#A17 INTEGERS 18 (2018)

GENERALIZED EULERIAN POLYNOMIALS AND SOME APPLICATIONS

Claudio Pita-Ruiz
Universidad Panamericana, Facultad de Ingeniería, Ciudad de México, Mexico
cpita@up.edu.mx

Received: 8/20/16, Revised: 6/22/17, Accepted: 2/28/18, Published: 3/9/18

Abstract

We consider polynomials $P_{a,b,r,p}(z)=\sum_{i=0}^{rp}A_{a,b,r}(p,i)\,z^{rp-i}$, in which the coefficients $A_{a,b,r}(p,i)$ are the generalized Eulerian numbers involved in the expansion $\binom{an+b}{r}^p=\sum_{i=0}^{rp}A_{a,b,r}(p,i)\binom{n+rp+i}{rp}$. The numbers $A_{a,b,r}(p,i)$ and their properties were studied in a previous work. The case $a=1,b=0,r=1$, corresponds to the standard Eulerian polynomials $P_{1,0,1,p}(z)$. We give generalizations for the known recurrences of $P_{1,0,1,p}(z)$. We also show some applications of polynomials $P_{a,b,r,p}(z)$, including explicit formulas for sums and alternating sums of powers of binomial coefficients. The main tool we use to obtain our results is the Z-transform of sequences.

1. Introduction

Since this work is a natural continuation of the previous work [18], where we studied generalized Eulerian numbers, the beginning has some intersections with it. Hence, there are some comments at the beginning of this section that appear also in [18].

Eulerian numbers and polynomials have been mathematical objects of interest to mathematicians along the years: beginning with Euler's work [10] in the eighteenth century, they are still considered as objects worthy of study [11, 12], including several generalizations [15, 16, 17, 19, 24]. Within the works of L. Carlitz we find (besides the expository work [4]) some generalizations of Eulerian numbers [7, 8], including q-generalizations of them [3, 5, 6], among other related works [9]. Eulerian numbers appear as coefficients of the sequence of the so-called Eulerian polynomials $1,\ z+1,\ z^2+4z+1,\ z^3+11z^2+11z+1,\ldots$, considered by L. Euler ([10], pp. 485,486). We will use the notation $A(p,i)$ for the corresponding Eulerian number in the p-th row and i-th column of the so-called Eulerian numbers triangle, with rows $p=1,2,\ldots$, and columns $i=0,1,2,\ldots$ (see Table 1).

Some shifted versions of the Table 1 appear in the literature also as Eulerian numbers triangle. We write $P_p(z)$ to denote the Eulerian polynomial $\sum_{i=0}^{p}A(p,i)\,z^{p-i}$.

$p \diagdown i$	0	1	2	3	4	5	$\cdots$
1	0	1					
2	0	1	1				$\cdots$
3	0	1	4	1			
4	0	1	11	11	1		$\cdots$
5	0	1	26	66	26	1	
$\vdots$		$\vdots$		$\vdots$		$\vdots$	

Table 1: Eulerian numbers triangle.

Observe that $P_p(z)$ is a $(p-1)$-th degree polynomial.

In Table 2 we show some important known facts about Eulerian numbers. In the formula for alternating row sums, B_{p+1} is the $(p+1)$-th Bernoulli number.

Eulerian numbers.	
Explicit Formula:	$A(p,i) = \sum_{j=0}^{i} (-1)^j \binom{p+1}{j} (i-j)^p$.
Symmetry:	$A(p,i) = A(p, p+1-i)$.
Recurrence:	$A(p,i) = iA(p-1,i) + (p+1-i)A(p-1,i-1)$.
Row Sums:	$\sum_{i=0}^{p} A(p,i) = p!$.
Alternating Row Sums:	$\sum_{i=0}^{p} (-1)^i A(p,i) = \frac{2^{p+1}\left(1-2^{p+1}\right)B_{p+1}}{p+1}$.

Table 2: Eulerian numbers properties.

Another remarkable fact of Eulerian numbers is that they are the coefficients appearing when we write n^p as a linear combination of the binomials $\binom{n}{p}, \binom{n+1}{p}, \ldots, \binom{n+p}{p}$, which form a basis of the vector space of n-polynomials of degree not exceeding p. This is the Worpitsky identity [23]:

$$n^p = \sum_{i=0}^{p} A(p,i) \binom{n+p-i}{p}. \tag{1}$$

In a recent work [18], we considered a generalization of Eulerian numbers (inspired by (1)), that includes the following situation: the function

$$f(n) = \binom{an+b}{r}^p \tag{2}$$

is a rp-th degree polynomial function. For $r, p \in \mathbb{N}$, the polynomial (2) can be written as a linear combination of the $rp+1$ binomials $\binom{n+rp-i}{rp}$, $i = 0, 1, \ldots, rp$, which form a basis of the vector space of n-polynomials of degree $\leq rp$. The resulting coefficients $A_{a,b,r}(p,i)$, $p \in \mathbb{N}$, $i = 0, 1, \ldots, rp$, are some of the generalized Eulerian numbers (GEN, for short) we studied in [18]. That is, we have

$$\binom{an+b}{r}^p = \sum_{i=0}^{rp} A_{a,b,r}(p,i) \binom{n+rp-i}{rp}, \tag{3}$$

and $A_{a,b,r}(p,i) = 0$ for $i < 0$ or $i > rp$. In the case $a = r = 1$ and $b = 0$, the polynomial (2) is n^p and the mentioned coefficients $A_{1,0,1}(p,i)$, $p \in \mathbb{N}$, $i = 0, 1, \ldots, p$, are the standard Eulerian numbers $A(p,i)$, and (3) reduces to (1).

The first two elements of the p-th row in the GEN $A_{a,b,r}(p,i)$ triangle (GENT, for short) are

$$A_{a,b,r}(p,0) = \binom{b}{r}^p, \tag{4}$$

$$A_{a,b,r}(p,1) = \binom{a+b}{r}^p - (rp+1)\binom{b}{r}^p. \tag{5}$$

In particular, if $0 \leq b < r$, we have $A_{a,b,r}(p,0) = 0$. This happens, for example, when $b = 0$. The last element of the p-th row is

$$A_{a,b,r}(p,rp) = \binom{a-b+r-1}{r}^p. \tag{6}$$

Two GENT are the following

GENT1: Generalized Eulerian Numbers $A_{1,0,3}(p,i)$.

Explicit Formula: $A_{1,0,3}(p,i) = \sum_{j=0}^{i} (-1)^j \binom{3p+1}{j} \binom{i-j}{3}^p$.

Expansion: $\binom{n}{3}^p = \sum_{i=3}^{3p} A_{1,0,3}(p,i) \binom{n+3p-i}{3p}$.

$p \diagdown i$	3	4	5	6	7	8	9	10	11	12	$\cdots$
1	1										
2	1	9	9	1							$\cdots$
3	1	54	405	760	405	54	1				
4	1	243	6750	49682	128124	128124	49682	6750	243	1	$\cdots$
$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$	

GENT2: Generalized Eulerian Numbers $A_{3,2,2}(p,i)$.									
Explicit Formula: $A_{3,2,2}(p,i)=\sum\limits_{j=0}^{i}(-1)^j\binom{2p+1}{j}\binom{3(i-j)+2}{2}^p$.									
Expansion: $\binom{3n+2}{2}^p=\sum\limits_{i=0}^{2p}A_{3,2,2}(p,i)\binom{n+2p-i}{2p}$.									
$p\diagdown i$	0	1	2	3	4	5	6	7	8	$\cdots$
1	1	7	1							
2	1	95	294	95	1					$\cdots$
3	1	993	14973	33676	14973	993	1			
4	1	9991	524692	3978637	7507078	3978637	524692	9991	1	$\cdots$
$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		

It turns out that the GEN $A_{1,0,r}(2,i)$ are squares of binomial coefficients (see the second row of GENT1). More precisely, for $k=0,1,\ldots,r$, we have

$$A_{1,0,r}(2,k+r)=\sum_{j=0}^{k}(-1)^j\binom{2r+1}{j}\binom{k+r-j}{r}^2=\binom{r}{k}^2, \tag{7}$$

(see identity (6.48) in [13]). Thus, the expansion $\binom{n}{r}^2=\sum_{i=r}^{2r}A_{1,0,r}(2,i)\binom{n+2r-i}{2r}$ can be written as follows

$$\binom{n}{r}^2=\sum_{k=0}^{r}\binom{r}{k}^2\binom{n+r-k}{2r}, \tag{8}$$

which is also a known fact (see identity (6.17) in [13]).

In [18] we obtained results for the GEN $A_{a,b,r}(p,i)$ that include generalizations of the properties of Table 2. For the reader's convenience, we quote some of them next.

- Explicit formula:

$$A_{a,b,r}(p,i)=\sum_{j=0}^{i}(-1)^j\binom{rp+1}{j}\binom{a(i-j)+b}{r}^p.$$

- Symmetries:

  (a) The GEN $A_{a,b,r}(p,i)$ have the symmetry

$$A_{a,b,r}(p,i)=A_{a,a-b+r-1,r}(p,rp-i),\quad i=0,1,\ldots,rp. \tag{9}$$

  (b) For odd $r\in\mathbb{N}$, the GEN $A_{2,1,r}(p,i)$ have the symmetry

$$A_{2,1,r}\left(p,i+\frac{r-1}{2}\right)=A_{2,1,r}(p,rp-i),\quad i=0,1,\ldots,rp-\frac{r-1}{2}. \tag{10}$$

- Recurrence: The recurrence for the GEN $A_{1,b,r}(p,i)$ is given by

$$A_{1,b,r}(p,i)=\sum_{k=0}^{r}\binom{i-k+b}{r-k}\binom{rp+k-i-b}{k}A_{1,b,r}(p-1,i-k), \tag{11}$$

where $i=0,1,\ldots,rp$.

- Row sums: The sum of the GEN $A_{a,b,r}(p,i)$, $i=0,1,\ldots,rp$, is given by

$$\sum_{i=0}^{rp}A_{a,b,r}(p,i)=a^{rp}\frac{(rp)!}{(r!)^{p}}. \tag{12}$$

- Alternating row sums:

(a) The alternating sum of GEN $A_{a,b,r}(p,j)$, $0\le j\le rp$, is given by

$$\begin{aligned}&\sum_{j=0}^{rp}(-1)^{j}A_{a,b,r}(p,j) \qquad (13)\\ &=\frac{2^{rp+1}}{(r!)^{p}}\sum_{i_r=0}^{p}\cdots\sum_{i_1=0}^{p}\left(\prod_{j=1}^{r}\binom{p}{i_j}(b-j+1)^{p-i_j}\right)\times\\ &\times\left(a^{\sum_{j=1}^{r}i_j}\frac{\left(1-2^{1+\sum_{j=1}^{r}i_j}\right)B_{1+\sum_{j=1}^{r}i_j}}{1+\sum_{j=1}^{r}i_j}\right).\end{aligned}$$

