_m^{2r} + 1} \le \prod_{i=m+1}^{\infty} \left(1 + \frac{1}{p_i^r}\right)$$

for all $m \in \{1,2,3,4,6,9\}$.

Proof. Let

$$F(m,r) = \frac{p_m^{2r} + p_m^r}{p_m^{2r} + 1} \prod_{i=1}^{m} \left(1 + \frac{1}{p_i^r}\right)$$

so that the inequality $\dfrac{p_m^{2r} + p_m^r}{p_m^{2r} + 1} \le \displaystyle\prod_{i=m+1}^{\infty} \left(1 + \frac{1}{p_i^r}\right)$ is equivalent to $F(m,r) \le \dfrac{\zeta(r)}{\zeta(2r)}$. Let $r \in (1,3]$. By Theorem 2, it suffices to show that if $F(m,r) \le \dfrac{\zeta(r)}{\zeta(2r)}$ for all $m \in \{1,2,3,4,6,9\}$, then $F(m,r) \le \dfrac{\zeta(r)}{\zeta(2r)}$ for all $m \in \mathbb{N}$. Therefore, assume that r is such that $F(m,r) \le \dfrac{\zeta(r)}{\zeta(2r)}$ for all $m \in \{1,2,3,4,6,9\}$.

We will show that $F(m+1,r) > F(m,r)$ for all $m \in \mathbb{N} \setminus \{1,2,3,4,6,9\}$. This will show that $(F(m,r))_{m=10}^{\infty}$ is an increasing sequence. As $\displaystyle\lim_{m\to\infty} F(m,r) = \frac{\zeta(r)}{\zeta(2r)}$, it will then follow that $F(m,r) < \dfrac{\zeta(r)}{\zeta(2r)}$ for all integers $m \ge 10$. Furthermore, we will see that $F(5,r) < F(6,r) \le \dfrac{\zeta(r)}{\zeta(2r)}$ and $F(7,r) < F(8,r) < F(9,r) \le \dfrac{\zeta(r)}{\zeta(2r)}$, which will complete the proof.

Let $m \in \mathbb{N} \setminus \{1,2,3,4,6,9\}$. By Lemma 2, $\dfrac{p_{m+1}}{p_m} < \sqrt[3]{2} \le \sqrt[r]{2}$. This shows that $p_{m+1}^r < 2p_m^r$, implying that $2p_m^{2r} > p_m^r p_{m+1}^r$. Therefore,

$$2p_m^{2r} + 2 > p_m^r p_{m+1}^r + \frac{p_m^r}{p_{m+1}^r} - p_{m+1}^r - \frac{1}{p_{m+1}^r} = \frac{(p_m^r - 1)(p_{m+1}^{2r} + 1)}{p_{m+1}^r}.$$

Multiplying each side of this inequality by $\dfrac{p_{m+1}^r}{(p_{m+1}^{2r}+1)(p_m^{2r}+1)}$ and adding 1 to each side, we get

$$1+\frac{2p_{m+1}^r}{p_{m+1}^{2r}+1} > 1+\frac{p_m^r-1}{p_m^{2r}+1},$$

which we may write as

$$\frac{(p_{m+1}^r+1)^2}{p_{m+1}^{2r}+1} > \frac{p_m^{2r}+p_m^r}{p_m^{2r}+1}.$$

Finally, we get

$$F(m+1,r)=\frac{p_{m+1}^{2r}+p_{m+1}^r}{p_{m+1}^{2r}+1}\prod_{i=1}^{m+1}\left(1+\frac{1}{p_i^r}\right)=\frac{(p_{m+1}^r+1)^2}{p_{m+1}^{2r}+1}\prod_{i=1}^{m}\left(1+\frac{1}{p_i^r}\right)$$

$$>\frac{p_m^{2r}+p_m^r}{p_m^{2r}+1}\prod_{i=1}^{m}\left(1+\frac{1}{p_i^r}\right)=F(m,r). \qquad \square$$

Now, let

$$V_m(r)=\log\left(\frac{p_m^{2r}+p_m^r}{p_m^{2r}+1}\right)-\sum_{i=m+1}^{\infty}\log\left(1+\frac{1}{p_i^r}\right).$$