(b) If p is even and r is odd, we have

$$\sum_{i=r}^{rp}(-1)^{i}A_{1,0,r}(p,i)=0, \tag{14}$$

and

$$\sum_{i=0}^{r(p-1)}(-1)^{i}A_{1,r,r}(p,i)=0. \tag{15}$$

(c) If r is odd, we have

$$\sum_{i=0}^{rp-\frac{r-1}{2}}(-1)^{i}A_{2,1,r}\left(p,i+\frac{r-1}{2}\right)=0, \tag{16}$$

in each of the following cases: (a) p even and $r\equiv 3 \mod 4$, and (b) p odd and $r\equiv 1 \mod 4$. In fact, in the case $r=1$ we have

$$\sum_{j=0}^{p}(-1)^{j}A_{2,1,1}(p,j)=2^{p}E_{p}. \tag{17}$$

In this article we consider the corresponding generalized Eulerian polynomials (GEP, for short) in which the coefficients are the GEN $A_{a,b,r}(p,i)$, $i=0,1,\ldots,rp$, that is, polynomials $P_{a,b,r,p}(z)=\sum_{i=0}^{rp}A_{a,b,r}(p,i)\,z^{rp-i}$. The main tool we use to study these polynomials is the Z-Transform. In Section 2, we give the definitions and the properties of this transformation that we will use in the remaining sections. It turns out that when one considers the Z-transform of the sequence n^p (via the Worpitsky identity (1)), Eulerian polynomials appear in a natural way. We use this fact to obtain, also in Section 2, some known results for the standard Eulerian polynomials, including recurrences for these polynomials. To generalize these results is one of the main goals of this work. In Section 3, we consider a first step of the GEP $P_{a,b,r,p}(z)$, namely, the case $r=1$. That is, we consider the GEP $P_{a,b,1,p}(z)=\sum_{i=0}^{p}A_{a,b,1}(p,i)\,z^{p-i}$, which appear in the Z-Transform of the sequence $(an+b)^p$. We obtain several kinds of recurrences for these polynomials, including generalizations of the known recurrences for the standard Eulerian polynomials. In Section 4, we consider the more general case, corresponding to GEP $P_{a,b,r,p}(z)=\sum_{i=0}^{p}A_{a,b,r}(p,i)\,z^{p-i}$, which appear in the Z-Transform of the sequence $\binom{an+b}{r}^p$, $r\in\mathbb{N}$, $r\geq 2$. We obtain a recurrence for these polynomials in the case $a=1$: by using the recurrence (11) for the GEN $A_{a,b,r}(p,i)$, together with other combinatorial identities, we demonstrate that

$$P_{1,b,r,p}(z)=\sum_{l=0}^{r}\sum_{t=0}^{l}\binom{r-b}{t}\binom{rp-2r+l+b}{l-t}\frac{z^{r-t}(1-z)^{r-l}}{(r-l)!}\frac{d^{r-l}}{dz^{r-l}}P_{1,b,r,p-1}(z).$$

This is the most challenging result of this work: Theorem 2. Finally, in Section 5, we show some applications where the GEP studied in the previous sections are involved. We obtain an explicit formula for some generalized telescoping sums, and we obtain explicit formulas for some sums of powers of binomial coefficients $\sum_{j=0}^{m}\binom{aj+b}{r}^p$ in terms of a convolution, and also explicit formulas for the alternating sums $\sum_{j=0}^{m}(-1)^j\binom{aj+b}{r}^p$.

2. Preliminaries: Z-Transform and Standard Eulerian Polynomials

The *Z-transform* is a map $\mathcal{Z}$ that takes complex sequences $a_n=(a_0,a_1,\ldots,a_n,\ldots)$ into complex functions $\mathcal{Z}(a_n)(z)$ or simply $\mathcal{Z}(a_n)$, given by the Laurent series $\mathcal{Z}(a_n)=\sum_{n=0}^{\infty}\frac{a_n}{z^n}$, defined for $|z|>R$, where $R>0$ is the radius of convergence of the Taylor series $\sum_{n=0}^{\infty}a_nz^n$. If $\mathcal{Z}(a_n)=A(z)$, we also say that the sequence a_n is the *inverse Z-transform* of the complex function $A(z)$, and we write $a_n=\mathcal{Z}^{-1}(A(z))$.

Remark 1. The references [14, 21, 22] are good examples of the fact that the audiences for Z-transforms and for generating functions are different (though not disjoint), with engineers for the former and mathematicians for the latter. By taking a look at the mentioned references, one realizes that there is some different flavor in the language used in each theory. But plainly the mathematical information contained in both tools is the same. However, we believe that Z-transforms give a more gentle environment for algebraic manipulations as those we face in this work.

We will recall now some basic facts about the Z transform that we will use throughout the work.

The sequence λ^n, where λ is a given non-zero complex number, has Z-transform

$$\mathcal{Z}(\lambda^n) = \sum_{n=0}^{\infty} \frac{\lambda^n}{z^n} = \frac{1}{1-\frac{\lambda}{z}} = \frac{z}{z-\lambda}, \tag{18}$$

defined for $|z| > |\lambda|$. In particular, the Z-transform of the constant sequence 1 and the alternating sequence $(-1)^n$ are

$$\mathcal{Z}(1) = \frac{z}{z-1} \quad \text{and} \quad \mathcal{Z}((-1)^n) = \frac{z}{z+1}, \tag{19}$$

respectively.

Five important properties of the Z-transform:

1. $\mathcal{Z}$ is linear and injective.

2. Advance-shifting property. If $\mathcal{Z}(a_n) = \mathcal{A}(z)$, and $k \in \mathbb{N}$ is given, then

$$\mathcal{Z}(a_{n+k}) = z^k \left(\mathcal{A}(z) - \sum_{j=0}^{k-1} \frac{a_j}{z^j} \right). \tag{20}$$

3. Multiplication by the sequence λ^n. If $\mathcal{Z}(a_n) = \mathcal{A}(z)$, then for a given $\lambda \in \mathbb{C}$, $\lambda \neq 0$,

$$\mathcal{Z}(\lambda^n a_n) = \mathcal{A}\left(\frac{z}{\lambda}\right).$$

   In particular, we have

$$\mathcal{Z}((-1)^n a_n) = \mathcal{A}(-z). \tag{21}$$

4. Multiplication by the sequence n. If $\mathcal{Z}(a_n) = \mathcal{A}(z)$, then

$$\mathcal{Z}(na_n) = -z\frac{d}{dz}\mathcal{A}(z). \tag{22}$$

   Formula (22) implies

$$\begin{aligned} \mathcal{Z}\left(n^2 a_n\right) &= z^2 \frac{d^2}{dz^2}\mathcal{A}(z) + z\frac{d}{dz}\mathcal{A}(z), & (23) \\ \mathcal{Z}\left(n^3 a_n\right) &= -z^3 \frac{d^3}{dz^3}\mathcal{A}(z) - 3z^2\frac{d^2}{dz^2}\mathcal{A}(z) - z\frac{d}{dz}\mathcal{A}(z), & (24) \end{aligned}$$

   and so on.

5. Convolution theorem. If $\mathcal{Z}(a_n) = \mathcal{A}(z)$ and $\mathcal{Z}(b_n) = \mathcal{B}(z)$, then

$$\mathcal{Z}(a_n * b_n) = \mathcal{A}(z)\mathcal{B}(z), \tag{25}$$

where $a_n * b_n$ is the *convolution* of a_n with b_n, defined as the sequence $a_n * b_n = \sum_{t=0}^{n} a_t b_{n-t}$.

Observe that, for a given sequence a_n, the convolution

$$a_n * 1 = \left(\sum_{i=0}^{n} a_i\right) = (a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots), \tag{26}$$

is the sequence of partial sums of a_n. In general, the multiple convolution $a_n *^k 1$, which means $a_n * 1 * 1 * \cdots * 1$, where the sequence 1 appears k times, is the sequence of k-th partial sums of the sequence a_n. For example, for $k = 2$ we have

$$a_n *^2 1 = (a_0, 2a_0 + a_1, 3a_0 + 2a_1 + a_2, \ldots).$$

If $\mathcal{A}(z)$ is the Z-transform of the sequence a_n, then the Z-transform of the sequence $a_n *^r 1$ (of the r-th partial sums of a_n) is, according to (19) and (25),

$$\mathcal{Z}(a_n *^r 1) = \left(\frac{z}{z-1}\right)^r \mathcal{A}(z). \tag{27}$$

Let $\mathcal{A}(z)$ be the Z-transform of the sequence a_n. The sequence $b_n = (0, \ldots, 0, a_0, a_1, \ldots)$, with k zeros at the beginning, is such that $b_{n+k} = a_n$ and $b_0 = \cdots = b_{k-1} = 0$. According to (20), we have $\mathcal{Z}(b_{n+k}) = z^k \mathcal{Z}(b_n)$, or

$$\mathcal{Z}(b_n) = z^{-k}\mathcal{A}(z). \tag{28}$$

In particular, observe that according to (27) and (28), we have

$$\mathcal{Z}^{-1}\left(\frac{z^{-k+1}}{z-1}\mathcal{A}(z)\right) = \left(\sum_{i=0}^{n-k} a_i\right) = (0, \ldots, 0, a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots). \tag{29}$$

From (19) and (22), we see that the Z-transform of the sequence n is

$$\mathcal{Z}(n) = -z\frac{d}{dz}\frac{z}{z-1} = \frac{z}{(z-1)^2}. \tag{30}$$

The same argument gives us

$$\mathcal{Z}(n^2) = -z\frac{d}{dz}\frac{z}{(z-1)^2} = \frac{z(z+1)}{(z-1)^3}, \tag{31}$$

$$\mathcal{Z}\left(n^3\right) = -z\frac{d}{dz}\frac{z\left(z+1\right)}{\left(z-1\right)^3} = \frac{z\left(z^2+4z+1\right)}{\left(z-1\right)^4}, \tag{32}$$

and so on. Also, it is immediately verifiable that

$$\mathcal{Z}\left(2n+1\right) = \frac{z\left(z+1\right)}{\left(z-1\right)^2}. \tag{33}$$

The Z-transform of the sequence $\binom{n}{r}$, where $r \in \mathbb{N}$ is given, is

$$\mathcal{Z}\left(\binom{n}{r}\right) = \frac{z}{\left(z-1\right)^{r+1}}. \tag{34}$$

The proof is an easy induction on r left to the reader.

According to the advance-shifting property (20), together with (34), we see that for $0 \le k \le r$, the Z-transform of the sequence $\binom{n+k}{r}$ is

$$\mathcal{Z}\left(\binom{n+k}{r}\right) = \frac{z^{k+1}}{\left(z-1\right)^{r+1}}. \tag{35}$$

Observe that, according to (26) and (34), we have

$$\mathcal{Z}\left(\sum_{j=0}^{n}\binom{j}{r}\right) = \frac{z}{z-1}\frac{z}{\left(z-1\right)^{r+1}} = \frac{z^2}{\left(z-1\right)^{r+2}}, \tag{36}$$

which gives us, by using (35), the well-known property of binomial coefficients

$$\sum_{j=0}^{n}\binom{j}{r} = \binom{n+1}{r+1}. \tag{37}$$

Observe also that, according to the Worpitsky identity (1) and formula (35), the Z-transform of the sequence n^p, where p is given, is

$$\mathcal{Z}\left(n^p\right) = \sum_{i=0}^{p} A\left(p,i\right)\frac{z^{p-i+1}}{\left(z-1\right)^{p+1}}, \tag{38}$$

which means that

$$\mathcal{Z}\left(n^p\right) = \frac{z\sum_{i=0}^{p} A\left(p,i\right)z^{p-i}}{\left(z-1\right)^{p+1}} = \frac{zP_p\left(z\right)}{\left(z-1\right)^{p+1}}. \tag{39}$$

That is, the numerator of $\mathcal{Z}\left(n^p\right)$ is z times the Eulerian polynomial $P_p\left(z\right)$. By setting $P_0\left(z\right) = 1$, formula (39) makes sense for all non-negative values of p, so we have $P_0\left(z\right) = P_1\left(z\right) = 1$, $P_2\left(z\right) = z+1$, and so on.

According to (22) and (39), we can write

$$\begin{aligned}\mathcal{Z}\left(n^{p}\right) &= \mathcal{Z}\left(nn^{p-1}\right)=-z\frac{d}{dz}\frac{zP_{p-1}\left(z\right)}{\left(z-1\right)^{p}}\\ &= -z\frac{\left(z-1\right)^{p}\left(zP_{p-1}'\left(z\right)+P_{p-1}\left(z\right)\right)-p\left(z-1\right)^{p-1}zP_{p-1}\left(z\right)}{\left(z-1\right)^{2p}}\\ &= z\frac{z\left(1-z\right)P_{p-1}'\left(z\right)+\left(1+\left(p-1\right)z\right)P_{p-1}\left(z\right)}{\left(z-1\right)^{p+1}}. \qquad (40)\end{aligned}$$

Comparing (40) with (39), we obtain that

$$P_{p}\left(z\right)=z\left(1-z\right)P_{p-1}'\left(z\right)+\left(1+\left(p-1\right)z\right)P_{p-1}\left(z\right), \qquad (41)$$

which is the well-known recurrence for Eulerian polynomials.

On the other hand, expression (39) together with (20) allow us to write, for $m\in\mathbb{N}$, that

$$\mathcal{Z}\left(\left(n+m\right)^{p}\right)=z^{m}\left(\frac{zP_{p}\left(z\right)}{\left(z-1\right)^{p+1}}-\sum_{j=0}^{m-1}\frac{j^{p}}{z^{j}}\right). \qquad (42)$$

The linearity of the Z-transform, and (39) again, give us that

$$\mathcal{Z}\left(\left(n+m\right)^{p}\right)=\mathcal{Z}\left(\sum_{k=0}^{p}\binom{p}{k}m^{p-k}n^{k}\right)=\sum_{k=0}^{p}\binom{p}{k}m^{p-k}\frac{zP_{k}\left(z\right)}{\left(z-1\right)^{k+1}}. \qquad (43)$$

Thus, from (42) and (43) we have that

$$\sum_{k=0}^{p}\binom{p}{k}m^{p-k}\frac{zP_{k}\left(z\right)}{\left(z-1\right)^{k+1}}=z^{m}\left(\frac{zP_{p}\left(z\right)}{\left(z-1\right)^{p+1}}-\sum_{j=0}^{m-1}\frac{j^{p}}{z^{j}}\right). \qquad (44)$$

Formula (44) can be written as follows

$$\sum_{k=0}^{p}\binom{p}{k}m^{p-k}\frac{P_{k}\left(z\right)}{\left(z-1\right)^{k+1}}=z^{m}\frac{P_{p}\left(z\right)}{\left(z-1\right)^{p+1}}-\sum_{j=0}^{m-1}j^{p}z^{m-1-j}. \qquad (45)$$

Observe that if we set $H_{j}=H_{j}\left(z\right)=\frac{P_{j}\left(z\right)}{\left(z-1\right)^{j+1}}$, then we can write (45) as follows

$$\left(H+m\right)^{p}=z^{m}H_{p}-\sum_{j=0}^{m-1}j^{p}z^{m-1-j}, \qquad (46)$$

where the left-hand side is understood as in Umbral Calculus: it is expanded by the binomial theorem, and the superscripts (exponents) of H are converted to subscripts, that is, H^{k} is $H_{k}=\frac{P_{k}\left(z\right)}{\left(z-1\right)^{k+1}}$, $k=0,1,\ldots,p$.

In the simplest case of $m = 1$, formula (45) looks as follows

$$\sum_{k=0}^{p} \binom{p}{k} \frac{P_k(z)}{(z-1)^{k+1}} = \frac{zP_p(z)}{(z-1)^{p+1}}, \tag{47}$$

and its version of (46) is

$$(H+1)^p = zH_p. \tag{48}$$

Observe that we can write (47) as $\sum_{k=0}^{p} \binom{p}{k} \frac{P_k(z)}{(z-1)^k} = \frac{zP_p(z)}{(z-1)^p}$, so we can define H_k by $H_k = \frac{P_k(z)}{(z-1)^k}$, and formula (48) remains valid. This version of (48) appears originally in Vandiver's paper on Bernoulli numbers ([20], formula 12, p. 506). (In passing, Carlitz [2] refers to (48) as "an interesting formula".)

From (47), we immediately obtain the known recurrence

$$P_p(z) = \sum_{k=0}^{p-1} \binom{p}{k} (z-1)^{p-1-k} P_k(z) \tag{49}$$

for Eulerian polynomials, which gives the polynomial $P_p(z)$ in terms of the previous p polynomials $P_k(z)$, $k = 0, 1, \ldots, p-1$, (see for example [11], formula (2.7), p. 12). For instance, we have $P_2(z) = (z-1) + 2 = z + 1$, $P_3(z) = (z-1)^2 + 3(z-1) + 3P_1(z) = z^2 + 4z + 1$, and so on.

We can write (44) as follows:

$$P_p(z) = \frac{1}{\sum_{j=0}^{m-1} z^j} \left(\sum_{k=0}^{p-1} \binom{p}{k} m^{p-k} (z-1)^{p-1-k} P_k(z) + (z-1)^p \sum_{j=0}^{m-1} j^p z^{m-j-1} \right), \tag{50}$$

where $p \geq 1$. That is, expression (50) is in fact *a family of recurrences for Eulerian polynomials*; the case $m = 1$ is (49). For example, for $m = 2, 3$ we have the recurrences

$$P_p(z) = \frac{1}{z+1} \left(\sum_{k=0}^{p-1} \binom{p}{k} 2^{p-k} (z-1)^{p-1-k} P_k(z) + (z-1)^p \right), \tag{51}$$

and

$$P_p(z) = \frac{1}{z^2+z+1} \left(\sum_{k=0}^{p-1} \binom{p}{k} 3^{p-k} (z-1)^{p-1-k} P_k(z) + (z+2^p)(z-1)^p \right), \tag{52}$$

respectively.

Expression (44) has more to offer: substitute z by $\frac{1}{z}$ to obtain

$$\sum_{k=0}^{p}\binom{p}{k}m^{p-k}\frac{z^k P_k\left(\frac{1}{z}\right)}{(1-z)^{k+1}}=\frac{1}{z^m}\left(\frac{z^p P_p\left(\frac{1}{z}\right)}{(1-z)^{p+1}}-\sum_{j=1}^{m-1}j^p z^j\right). \tag{53}$$

Now we use the fact that $z^k P_k\left(\frac{1}{z}\right)=zP_k(z)$ for any $k\in\mathbb{N}$, and write expression (53) modifying its left-hand side (avoiding the term $k=0$ of the sum, for which the relation $z^k P_k\left(\frac{1}{z}\right)=zP_k(z)$ is false), as follows

$$m^p z^m+z^m\frac{zP_p(z)}{(1-z)^{p+1}}+z^m\sum_{k=0}^{p-1}\binom{p}{k}m^{p-k}\frac{zP_k(z)}{(1-z)^{k+1}}=\frac{zP_p(z)}{(1-z)^{p+1}}-\sum_{j=1}^{m-1}j^p z^j, \tag{54}$$

from which we finally obtain the following closed formula for the weighted sum of the p-th powers of the first m positive integers, $\sum_{j=1}^{m}j^p z^j$, where $z\neq 0,1$ is the weight (see [11], formula (2.8), p. 12):

$$\sum_{j=1}^{m}z^j j^p=\frac{(1-z^m)\,zP_p(z)}{(1-z)^{p+1}}-\sum_{k=0}^{p-1}\binom{p}{k}m^{p-k}\frac{z^{m+1}P_k(z)}{(1-z)^{k+1}}. \tag{55}$$

In the original work of Euler [10] we find results involving alternating sums of powers $\sum_{j=1}^{m}(-1)^j j^p$; see also [1] for a different approach.

By using (49), we can write (55) as follows

$$\sum_{j=1}^{m}z^{j-1}j^p=\sum_{k=0}^{p-1}\binom{p}{k}\frac{(-1)^{p+k+1}(1-z^m)-m^{p-k}z^m(1-z)}{(1-z)^{k+2}}P_k(z). \tag{56}$$

For example, we have

$$\sum_{j=1}^{m}(-1)^{j-1}j^p=\sum_{k=0}^{p-1}\binom{p}{k}\frac{(-1)^{p+k+1}(1-(-1)^m)-2m^{p-k}(-1)^m}{2^{k+2}}P_k(-1),$$

or

$$\sum_{j=1}^{2m}(-1)^j j^p=\sum_{k=0}^{p-1}\binom{p}{k}\frac{(2m)^{p-k}}{2^{k+1}}P_k(-1).$$

By using the alternating row sums formula for Eulerian numbers (see Table 1) we can write

$$\sum_{j=1}^{2m}(-1)^j j^p=\sum_{k=0}^{p-1}\binom{p}{k}(-1)^k(2m)^{p-k}\frac{\left(1-2^{k+1}\right)B_{k+1}}{k+1}.$$

More generally, for $\rho\in\mathbb{N}$ given, let $\omega\in\mathbb{C}$ be a ρ-th root of 1, $\omega\neq 1$. Then, according to (56), we have

$$\sum_{j=1}^{\rho m}\omega^{j-1}j^{p}=-\sum_{k=0}^{p-1}\binom{p}{k}\frac{(\rho m)^{p-k}}{(1-\omega)^{k+1}}P_{k}(\omega).$$

For example, we have

$$\sum_{j=1}^{4m}i^{j-1}j^{p}=-\sum_{k=0}^{p-1}\binom{p}{k}\left(\frac{1}{2}+\frac{1}{2}i\right)^{k+1}(4m)^{p-k}P_{k}(i),$$

where $i=\sqrt{-1}$.

3. Generalized Eulerian Polynomials I: $r = 1$

In this section we consider GEP of the form

$$P_{a,b,1,p}(z)=\sum_{i=0}^{p}A_{a,b,1}(p,i)\,z^{p-i}, \tag{57}$$

where the coefficients are the GEN $A_{a,b,1}(p,i)=\sum_{j=0}^{i}(-1)^{j}\binom{p+1}{j}(a(i-j)+b)^{p}$. Some examples are

$$\begin{aligned} P_{a,b,1,1}(z) &= bz+a-b, \\ P_{a,b,1,2}(z) &= b^{2}z^{2}+\left(a^{2}+2ab-2b^{2}\right)z+(a-b)^{2}, \\ P_{a,b,1,3}(z) &= b^{3}z^{3}+\left(a^{3}+3a^{2}b+3ab^{2}-3b^{3}\right)z^{2} \\ &\quad +\left(4a^{3}-6ab^{2}+3b^{3}\right)z+(a-b)^{3}. \end{aligned} \tag{58}$$

The case $a=1$, $b=0$ corresponds to the standard Eulerian polynomials $P_{1,0,1,1}(z)=1$, $P_{1,0,1,2}(z)=z+1$, $P_{1,0,1,3}(z)=z^{2}+4z+1$, and so on. Observe that $P_{a,b,1,p}(z)$ is a p-th degree polynomial if and only if $b\neq 0$, otherwise it is a $(p-1)$-th degree polynomial, as in the case of the standard Eulerian polynomials $P_{1,0,1,p}(z)$. We set $P_{a,b,1,0}(z)=1$, for any $b\in\mathbb{R}$. Observe also that $P_{a,a,1,p}(z)=a^{p}zP_{1,0,1,p}(z)$.