Equivalently, $V_m(r)=\log(F(m,r))-\log\left(\dfrac{\zeta(r)}{\zeta(2r)}\right)$, where F is the function defined in the proof of Theorem 3. Observe that

$$\frac{p_m^{2r}+p_m^r}{p_m^{2r}+1}\le\prod_{i=m+1}^{\infty}\left(1+\frac{1}{p_i^r}\right)$$

if and only if $V_m(r)\le 0$. If we let $J_m(r)=\displaystyle\sum_{i=m+1}^{m+6}\frac{1}{p_i^r+1}-\frac{p_m^{2r}-2p_m^r-1}{(p_m^r+1)(p_m^{2r}+1)}$, then we have

$$\frac{\partial}{\partial r}J_m(r)=\frac{p_m^r((p_m^r-1)^4-12p_m^{2r})\log p_m}{(p_m^r+1)^2(p_m^{2r}+1)^2}-\sum_{i=m+1}^{m+6}\frac{p_i^r\log p_i}{(p_i^r+1)^2}.$$

It is not difficult to verify that $\dfrac{p_m^r((p_m^r-1)^4-12p_m^{2r})\log p_m}{(p_m^r+1)^2(p_m^{2r}+1)^2}\ge -1$ for all $r\in[1,2]$ and $m\in\{1,2,3,4,6,9\}$. Therefore, when $r\in[1,2]$ and $m\in\{1,2,3,4,6,9\}$, we have

$$\frac{\partial}{\partial r}J_m(r)\ge -1-\sum_{i=m+1}^{m+6}\frac{p_i^r\log p_i}{(p_i^r+1)^2}\ge -1-\sum_{i=m+1}^{m+6}\frac{\log p_i}{p_i^r}>-7.$$

Numerical calculations show that $J_m(r) > \dfrac{1}{400}$ for all $m \in \{1,2,3,4,6,9\}$ and

$$r \in \left\{1 + \frac{n}{2800} : n \in \{0,1,2,\dots,2800\}\right\}.$$

Because each function J_m is continuous in r for $r \in [1,2]$, we see that

$$J_m(r) > \frac{1}{400} - 7\left(\frac{1}{2800}\right) = 0$$

for all $r \in [1,2]$ and $m \in \{1,2,3,4,6,9\}$.

We introduced the functions J_m so that we could write

$$\frac{\partial}{\partial r}V_m(r) = \sum_{i=m+1}^{\infty} \frac{\log p_i}{p_i^r + 1} - \frac{(p_m^{2r} - 2p_m^r - 1)\log p_m}{(p_m^r + 1)(p_m^{2r} + 1)} > (\log p_m) J_m(r) > 0$$

for all $m \in \{1,2,3,4,6,9\}$ and $r \in [1,2]$. A quick numerical calculation shows that $V_2(1.5) < 0 < V_2(2)$, so the function V_2 has exactly one root, which we will call η^*, in the interval $(1,2]$. Further calculations show that $V_m(2) < 0$ for all $m \in \{1,3,4,6,9\}$. Hence, $V_m(r) \le 0$ for all $m \in \{1,2,3,4,6,9\}$ and $r \in (1, \eta^*]$. By Theorem 3, this means that if $r \in (1,2]$, then $\overline{\sigma^*_{-r}(\mathbb{N})}\,[1, \zeta(r)/\zeta(2r)]$ if and only if $r \le \eta^*$.

Next, note that

$$\frac{\partial}{\partial r}V_2(r) = \sum_{i=3}^{\infty} \frac{\log p_i}{p_i^r + 1} - \frac{(3^{2r} - 2\cdot 3^r - 1)\log 3}{(3^{2r}+1)(3^r+1)} > -\frac{(3^{2r} - 2\cdot 3^r - 1)\log 3}{(3^{2r}+1)(3^r+1)}$$

$$> -\frac{(3^{2r}+1)\log 3}{(3^{2r}+1)(3^r+1)} \ge -\frac{\log 3}{3^2+1} > -1.1$$

for all $r \in [2,3]$. Let $A = \left\{2 + \dfrac{n}{400} : n \in \{0,1,2,\dots,400\}\right\}$. With a computer program, one may verify that $V_2(r) > 0.003$ for all $r \in A$. Because V_2 is continuous, this shows that $V_2(r) > 0.003 - 1.1\left(\dfrac{1}{400}\right) > 0$ for all $r \in [2,3]$. Consequently, $\overline{\sigma^*_{-r}(\mathbb{N})} \ne [1, \zeta(r)/\zeta(2r))$ if $r \in [2,3]$.