From (4), we see that the leading coefficient of $P_{a,b,1,p}(z)$ is b^{p}. From (6), we see that the independent term of $P_{a,b,1,p}(z)$ is

$$P_{a,b,1,p}(0)=A_{a,b,1}(p,p)=(a-b)^{p}, \tag{59}$$

and then $z=0$ is a zero of $P_{a,b,1,p}(z)$ if and only if $a=b$ (which is the case when the polynomial $P_{a,b,1,p}(z)$ is $a^{p}z$ times the standard Eulerian polynomial $P_{1,0,1,p}(z)$, as we noticed before).

From (12), with $r = 1$, we see that the value of $P_{a,b,1,p}(z)$ at $z = 1$ is

$$P_{a,b,1,p}(1) = \sum_{i=0}^{p} A_{a,b,1}(p,i) = a^p p!, \tag{60}$$

and from (13), with $r = 1$, we see that the value of $P_{a,b,1,p}(z)$ at $z = -1$ is

$$P_{a,b,1,p}(-1) = (-2)^{p+1} \sum_{i=0}^{p} \binom{p}{i} b^{p-i} a^i \frac{\left(2^{i+1}-1\right) B_{i+1}}{i+1}. \tag{61}$$

According to the expansion

$$(an+b)^p = \sum_{i=0}^{p} A_{a,b,1}(p,i) \binom{n+p-i}{p}, \tag{62}$$

together with (35) and (57), it is clear that the generalized Eulerian polynomial $P_{a,b,1,p}(z)$ appears in the numerator of the Z-transform of the sequence $(an+b)^p$, namely

$$\mathcal{Z}\left((an+b)^p\right) = \frac{zP_{a,b,1,p}(z)}{(z-1)^{p+1}}. \tag{63}$$

We can mimic what we did to obtain the recurrence (40) for standard Eulerian polynomials, in order to obtain a similar formula for GEP $P_{a,b,1,p}(z)$ (see also [24]).

Proposition 1. *We have the following recurrence for generalized Eulerian Polynomials $P_{a,b,1,p}(z)$*

$$\begin{aligned} P_{a,b,1,p}(z) &= \\ & az(1-z)P'_{a,b,1,p-1}(z) + (a-b+(a(p-1)+b)z)P_{a,b,1,p-1}(z). \end{aligned} \tag{64}$$

Proof. We use (22) and (63) to write

$$\begin{aligned} & \mathcal{Z}\left((an+b)^p\right) \\ = \; & \mathcal{Z}\left((an+b)(an+b)^{p-1}\right) \\ = \; & -az\frac{d}{dz}\frac{zP_{a,b,1,p-1}(z)}{(z-1)^p} + b\frac{zP_{a,b,1,p-1}(z)}{(z-1)^p} \\ = \; & -az\frac{(z-1)^p\left(zP'_{a,b,1,p-1}(z)+P_{a,b,1,p-1}(z)\right) - p(z-1)^{p-1}zP_{a,b,1,p-1}(z)}{(z-1)^{2p}} \\ & +b\frac{zP_{a,b,1,p-1}(z)}{(z-1)^p} \\ = \; & z\frac{az(1-z)P'_{a,b,1,p-1}(z)+(a-b+(a(p-1)+b)z)P_{a,b,1,p-1}(z)}{(z-1)^{p+1}}. \end{aligned} \tag{65}$$

Comparing (63) and (65) we obtain the desired conclusion (64). $\square$

On the other hand, observe that the advance-shifting property (20) tells us that, for any non-negative integer m,

$$\mathcal{Z}\left(\left(a\left(n+m\right)+b\right)^{p}\right)=z^{m}\left(\mathcal{Z}\left(\left(an+b\right)^{p}\right)-\sum_{j=0}^{m-1}\left(aj+b\right)^{p}z^{-j}\right). \tag{66}$$

That is, we have

$$\begin{aligned}z^{m}\left(\frac{zP_{a,b,1,p}\left(z\right)}{\left(z-1\right)^{p+1}}-\sum_{j=0}^{m-1}\left(aj+b\right)^{p}z^{-j}\right)&\\ &=\mathcal{Z}\left(\left(an+b+ma\right)^{p}\right)\\ &=\mathcal{Z}\left(\sum_{k=0}^{p}\binom{p}{k}\left(ma\right)^{p-k}\left(an+b\right)^{k}\right)\\ &=\sum_{k=0}^{p}\binom{p}{k}\left(ma\right)^{p-k}\frac{zP_{a,b,1,k}\left(z\right)}{\left(z-1\right)^{k+1}}.\end{aligned} \tag{67}$$

We can write (67) as follows

$$z^{m}\frac{P_{a,b,1,p}\left(z\right)}{\left(z-1\right)^{p+1}}=\sum_{k=0}^{p}\binom{p}{k}\left(ma\right)^{p-k}\frac{P_{a,b,1,k}\left(z\right)}{\left(z-1\right)^{k+1}}+\sum_{j=0}^{m-1}\left(aj+b\right)^{p}z^{m-1-j}, \tag{68}$$

and again, as in (45), we have the following version: set $H_j = H_j(z) = P_{a,b,1,j}(z)/(z-1)^{j+1}$. Then we can write (68) as follows

$$z^{m}H_{p}=\left(H+ma\right)^{p}+\sum_{j=0}^{m-1}\left(aj+b\right)^{p}z^{m-1-j}, \tag{69}$$

where $(H+ma)^p$ is expanded by the binomial theorem, and the exponents of H are converted to subscripts. In particular, when $m=1$, formula (68) is the following

$$\frac{zP_{a,b,1,p}\left(z\right)}{\left(z-1\right)^{p+1}}=\sum_{k=0}^{p}\binom{p}{k}\frac{P_{a,b,1,k}\left(z\right)}{\left(z-1\right)^{k+1}}a^{p-k}+b^{p}, \tag{70}$$

and its version (69) is

$$zH_{p}=\left(H+a\right)^{p}+b^{p}. \tag{71}$$

Thus, (71) is a generalization of Vandiver's formula (47), which is the case $a=1$, $b=0$ of (69), with $H_j = P_{1,0,1,j}\left(z\right)/\left(z-1\right)^{j}$.

By writing (70) as follows

$$P_{a,b,1,p}(z) = \tag{72}$$
$$\frac{1}{\sum_{j=0}^{m-1} z^j}\left(\sum_{k=0}^{p-1}\binom{p}{k}(ma)^{p-k}(z-1)^{p-1-k}P_{a,b,1,k}(z)+(z-1)^p\sum_{j=0}^{m-1}(aj+b)^p z^{m-1-j}\right)$$

we see that this is an infinite m-family of recurrences for the GEP $P_{a,b,1,p}(z)$. The cases $m=1$ and $m=2$ of (72) are as follows:

$$P_{a,b,1,p}(z)=\sum_{k=0}^{p-1}\binom{p}{k}a^{p-k}(z-1)^{p-k-1}P_{a,b,1,k}(z)+b^p(z-1)^p, \tag{73}$$

$$P_{a,b,1,p}(z) = \tag{74}$$
$$\frac{1}{z+1}\left(\sum_{k=0}^{p-1}\binom{p}{k}(2a)^{p-k}(z-1)^{p-1-k}P_{a,b,1,k}(z)+(z-1)^p\left(b^p z+(a+b)^p\right)\right).$$

Observe that the symmetry relation $A_{a,b,1}(p,i) = A_{a,a-b,1}(p,p-i)$ (see (9)), allows us to write

$$\begin{aligned} P_{a,a-b,1,p}(z) &= \sum_{i=0}^{p} A_{a,a-b,1}(p,i)\, z^{p-i} \\ &= \sum_{i=0}^{p} A_{a,a-b,1}(p,p-i)\, z^{i} \\ &= \sum_{i=0}^{p} A_{a,b,1}(p,i)\, z^{i}. \end{aligned}$$

Thus, $P_{a,a-b,1,p}(z)$ is the reciprocal polynomial of $P_{a,b,1,p}(z)=\sum_{i=0}^{p}A_{a,b,1}(p,i)z^{p-i}$. In other words, we have

$$P_{a,a-b,1,p}(z)=z^p P_{a,b,1,p}\left(z^{-1}\right). \tag{75}$$

From

$$\frac{zP_{a,a-b,1,p}(z)}{(z-1)^{p+1}}=\mathcal{Z}\left((a(n+1)-b)^p\right)=z\left(\frac{zP_{a,-b,1,p}(z)}{(z-1)^{p+1}}-(-b)^p\right),$$

we see that

$$P_{a,a-b,1,p}(z) = zP_{a,-b,1,p}(z) - (-b)^p (z-1)^{p+1}.$$

In particular, we have that $P_{a,a,1,p}(z) = zP_{a,0,1,p}(z) = a^p zP_{1,0,1,p}(z)$ (as we noticed before), and that $P_{1,1,1,p}(z) = zP_{1,0,1,p}(z) = z^p P_{1,0,1,p}(z^{-1})$ is the reciprocal polynomial of $P_{1,0,1,p}(z)$. Observe that, by using (75), we can write (63) as follows

$$\mathcal{Z}((an+b)^p) = \frac{z^{p+1} P_{a,a-b,1,p}(z^{-1})}{(z-1)^{p+1}}. \tag{76}$$

Thus, proceeding as in (65) and using again (75), we can obtain "mixed" recurrences such as

$$\begin{aligned} P_{a,b,1,p}(z) = \\ a(z-1)z^{p-2}P'_{a,a-b,1,p-1}(z^{-1}) + (ap + bz(z-1))P_{a,b,1,p-1}(z), \end{aligned} \tag{77}$$

which gives us, in particular, the following recurrences involving the standard Eulerian polynomials $P_{1,0,1,p}(z)$ and their reciprocal polynomials $P_{1,1,1,p}(z)$:

$$P_{1,0,1,p}(z) = (z-1)z^{p-2}P'_{1,1,1,p-1}(z^{-1}) + pP_{1,0,1,p-1}(z),$$

$$P_{1,1,1,p}(z) = (z-1)z^{p-2}P'_{1,0,1,p-1}(z^{-1}) + (p + z(z-1))P_{1,1,1,p-1}(z).$$

Next, we explore the relation among GEP $P_{a,b,1,p}(z)$ for different values of the parameters a, b. The main result is given in the following proposition.

Proposition 2. *Let a, b, c, d be given complex numbers, $a, c \neq 0$. The generalized Eulerian polynomial $P_{a,b,1,p}(z)$ can be written in terms of the generalized Eulerian polynomials $P_{c,d,1,k}(z)$, $k = 0, 1, \ldots, p$, according to*

$$\begin{aligned} P_{a,b,1,p}(z) &= z^m \sum_{k=0}^{p} \binom{p}{k} \left(\frac{a}{c}\right)^k \left(\left(b - am - \frac{ad}{c}\right)(z-1)\right)^{p-k} P_{c,d,1,k}(z) \\ &\quad - (z-1)^{p+1} \sum_{j=0}^{m-1} (b - a(m-j))^p z^{m-1-j}, \end{aligned} \tag{78}$$

where m is a given non-negative integer.

Proof. We have

$$\begin{aligned}
&\frac{zP_{a,b,1,p}(z)}{(z-1)^{p+1}} = \mathcal{Z}\left((an+b)^p\right)\\
&=\left(\frac{a}{c}\right)^p \mathcal{Z}\left(\left(c(n+m)+d+\frac{bc-ad}{a}-cm\right)^p\right)\\
&=\left(\frac{a}{c}\right)^p \mathcal{Z}\left(\sum_{k=0}^{p}\binom{p}{k}\left(\frac{bc-ad}{a}-cm\right)^{p-k}(c(n+m)+d)^k\right)\\
&=\left(\frac{a}{c}\right)^p \sum_{k=0}^{p}\binom{p}{k}\left(\frac{bc-ad}{a}-cm\right)^{p-k} z^m\left(\frac{zP_{c,d,1,k}(z)}{(z-1)^{k+1}}-\sum_{j=0}^{m-1}(cj+d)^k z^{-j}\right)\\
&=\sum_{k=0}^{p}\binom{p}{k}\left(\frac{a}{c}\right)^k\left(b-am-\frac{ad}{c}\right)^{p-k} z^m\frac{zP_{c,d,1,k}(z)}{(z-1)^{k+1}}\\
&\quad-\sum_{k=0}^{p}\binom{p}{k}\left(\frac{a}{c}\right)^k\left(b-am-\frac{ad}{c}\right)^{p-k}\sum_{j=0}^{m-1}(cj+d)^k z^{m-j}\\
&=\sum_{k=0}^{p}\binom{p}{k}\left(\frac{a}{c}\right)^k\left(b-am-\frac{ad}{c}\right)^{p-k} z^m\frac{zP_{c,d,1,k}(z)}{(z-1)^{k+1}}\\
&\quad-\sum_{j=0}^{m-1}(b-a(m-j))^p z^{m-j},
\end{aligned}$$

from which the conclusion (78) follows. ☐

Note that (78) implies $P_{a,b,1,p}(1)=\left(\frac{a}{c}\right)^p P_{c,d,1,p}(1)$, that is, $c^p\sum\limits_{i=0}^{p}A_{a,b,1}(p,i)=a^p\sum\limits_{i=0}^{p}A_{c,d,1}(p,i)$. We already knew this: the complete story is $c^p\sum_{i=0}^{p}A_{a,b,1}(p,i)=a^p\sum_{i=0}^{p}A_{c,d,1}(p,i)=a^pc^pp!$, and is a consequence of (12).

The simplest case $m=0$, from (78), looks as follows

$$P_{a,b,1,p}(z)=\sum_{k=0}^{p}\binom{p}{k}\left(\frac{a}{c}\right)^k\left(\left(b-\frac{ad}{c}\right)(z-1)\right)^{p-k}P_{c,d,1,k}(z). \tag{79}$$

If we set $c=2, d=1$ and $z=-1$, we can use (17) to obtain from (79) that

$$P_{a,b,1,p}(-1)=(-1)^p\sum_{k=0}^{p}\binom{p}{k}a^k(2b-a)^{p-k}E_k. \tag{80}$$

Compare (80) with (61).

Two particular cases of (78) are the following:

1. Set $c=1, d=0$ in (78) to obtain

$$
\begin{aligned}
&P_{a,b,1,p}\left(z\right) \qquad (81)\\
&=z^{m}\sum_{k=0}^{p}\binom{p}{k}a^{k}\left(\left(b-am\right)\left(z-1\right)\right)^{p-k}P_{1,0,1,k}\left(z\right)\\
&-\left(z-1\right)^{p+1}\sum_{j=0}^{m-1}\left(b-a\left(m-j\right)\right)^{p}z^{m-1-j},
\end{aligned}
$$

which shows that the GEP $P_{a,b,1,p}\left(z\right)$ can be written in terms of the standard Eulerian polynomials $P_{1,0,1,k}\left(z\right)$, $k=0,1,\ldots,p$. For example, with $m=0,1$, we have

$$
P_{a,b,1,p}\left(z\right)=\sum_{k=0}^{p}\binom{p}{k}a^{k}\left(b\left(z-1\right)\right)^{p-k}P_{1,0,1,k}\left(z\right),
$$

$$
P_{a,b,1,p}\left(z\right)=z\sum_{k=0}^{p}\binom{p}{k}a^{k}\left(\left(b-a\right)\left(z-1\right)\right)^{p-k}P_{1,0,1,k}\left(z\right)-\left(z-1\right)^{p+1}\left(b-a\right)^{p}.
$$

(See formula (36) in [24].)

2. Set $a=1, b=0$ in (78), and rename c, d as a, b to obtain

$$
\begin{aligned}
&P_{1,0,1,p}\left(z\right) \qquad (82)\\
&=a^{-p}z^{m}\sum_{k=0}^{p}\binom{p}{k}\left(-\left(b+am\right)\left(z-1\right)\right)^{p-k}P_{a,b,1,k}\left(z\right)\\
&-\left(z-1\right)^{p+1}\sum_{j=0}^{m-1}\left(j-m\right)^{p}z^{m-1-j},
\end{aligned}
$$

which shows that the standard Eulerian polynomial $P_{1,0,1,p}\left(z\right)$ can be written in terms of the GEP $P_{a,b,1,k}\left(z\right)$, $k=0,1,\ldots,p$. For example, with $m=0,1$, we have

$$
\begin{aligned}
P_{1,0,1,p}\left(z\right) &= a^{-p}\sum_{k=0}^{p}\binom{p}{k}\left(-b\left(z-1\right)\right)^{p-k}P_{a,b,1,k}\left(z\right),\\
&= a^{-p}z\sum_{k=0}^{p}\binom{p}{k}\left(-\left(b+a\right)\left(z-1\right)\right)^{p-k}P_{a,b,1,k}\left(z\right)+\left(1-z\right)^{p+1}.
\end{aligned}
$$

From formula (78) we can obtain two new families of recurrences for the GEP (besides the known family of recurrences (72)). We show them together in the following corollary.

Corollary 1. *For any given $m \in \mathbb{N}$, we have the following recurrences for the generalized Eulerian polynomial $P_{a,b,1,p}(z)$:*

$$P_{a,b,1,p}(z) = \frac{1}{\sum_{j=0}^{m-1} z^j}\left((z-1)^p \sum_{j=0}^{m-1}(aj+b)^p z^{m-1-j} + \sum_{k=0}^{p-1}\binom{p}{k}(am)^{p-k}(z-1)^{p-1-k} P_{a,b,1,k}(z)\right), \tag{83}$$

$$P_{a,b,1,p}(z) = \frac{1}{\sum_{j=0}^{m-1} z^j}\left((z-1)^p \sum_{j=0}^{m-1}(b-a(m-j))^p z^{m-1-j} - z^m \sum_{k=0}^{p-1}\binom{p}{k}(-am)^{p-k}(z-1)^{p-1-k} P_{a,b,1,k}(z)\right), \tag{84}$$

$$P_{a,b,1,p}(z) = a^p P_{1,0,1,p}(z) + \sum_{k=0}^{p-1}\binom{p}{k}(z-1)^{p-k}\left(a^p(-m)^{p-k} P_{1,0,1,k}(z) - (-am-b)^{p-k} P_{a,b,1,k}(z)\right). \tag{85}$$

Proof. Expression (83) is the already known recurrence (72). Let us prove (84) and (85). Observe that the polynomial

$$\sum_{k=0}^{p}\binom{p}{k}\left(\frac{a}{c}\right)^k\left(\left(b-am-\frac{ad}{c}\right)(z-1)\right)^{p-k} P_{c,d,1,k}(z), \tag{86}$$

of the right-hand side of (78), does not depend on c or d. That is, we have

$$\begin{aligned}
&\sum_{k=0}^{p}\binom{p}{k}\left(\frac{a}{c}\right)^k\left(\left(b-am-\frac{ad}{c}\right)(z-1)\right)^{p-k} P_{c,d,1,k}(z)\\
&= \sum_{k=0}^{p}\binom{p}{k}a^k\left((b-am)(z-1)\right)^{p-k} P_{1,0,1,k}(z) &(87)\\
&= \sum_{k=0}^{p}\binom{p}{k}\left(-am(z-1)\right)^{p-k} P_{a,b,1,k}(z) &(88)
\end{aligned}$$

corresponding to (86) with $c = 1$ and $d = 0$, and (86) with $c = a$ and $d = b$, respectively.

By using (88) we can write (78) as

$$\begin{aligned} P_{a,b,1,p}(z) =& z^m \sum_{k=0}^{p} \binom{p}{k} (-am(z-1))^{p-k} P_{a,b,1,k}(z) \\ & - (z-1)^{p+1} \sum_{j=0}^{m-1} (b - a(m-j))^p z^{m-1-j}, \end{aligned} \tag{89}$$

from which (84) follows. On the other hand, expression (87) can be written as

$$\begin{aligned} P_{c,d,1,p}(z) =& c^p P_{1,0,1,p}(z) \\ & + \sum_{k=0}^{p-1} \binom{p}{k} \left(\frac{z-1}{a}\right)^{p-k} \times \\ & \times \left(c^p (b-am)^{p-k} P_{1,0,1,k}(z) - (bc - acm - ad)^{p-k} P_{c,d,1,k}(z)\right), \end{aligned} \tag{90}$$

and then the polynomial

$$\sum_{k=0}^{p-1} \binom{p}{k} \left(\frac{z-1}{a}\right)^{p-k} \left(c^p (b-am)^{p-k} P_{1,0,1,k}(z) - (bc - acm - ad)^{p-k} P_{c,d,1,k}(z)\right), \tag{91}$$

from the right-hand side of (90), does not depend on a or b. Then we have, from (91) with $a = 1$ and $b = 0$, that

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{p}{k} \left(\frac{z-1}{a}\right)^{p-k} \times \\ & \qquad \times \left(c^p (b-am)^{p-k} P_{1,0,1,k}(z) - (bc - acm - ad)^{p-k} P_{c,d,1,k}(z)\right) \\ & = \sum_{k=0}^{p-1} \binom{p}{k} (z-1)^{p-k} \left(c^p (-m)^{p-k} P_{1,0,1,k}(z) - (-cm - d)^{p-k} P_{c,d,1,k}(z)\right). \end{aligned} \tag{92}$$

Thus, expression (92) allows us to write (90) as follows:

$$\begin{aligned} P_{c,d,1,p}(z) = & c^p P_{1,0,1,p}(z) \\ & + \sum_{k=0}^{p-1} \binom{p}{k} (z-1)^{p-k} \left(c^p (-m)^{p-k} P_{1,0,1,k}(z) - (-cm - d)^{p-k} P_{c,d,1,k}(z)\right). \end{aligned} \tag{93}$$

Rename c, d as a, b to obtain (85). $\square$

Observe that (85) makes sense for $m = 0$, giving the recurrence

$$P_{a,b,1,p}(z) = a^p P_{1,0,1,p}(z) - \sum_{k=0}^{p-1} \binom{p}{k} (b(1-z))^{p-k} P_{a,b,1,k}(z).$$

When $a = 1, b = 0$, formula (83) becomes the known family of recurrences (50) for the standard Eulerian polynomials, and (85) becomes a trivial formula. However, formula (84) gives a new family of recurrences for the standard Eulerian polynomials $P_{1,0,1,p}(z)$, namely

$$P_{1,0,1,p}(z) = \frac{1}{\sum_{j=0}^{m-1} z^j} \left((z-1)^p \sum_{j=0}^{m-1} (-(m-j))^p z^{m-1-j} - z^m \sum_{k=0}^{p-1} \binom{p}{k} (-m)^{p-k} (z-1)^{p-1-k} P_{1,0,1,k}(z) \right). \tag{94}$$

For example, the case $m = 1$ of (94) is

$$P_{1,0,1,p}(z) = (-1)^p (z-1)^p - z \sum_{k=0}^{p-1} \binom{p}{k} (-1)^{p-k} (z-1)^{p-1-k} P_{1,0,1,k}(z). \tag{95}$$

Compare with (49).