We are now in a position to prove Theorem 1. Note that the equation defining η^* in the statement of this theorem is simply a rearrangement of the equation $V_2(\eta^*) = 0$. Therefore, we have shown that the theorem is true for $r \in (1,3]$. In order to prove the theorem for $r > 3$, it suffices (by Theorem 3) to show that $F(1,r) > \dfrac{\zeta(r)}{\zeta(2r)}$ for all $r > 3$. If $r > 3$, then

$$F(1,r) = \frac{(2^r+1)^2}{2^{2r}+1} = \frac{2^{2r} + 2^{r+1} + 1}{2^{2r}+1} > \frac{2^{2r} + 2^r + \frac{2^{r+1}}{r-1}}{2^{2r}+1} = \frac{1 + \frac{1}{2^r} + \frac{1}{(r-1)2^{r-1}}}{1 + \frac{1}{2^{2r}}}$$

$$> \frac{1 + \frac{1}{2^r} + \frac{1}{(r-1)2^{r-1}}}{\zeta(2r)} = \frac{1 + \frac{1}{2^r} + \int_2^\infty x^{-r}\,dx}{\zeta(2r)} > \frac{\zeta(r)}{\zeta(2r)}.$$

3. Future Directions

Let $\mathcal{N}^*(t)$ denote the number of connected components of $\overline{\sigma_t^*(\mathbb{N})}$. It would be interesting to obtain analogues of Zubrilina's results [17] by finding asymptotic estimates for $\mathcal{N}^*(-r)$ as $r \to \infty$. Let

$$E_m^* = \{t \in \mathbb{R} \colon \mathcal{N}^*(t) = m\}.$$

Theorem 1 tells us that $E_1^* = [-\eta^*, 0)$. The sets E_m^* are the natural unitary analogues of the sets E_m defined in [5, Section 4]. Continuing the analogy, we say a positive integer m is a *unitary Zubrilina number* if $E_m^* = \emptyset$ (the name comes from Zubrilina's result that $E_4 = \emptyset$). We do not have any specific examples of unitary Zubrilina numbers, but we still make the following conjectures.

Conjecture 1. There are infinitely many unitary Zubrilina numbers.

Conjecture 2. For $r > 1$, $\mathcal{N}^*(-r)$ is monotonically increasing as a function of r.

Note that Conjecture 2 implies that the sets E_m^* are intervals.

Acknowledgements. The author thanks the referee for carefully reading the manuscript and providing very helpful suggestions.

References

[1] K. Alladi, On arithmetic functions and divisors of higher order, *J. Austral. Math. Soc. Ser. A.* **23** (1997), 9–27.

[2] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, 1976.

[3] E. Cohen, Arithmetical functions associated with the unitary divisors of an integer, *Math. Z.* **74** (1960), 66–80.

[4] C. Defant, Complex divisor functions, *Anal. Geom. Number Theory* **1** (2016), 22–48.

[5] C. Defant, Connected components of complex divisor functions, arXiv:1711.04244.

[6] C. Defant, On ranges of variants of the divisor functions that are dense, *Int. J. Number Theory* **11** (2015), 1905–1912.

[7] C. Defant, On the density of ranges of generalized divisor functions, *Notes on Number Theory and Discrete Mathematics* **21** (2015), 80–87.

[8] C. Defant, On the density of ranges of generalized divisor functions with restricted domains, *Unif. Distrib. Theory* **10** (2015), 19–33.

[9] R. K. Guy, Unsolved Problems in Number Theory, Springer-Verlag, 1981.

[10] G. H. Hardy, P. V. Seshu Aiyar, and B. M. Wilson, *Collected Papers of Srinivasa Ramanujan*, Cambridge: University Press, 1927.

[11] R. Laatsch, Measuring the abundancy of integers, *Math. Mag.* **59** (1986), 84–92.

[12] D. S. Mitrinović, J. Sándor, and B. Crstici, *Handbook of Number Theory, Mathematics and its Applications, Vol. 351, Chapter III*, Kluwer Academic Publishers Group, Dordrecht, 1996.

[13] J. Nagura, On the interval containing at least one prime number, *Proc. Japan Acad.* **28** (1952), 177–181.

[14] P. Pollack and C. Pomerance, Some problems of Erdös on the sum-of-divisors function. *Trans. Amer. Math. Soc.* **3** (2016), 1–26.

[15] C. Sanna, On the closure of the image of the generalized divisor function, *Unif. Distrib. Theory* **12** (2017), 77–90.

[16] P. A. Weiner, The abundancy ratio, a measure of perfection, *Math. Mag.* **73** (2000), 307–310.

[17] N. Zubrilina, On the number of connected components of ranges of divisor functions, arXiv:1711.02871.