4. Generalized Eulerian Polynomials II: $r > 1$

In this section we consider the GEP

$$P_{a,b,r,p}(z) = \sum_{i=0}^{rp} A_{a,b,r}(p,i) z^{rp-i}, \tag{96}$$

in which the coefficients are the GEN

$$A_{a,b,r}(p,i) = \sum_{j=0}^{i} (-1)^j \binom{rp+1}{j} \binom{a(i-j)+b}{r}^p, \tag{97}$$

involved in the expansion (3), where $r > 1$.

The leading coefficient of the polynomial $P_{a,b,r,p}(z)$ is $\binom{b}{r}^p$ (see (4)), and then, if $b \geq r$, the polynomial $P_{a,b,r,p}(z)$ has degree rp. If $b = 0$, the degree of $P_{a,0,r,p}(z)$ is $r(p-1)$ (this is the case of the GEP $P_{1,0,r,p}(z)$). If $b = 0$, we set $P_{1,0,r,1}(z) = 1$.

Observe that the polynomial $P_{1,0,r,2}(z)$ can be written, by using (7), as follows

$$P_{1,0,r,2}(z) = \sum_{k=0}^{r} \binom{r}{k}^2 z^{r-k}. \tag{98}$$

From the expansion (3), we see at once that the GEP $P_{a,b,r,p}(z)$ appear in the numerator of the Z-transform of the sequence $\binom{an+b}{r}^p$. In fact, we have

$$\mathcal{Z}\left(\binom{an+b}{r}^p\right) = \frac{zP_{a,b,r,p}(z)}{(z-1)^{rp+1}}. \tag{99}$$

The reciprocal polynomial of $P_{a,b,r,p}(z)$ is $P_{a,a-b+r-1,r,p}(z)$. That is, we have

$$z^{rp}P_{a,b,r,p}\left(z^{-1}\right) = P_{a,a-b+r-1,r,p}(z). \tag{100}$$

In fact, by using (9), we have that

$$\begin{aligned}
z^{rp}P_{a,b,r,p}\left(z^{-1}\right) &= z^{rp}\sum_{i=0}^{rp} A_{a,b,r}(p,i)\left(z^{-1}\right)^{rp-i} \\
&= \sum_{i=0}^{rp} A_{a,a-b+r-1,r}(p,rp-i)\,z^{i} \\
&= \sum_{i=0}^{rp} A_{a,a-b+r-1,r}(p,i)\,z^{rp-i} \\
&= P_{a,a-b+r-1,r,p}(z),
\end{aligned}$$

which proves our claim (100).

We would like to obtain a recurrence for GEP $P_{a,b,r,p}(z)$ that generalizes the recurrence (64), corresponding to the case $r = 1$, which in turn generalizes the known recurrence (41) for the standard Eulerian polynomials. As we will see next, this is a very difficult task. However, we will see also that there is a particular case with a nice formula for that recurrence.

Let us consider first the case $r = 2$. Beginning with the Z-transform of the sequence $\binom{an+b}{2}^p$, namely formula (99) with $r = 2$,

$$\mathcal{Z}\left(\binom{an+b}{2}^p\right) = \frac{zP_{a,b,2,p}(z)}{(z-1)^{2p+1}}, \tag{101}$$

we will mimic the procedure we used to obtain the recurrence (64). First we write $\binom{an+b}{2}^p$ as $\binom{an+b}{2}\binom{an+b}{2}^{p-1}$. Then we expand $\binom{an+b}{2}$ as $\frac{1}{2}a^2n^2 + a\left(b-\frac{1}{2}\right)n + \frac{1}{2}b(b-1)$. Next we use (30) and (31) to obtain

$$
\begin{aligned}
& \mathcal{Z}\left(\binom{an+b}{2}^{p}\right) && (102)\\
= \;& \mathcal{Z}\left(\left(\frac{1}{2}a^2n^2+a\left(b-\frac{1}{2}\right)n+\frac{1}{2}b\left(b-1\right)\right)\binom{an+b}{2}^{p-1}\right)\\
= \;& \frac{1}{2}a^2\mathcal{Z}\left(n^2\binom{an+b}{2}^{p-1}\right)+a\left(b-\frac{1}{2}\right)\mathcal{Z}\left(n\binom{an+b}{2}^{p-1}\right)\\
& +\frac{1}{2}b\left(b-1\right)\mathcal{Z}\left(\binom{an+b}{2}^{p-1}\right)\\
= \;& \frac{1}{2}a^2\left(z^2\frac{d^2}{dz^2}\frac{zP_{a,b,2,p-1}\left(z\right)}{\left(z-1\right)^{2p-1}}+z\frac{d}{dz}\frac{zP_{a,b,2,p-1}\left(z\right)}{\left(z-1\right)^{2p-1}}\right)\\
& -az\left(b-\frac{1}{2}\right)\frac{d}{dz}\frac{zP_{a,b,2,p-1}\left(z\right)}{\left(z-1\right)^{2p-1}}+\frac{1}{2}b\left(b-1\right)\frac{zP_{a,b,2,p-1}\left(z\right)}{\left(z-1\right)^{2p-1}}.
\end{aligned}
$$

Now we have to do some algebraic work (that we omit), and finally we get, from (101) and (102), the desired recurrence for the GEP $P_{a,b,2,p}\left(z\right)$:

$$
\begin{aligned}
& P_{a,b,2,p}\left(z\right)= && (103)\\
& \frac{1}{2}a^2z^2\left(z-1\right)^2P''_{a,b,2,p-1}\left(z\right)\\
& -\frac{a}{2}z\left(z-1\right)\left(3a-2b+1+\left(a\left(4p-5\right)+2b-1\right)z\right)P'_{a,b,2,p-1}\left(z\right)\\
& +\left(\binom{b+2a\left(p-1\right)}{2}z^2+\left(a^2\left(6p-5\right)+a\left(1-2b\right)\left(2p-3\right)+2b\left(1-b\right)\right)\frac{z}{2}\right.\\
& \qquad\left.+\binom{b-a}{2}\right)P_{a,b,2,p-1}\left(z\right).
\end{aligned}
$$

For example, if $a=2,b=1$, the recurrence (103) for the GEP $P_{2,1,2,p}\left(z\right)$ is

$$
\begin{aligned}
P_{2,1,2,p}\left(z\right) &= 2z^2\left(z-1\right)^2P''_{2,1,2,p-1}\left(z\right) && (104)\\
&\quad -z\left(z-1\right)\left(5+\left(8p-9\right)z\right)P'_{2,1,2,p-1}\left(z\right)\\
&\quad +\left(\binom{4p-3}{2}z^2+\left(10p-7\right)z+1\right)P_{2,1,2,p-1}\left(z\right).
\end{aligned}
$$

Observe that, by taking $P_{2,1,2,0}\left(z\right)=1$, the right-hand side of (104) makes sense with $p=1$:

$$2z^2\left(z-1\right)^2\left(1\right)''-z\left(z-1\right)\left(5-z\right)\left(1\right)'+\left(3z+1\right)\left(1\right)=3z+1,$$

which is the polynomial $P_{2,1,2,1}\left(z\right)$. With $p=2$, we use the polynomial $P_{2,1,2,1}\left(z\right)=$

$3z+1$ in the right-hand side of (104), to obtain that

$$2z^2(z-1)^2(3z+1)'' - z(z-1)(5+7z)(3z+1)' + \left(10z^2+13z+1\right)(3z+1) \\ = 9z^3+55z^2+31z+1,$$

which is the polynomial $P_{2,1,2,2}(z)$. With $p=3$, we use $P_{2,1,2,2}(z) = 9z^3+55z^2+31z+1$ in the right-hand side of (104), to obtain that

$$2z^2(z-1)^2\left(9z^3+55z^2+31z+1\right)'' - z(z-1)(5+15z)\left(9z^3+55z^2+31z+1\right)' \\ + \left(36z^2+23z+1\right)\left(9z^3+55z^2+31z+1\right) \\ = 27z^5+811z^4+2828z^3+1884z^2+209z+1.$$

which is the polynomial $P_{2,1,2,3}(z)$, and so on.

If we pursue a general recurrence for the GEP $P_{a,b,r,p}(z)$, for any $r\in\mathbb{N}$, the next step should be to repeat, in the case $r=3$, the procedure of the cases $r=1$ and $r=2$. Then, we would hopefully be able to figure out the form of the recurrence in the general case, in order to have a conjecture for the desired recurrence, and finally, of course, to prove the conjecture. But this seems likely to be impossible, except if the parameter a is equal to 1. In this case, formula (103) can be written as follows:

$$P_{1,b,2,p}(z) = \tag{105}$$
$$\frac{1}{2}z^2(z-1)^2 P''_{1,b,2,p-1}(z) - z(z-1)(2-b+(2p+b-3)z)P'_{1,b,2,p-1}(z) \\ + \left(\binom{b+2(p-1)}{2}z^2 - (b-2)(b+2p-2)z + \binom{b-1}{2}\right)P_{1,b,2,p-1}(z),$$

and we were able to identify that (105) is the case $r=2$ of

$$P_{1,b,r,p}(z) = \tag{106}$$
$$\sum_{l=0}^{r}\sum_{t=0}^{l}\binom{r-b}{t}\binom{rp-2r+l+b}{l-t}\frac{z^{r-t}(1-z)^{r-l}}{(r-l)!}\frac{d^{r-l}}{dz^{r-l}}P_{1,b,r,p-1}(z).$$

Observe that the case $r=1$ of (106) is

$$P_{1,b,1,p}(z) = z(1-z)P'_{1,b,1,p-1}(z) + (1-b+(p-1+b)z)P_{1,b,1,p-1}(z),$$

which is formula (64) with $a=1$.

The rest of this section is devoted to proving the recurrence (106). We begin with two lemmas containing some combinatorial identities that have some interest on their own.

Lemma 1. *Let $i,\alpha,r,s,$ be given non-negative integers, $0\le s\le r$. We have the identity*

$$\sum_{t=0}^{r}(-1)^t\binom{t}{s}\binom{i}{t}\binom{\alpha-t}{r-t} = (-1)^s\binom{i}{s}\binom{\alpha-i}{r-s}. \tag{107}$$

Proof. We proceed by induction on i. The case $i=0$ is clear: both sides are equal to $\binom{\alpha}{r}$. Indeed, in the case $i=1$ it is also easy to see that the left-hand side $\binom{0}{s}\binom{\alpha}{r}-\binom{1}{s}\binom{\alpha-1}{r-1}$ is equal to $(-1)^s\binom{1}{s}\binom{\alpha-1}{r-s}$. If (107) is true for a given $i\in\mathbb{N}$, let us prove that it is true for $i+1$. We have

$$\begin{aligned}
&\sum_{t=0}^{r}(-1)^t\binom{t}{s}\binom{i+1}{t}\binom{\alpha-t}{r-t}\\
= \;& \sum_{t=0}^{r}(-1)^t\binom{t}{s}\binom{i}{t}\binom{\alpha-t}{r-t}+\sum_{t=1}^{r}(-1)^t\binom{t}{s}\binom{i}{t-1}\binom{\alpha-t}{r-t}\\
= \;& (-1)^s\binom{i}{s}\binom{\alpha-i}{r-s}+\sum_{t=0}^{r-1}(-1)^{t+1}\binom{t+1}{s}\binom{i}{t}\binom{\alpha-t-1}{r-t-1}\\
= \;& (-1)^s\binom{i}{s}\binom{\alpha-i}{r-s}+\sum_{t=0}^{r-1}(-1)^{t+1}\binom{t}{s}\binom{i}{t}\binom{\alpha-t-1}{r-t-1}\\
& +\sum_{t=0}^{r-1}(-1)^{t+1}\binom{t}{s-1}\binom{i}{t}\binom{\alpha-t-1}{r-t-1}\\
= \;& (-1)^s\binom{i}{s}\binom{\alpha-i}{r-s}-(-1)^s\binom{i}{s}\binom{\alpha-1-i}{r-1-s}\\
& -(-1)^{s+1}\binom{i}{s-1}\binom{\alpha-1-i}{r-s}\\
= \;& (-1)^s\binom{i}{s}\left(\binom{\alpha-i}{r-s}-\binom{\alpha-1-i}{r-1-s}\right)+(-1)^s\binom{i}{s-1}\binom{\alpha-1-i}{r-s}\\
= \;& (-1)^s\binom{i}{s}\binom{\alpha-i-1}{r-s}+(-1)^s\binom{i}{s-1}\binom{\alpha-1-i}{r-s}\\
= \;& (-1)^s\left(\binom{i}{s}+\binom{i}{s-1}\right)\binom{\alpha-i-1}{r-s}\\
= \;& (-1)^s\binom{i+1}{s}\binom{\alpha-i-1}{r-s},
\end{aligned}$$

as desired. □

Lemma 2. *Let $i,\alpha,r,s,m,$ be given non-negative integers, $0\le s\le r$. We have the identity*

$$\sum_{t=0}^{r}\sum_{l=0}^{t}(-1)^t\binom{l}{t-s}\binom{m}{t-l}\binom{i-m}{l}\binom{\alpha-l-m}{r-t}=(-1)^s\binom{i}{s}\binom{\alpha-i}{r-s}. \quad (108)$$

Proof. We proceed by induction on m. For $m=0$ the result is true by (107). If it is true for $m\in\mathbb{N}$, let us prove that it is also true for $m+1$. First, note that the induction hypothesis with α,i,r and s replaced by $\alpha-1,i-1,r-1$ and $s-1$, respectively, says that

$$\sum_{t=0}^{r-1}\sum_{l=0}^{t}(-1)^t\binom{l}{t-s+1}\binom{m}{t-l}\binom{i-m-1}{l}\binom{\alpha-l-m-1}{r-1-t}$$
$$=(-1)^{s-1}\binom{i-1}{s-1}\binom{\alpha-i}{r-s},$$

or

$$\sum_{t=1}^{r}\sum_{l=0}^{t-1}(-1)^t\binom{l}{t-s}\binom{m}{t-l-1}\binom{i-m-1}{l}\binom{\alpha-l-m-1}{r-t}$$
$$=(-1)^{s}\binom{i-1}{s-1}\binom{\alpha-i}{r-s}.$$

This can be written as

$$\sum_{t=0}^{r}\sum_{l=0}^{t}(-1)^t\binom{l}{t-s}\binom{m}{t-l-1}\binom{i-m-1}{l}\binom{\alpha-l-m-1}{r-t} \tag{109}$$
$$=(-1)^{s}\binom{i-1}{s-1}\binom{\alpha-i}{r-s}.$$

Then, by using the induction hypothesis and (109), we have

$$\begin{aligned}
&\sum_{t=0}^{r}\sum_{l=0}^{t}(-1)^t\binom{l}{t-s}\binom{m+1}{t-l}\binom{i-m-1}{l}\binom{\alpha-l-m-1}{r-t}\\
=&\sum_{t=0}^{r}\sum_{l=0}^{t}(-1)^t\binom{l}{t-s}\left(\binom{m}{t-l}+\binom{m}{t-l-1}\right)\binom{i-m-1}{l}\binom{\alpha-l-m-1}{r-t}\\
=&\sum_{t=0}^{r}\sum_{l=0}^{t}(-1)^t\binom{l}{t-s}\binom{m}{t-l}\binom{i-m-1}{l}\binom{\alpha-l-m-1}{r-t}\\
&+\sum_{t=0}^{r}\sum_{l=0}^{t}(-1)^t\binom{l}{t-s}\binom{m}{t-l-1}\binom{i-m-1}{l}\binom{\alpha-l-m-1}{r-t}\\
=&(-1)^s\binom{i-1}{s}\binom{\alpha-i}{r-s}+(-1)^s\binom{i-1}{s-1}\binom{\alpha-i}{r-s}\\
=&(-1)^s\binom{i}{s}\binom{\alpha-i}{r-s},
\end{aligned}$$

as desired. □

After interchanging indices and replacing i and s by $\alpha-i$ and $r-s$, respectively, expression (108) can be written as follows

$$\sum_{l=0}^{r}\sum_{t=0}^{r-l}(-1)^{t}\binom{l}{s-t}\binom{m}{r-l-t}\binom{\alpha-i-m}{l}\binom{\alpha-l-m}{t} = (-1)^{s}\binom{i}{s}\binom{\alpha-i}{r-s}. \tag{110}$$

This is the formula we will use in the proof of the main result of this section (Theorem 2 below). We mention in passing the nice identity

$$\sum_{i=s}^{\alpha-r+s}\sum_{t=0}^{r}\sum_{l=0}^{t}(-1)^{s+t}\binom{l}{t-s}\binom{m}{t-l}\binom{i-m}{l}\binom{\alpha-l-m}{r-t}=\binom{\alpha+1}{r+1}, \tag{111}$$

(valid for any $s=0,1,\ldots,r$, and any non-negative integers α,r,m), obtained from (108) together with

$$\sum_{i=s}^{\alpha-r+s}\binom{i}{s}\binom{\alpha-i}{r-s}=\binom{\alpha+1}{r+1}, \tag{112}$$

which is identity (3.3) of [13].

Expressions (110) and (111) are, in fact, infinite families of identities. For example, if in (110) we set $m=0$, $m=\alpha$, and $m=r$, we obtain the identities

$$\begin{aligned}
&\sum_{l=0}^{r}(-1)^{r-l}\binom{l}{s-r+l}\binom{\alpha-i}{l}\binom{\alpha-l}{r-l}\\
&\qquad=\sum_{l=0}^{r}\sum_{t=0}^{r-l}(-1)^{l}\binom{l}{s-t}\binom{\alpha}{r-l-t}\binom{i+l-1}{l}\binom{l+t-1}{t}\\
&\qquad=\sum_{l=0}^{r}\sum_{t=0}^{r-l}(-1)^{t}\binom{l}{s-t}\binom{r}{l+t}\binom{\alpha-i-r}{l}\binom{\alpha-l-r}{t}\\
&\qquad=(-1)^{s}\binom{i}{s}\binom{\alpha-i}{r-s},
\end{aligned}$$

respectively. Some of the identities contained in (111) are

$$\begin{aligned}
&\sum_{i=s}^{\alpha-r+s}\sum_{t=0}^{r}(-1)^{s+t}\binom{l}{t-s}\binom{i}{t}\binom{\alpha-t}{r-t}\\
&\qquad=\sum_{i=s}^{\alpha-r+s}\sum_{t=0}^{r}\sum_{l=0}^{t}(-1)^{s+r}\binom{l}{t-s}\binom{\alpha}{t-l}\binom{i-\alpha}{l}\binom{l+r-t-1}{r-t}\\
&\qquad=\sum_{i=r}^{\alpha}\sum_{l=0}^{r}\binom{m}{r-l}\binom{i-m}{l}\\
&\qquad=\binom{\alpha+1}{r+1}.
\end{aligned}$$

Now we are ready to prove the main result of this section, namely, recurrence (106), stated in the following theorem. We again write the corresponding formula (106) to be demonstrated.

Theorem 2. *We have the following recurrence for the generalized Eulerian polynomials $P_{1,b,r,p}(z)$:*

$$P_{1,b,r,p}(z) = \sum_{l=0}^{r}\sum_{t=0}^{l}\binom{r-b}{t}\binom{rp-2r+l+b}{l-t}\frac{z^{r-t}(1-z)^{r-l}}{(r-l)!}\frac{d^{r-l}}{dz^{r-l}}P_{1,b,r,p-1}(z). \tag{113}$$

Proof. If $p=1$, formula (113) is clearly true, so we can suppose that $p>1$. We want to show that

$$\begin{aligned}
&\sum_{i=0}^{rp}\sum_{j=0}^{i}(-1)^j\binom{rp+1}{j}\binom{i-j+b}{r}^p z^{rp-i} \qquad (114)\\
=\;& \sum_{l=0}^{r}\left(\sum_{t=0}^{l}\binom{r-b}{t}\binom{rp-2r+l+b}{l-t}z^{l-t}\right)\frac{1}{(r-l)!}z^{r-l}(1-z)^{r-l}\\
&\times\frac{d^{r-l}}{dz^{r-l}}\sum_{i=0}^{r(p-1)}\sum_{j=0}^{i}(-1)^j\binom{r(p-1)+1}{j}\binom{i-j+b}{r}^{p-1}z^{r(p-1)-i}.
\end{aligned}$$

Beginning with the right-hand side of (114) we have

$$\begin{aligned}
&\sum_{l=0}^{r}\left(\sum_{t=0}^{l}\binom{r-b}{t}\binom{rp-2r+l+b}{l-t}z^{l-t}\right)\frac{1}{(r-l)!}z^{r-l}(1-z)^{r-l}\\
&\quad\times\frac{d^{r-l}}{dz^{r-l}}\sum_{i=0}^{r(p-1)}\sum_{j=0}^{i}(-1)^j\binom{r(p-1)+1}{j}\binom{i-j+b}{r}^{p-1}z^{r(p-1)-i}\\
&=\sum_{l=0}^{r}\left(\sum_{t=0}^{l}\binom{r-b}{t}\binom{rp-2r+l+b}{l-t}z^{l-t}\right)(1-z)^{r-l}\\
&\quad\times\sum_{i=0}^{r(p-1)}\left(\sum_{j=0}^{i}(-1)^j\binom{r(p-1)+1}{j}\binom{i-j+b}{r}^{p-1}\right)\binom{r(p-1)-i}{r-l}z^{r(p-1)-i}\\
&=\sum_{i=0}^{r(p-1)}\left(\sum_{j=0}^{i}(-1)^j\binom{r(p-1)+1}{j}\binom{i-j+b}{r}^{p-1}z^{rp-i}\right)\\
&\quad\times\sum_{l=0}^{r}\left(\sum_{t=0}^{l}\binom{r-b}{t}\binom{rp-2r+l+b}{l-t}z^{-t}\right)\binom{r(p-1)-i}{r-l}(1-z)^{r-l}z^{-(r-l)}.
\end{aligned} \tag{115}$$

Observe that

$$
\begin{aligned}
&\sum_{l=0}^{r}\left(\sum_{t=0}^{l}\binom{r-b}{t}\binom{rp-2r+l+b}{l-t}z^{-t}\right)\binom{r\,(p-1)-i}{r-l}(1-z)^{r-l}\,z^{-(r-l)}\\
=&\sum_{l=0}^{r}\left(\sum_{t=0}^{r-l}\binom{r-b}{t}\binom{rp-r-l+b}{r-l-t}z^{-t}\right)\binom{r\,(p-1)-i}{l}(1-z)^{l}\,z^{-l}\\
=&\sum_{l=0}^{r}\left(\sum_{t=0}^{r-l}\binom{r-b}{t}\binom{r\,(p-1)-l+b}{r-t-l}\right)\binom{r\,(p-1)-i}{l}z^{-l-t}\sum_{k=0}^{l}\binom{l}{k}(-z)^{k}\\
=&z^{-r}\sum_{l=0}^{r}\sum_{t=0}^{r-l}\sum_{k=0}^{l}(-1)^{k}\binom{l}{k}\binom{r-b}{r-l-t}\binom{r\,(p-1)-l+b}{t}\binom{r\,(p-1)-i}{l}z^{k+t}\\
=&z^{-r}\sum_{s=0}^{r}(-1)^{s}\sum_{l=0}^{r}\sum_{t=0}^{r-l}(-1)^{t}\binom{l}{s-t}\binom{r-b}{r-l-t}\binom{r\,(p-1)-l+b}{t}\times\\
&\times\binom{r\,(p-1)-i}{l}z^{s},
\end{aligned}
\tag{116}
$$

where in the last step we substituted $k+t$ with s. In (110) set $\alpha = rp$ and $m = r-b$, and replace i by $i+b$, to obtain

$$
\begin{aligned}
&\sum_{l=0}^{r}\sum_{t=0}^{r-l}(-1)^{t}\binom{l}{s-t}\binom{r-b}{r-l-t}\binom{r\,(p-1)-i}{l}\binom{r\,(p-1)-l+b}{t}\\
&\qquad=(-1)^{s}\binom{i+b}{s}\binom{rp-i-b}{r-s}.
\end{aligned}
\tag{117}
$$

Thus, the right-hand side of (114) can be written, according to (115), (116) and (117), as follows:

$$
\begin{aligned}
&\sum_{l=0}^{r}\left(\sum_{t=0}^{l}\binom{r-b}{t}\binom{rp-2r+l+b}{l-t}z^{l-t}\right)\frac{1}{(r-l)!}z^{r-l}(1-z)^{r-l} \qquad (118)\\
&\quad\times\frac{d^{r-l}}{dz^{r-l}}\sum_{i=0}^{r(p-1)}\sum_{j=0}^{i}(-1)^{j}\binom{r\,(p-1)+1}{j}\binom{i-j+b}{r}^{p-1}z^{r(p-1)-i}\\
&=\sum_{i=0}^{r(p-1)}\left(\sum_{j=0}^{i}(-1)^{j}\binom{r\,(p-1)+1}{j}\binom{i-j+b}{r}^{p-1}z^{rp-i}\right)\times\\
&\quad\times\sum_{s=0}^{r}\binom{i+b}{s}\binom{rp-i-b}{r-s}z^{s-r}.
\end{aligned}
$$

To end the proof, according to (114) and (118), we have to show that the following polynomial identity is true:

$$\begin{aligned}\sum_{i=0}^{rp}\sum_{j=0}^{i}(-1)^j\binom{rp+1}{j}\binom{i-j+b}{r}^p z^{rp-i} = \\ \sum_{i=0}^{r(p-1)}\left(\sum_{j=0}^{i}(-1)^j\binom{r(p-1)+1}{j}\binom{i-j+b}{r}^{p-1} z^{rp-i}\right)\times \\ \times\sum_{s=0}^{r}\binom{i+b}{s}\binom{rp-i-b}{r-s}z^{s-r}.\end{aligned} \tag{119}$$

Let us change the sum of indices $i+s$ by the new index i (which runs from 0 up to rp). Expression (119) is then written as follows

$$\begin{aligned}\sum_{i=0}^{rp}\sum_{j=0}^{i}(-1)^j\binom{rp+1}{j}\binom{i-j+b}{r}^p z^{rp-i} = \\ \sum_{i=0}^{rp}\sum_{s=0}^{r}\sum_{j=0}^{i-s}(-1)^j\binom{r(p-1)+1}{j}\binom{i-s-j+b}{r}^{p-1}\times \\ \times\binom{i-s+b}{r-s}\binom{rp-i+s-b}{s}z^{rp-i}.\end{aligned} \tag{120}$$

Plainly, expression (120) is true if and only if, for $i=0,1,\ldots,rp$, we have that

$$\begin{aligned}\sum_{j=0}^{i}(-1)^j\binom{rp+1}{j}\binom{i-j+b}{r}^p = \\ \sum_{s=0}^{r}\sum_{j=0}^{i-s}(-1)^j\binom{r(p-1)+1}{j}\binom{i-s-j+b}{r}^{p-1}\binom{i-s+b}{r-s}\binom{rp-i+s-b}{s}.\end{aligned} \tag{121}$$

But (121) is precisely the recurrence (11) for the GEN $A_{1,b,r}(p,i)$. Thus, the proof is complete. $\square$

As an example, let us consider the case $r=3$, $b=0$. The coefficients of the GEP $P_{1,0,3,p}(z)$, are the GEN $A_{1,0,3}(p,i)$ (see GENT1 in Section 1). In this case the recurrence (113) can be written as follows:

$$\begin{aligned}P_{1,0,3,p}(z) &= \frac{z^3(1-z)^3}{6}\frac{d^3}{dz^3}P_{1,0,3,p-1}(z) \\ &+ ((3p-5)z+3)\frac{z^2(1-z)^2}{2}\frac{d^2}{dz^2}P_{1,0,3,p-1}(z) \\ &+ \left(\binom{3p-4}{2}z^2+3(3p-4)z+3\right)z(1-z)\frac{d}{dz}P_{1,0,3,p-1}(z) \\ &+ \left(\binom{3p-3}{3}z^3+3\binom{3p-3}{2}z^2+3\binom{3p-3}{1}z+1\right)P_{1,0,3,p-1}(z).\end{aligned} \tag{122}$$

If we set $p=3$ in the right-hand side of (122), and use the polynomial $P_{1,0,3,2}(z) = z^3+9z^2+9z+1$, we obtain

$$\begin{aligned}& \frac{z^3(1-z)^3}{6}(6)+(4z+3)\frac{z^2(1-z)^2}{2}(6z+18) \\ & + \left(\binom{5}{2}z^2+15z+3\right)z(1-z)\left(3z^2+18z+9\right) \\ & + \left(\binom{6}{3}z^3+3\binom{6}{2}z^2+3\binom{6}{1}z+1\right)\left(z^3+9z^2+9z+1\right) \\ = {} & z^6+54z^5+405z^4+760z^3+405z^2+54z+1,\end{aligned}$$

which is $P_{1,0,3,p}(z)$.

Observe that, by using expression (98) for $P_{1,0,r,2}(z)$ together with the recurrence (113), we can write the following explicit formula for the GEP $P_{1,0,r,3}(z)$:

$$P_{1,0,r,3}(z) = \sum_{l=0}^{r}\sum_{i=0}^{l}\sum_{k=0}^{r}\binom{r}{i}\binom{r+l}{l-i}\binom{r-k}{r-l}\binom{r}{k}^2(1-z)^{r-l}z^{r+l-k-i}. \tag{123}$$

5. Some Applications

We present two applications: (1) General telescoping sums, in which the GEP $P_{a,b,1,p}(z)$ (studied in Section 3) are involved, and (2) Sums and alternating sums of powers of binomial coefficients, in which the GEP $P_{a,b,r,p}(z)$ (studied in Section 4) are involved.

5.1. General Telescoping Sums

The main result in this subsection is the following:

Proposition 3. *Let m be a given positive integer, and let ω_k, $k = 0, 1, \ldots, m-1$, be the m-th roots of 1, namely $\omega_k = \exp \frac{2\pi ik}{m}$. The sequence $(an+b)^p$ can be written as an m-telescoping sum according to*

$$\begin{aligned}(an+b)^p &= \sum_{k=0}^{m-1} \alpha_k \omega_k^n \\ &\quad + \sum_{t=0}^{n-1} \left(\sum_{k=0}^{m-1} \beta_k \omega_k^{n-1-t} \right) \left((a(t+m)+b)^p - (at+b)^p \right),\end{aligned} \tag{124}$$

where

$$\alpha_k = \lim_{z\to\omega_k} \frac{(z-\omega_k)\sum_{j=0}^{m-1} (aj+b)^p z^{m-1-j}}{z^m-1}, \tag{125}$$

$$\beta_k = \lim_{z\to\omega_k} \frac{z-\omega_k}{z^m-1}. \tag{126}$$

Proof. Substitute (72) in (63) to obtain

$$\mathcal{Z}((an+b)^p) = \tag{127}$$

$$\underbrace{\frac{z}{z^m-1} \sum_{j=0}^{m-1} (aj+b)^p z^{m-1-j}}_{\text{(A)}} + \underbrace{\frac{1}{z^m-1} \sum_{k=0}^{p-1} \binom{p}{k} (ma)^{p-k} \frac{zP_{a,b,1,k}(z)}{(z-1)^{k+1}}}_{\text{(B)}}.$$

Since $z^m - 1 = \prod_{k=0}^{m-1} (z-\omega_k)$, we can expand in partial fractions the rational function $\frac{\sum_{j=0}^{m-1} (aj+b)^p z^{m-1-j}}{z^m-1}$ to obtain

$$\frac{1}{z^m-1} \sum_{j=0}^{m-1} (aj+b)^p z^{m-1-j} = \sum_{k=0}^{m-1} \frac{\alpha_k}{z-\omega_k},$$

where α_k is given in (125). Then, expression (A) can be written as follows

$$\frac{z}{z^m-1} \sum_{j=0}^{m-1} (aj+b)^p z^{m-1-j} = \sum_{k=0}^{m-1} \alpha_k \frac{z}{z-\omega_k} = \mathcal{Z}\left(\sum_{k=0}^{m-1} \alpha_k \omega_k^n \right). \tag{128}$$

Similarly, we have the partial fraction decomposition

$$\frac{1}{z^m-1} = \sum_{k=0}^{m-1} \frac{\beta_k}{z-\omega_k}, \tag{129}$$

where β_k is given in (126). Then

$$\frac{1}{z^m-1} = \frac{1}{z} \sum_{k=0}^{m-1} \beta_k \frac{z}{z-\omega_k} = \frac{1}{z} \mathcal{Z}\left(\sum_{k=0}^{m-1} \beta_k \omega_k^n \right).$$

Thus, expression (B) can be written as follows

$$\begin{aligned}
& \frac{1}{z^m-1}\sum_{k=0}^{p-1}\binom{p}{k}(ma)^{p-k}\frac{zP_{a,b,1,k}(z)}{(z-1)^{k+1}} \\
= \quad & \frac{1}{z}\mathcal{Z}\left(\sum_{k=0}^{m-1}\beta_k\omega_k^n\right)\mathcal{Z}\left(\sum_{k=0}^{p}\binom{p}{k}(ma)^{p-k}(an+b)^k-(an+b)^p\right) \\
= \quad & \frac{1}{z}\sum_{k=0}^{m-1}\beta_k\mathcal{Z}(\omega_k^n)\,\mathcal{Z}\left((a(n+m)+b)^p-(an+b)^p\right). \qquad (130)
\end{aligned}$$

The Convolution theorem (25), together with (28), give us from (130) that expression (B) is

$$\begin{aligned}
& \frac{1}{z^m-1}\sum_{k=0}^{p-1}\binom{p}{k}(ma)^{p-k}\frac{zP_{a,b,1,k}(z)}{(z-1)^{k+1}} \\
= \quad & \mathcal{Z}\left(\sum_{t=0}^{n-1}\left(\sum_{k=0}^{m-1}\beta_k\omega_k^{n-1-t}\right)\left((a(t+m)+b)^p-(at+b)^p\right)\right). \qquad (131)
\end{aligned}$$

Finally, the desired conclusion (124) comes from (127), (128) and (131). □

There is a different form of writing the term (B) in (127). Observe that

$$\begin{aligned}
& \frac{1}{z^m-1}\sum_{k=0}^{p-1}\binom{p}{k}(ma)^{p-k}\frac{zP_{a,b,1,k}(z)}{(z-1)^{k+1}} \\
= \quad & z^{-m}\left(\prod_{k=0}^{m-1}\frac{z}{z-\omega_k}\right)\left(\sum_{k=0}^{p}\binom{p}{k}(ma)^{p-k}\frac{zP_{a,b,1,k}(z)}{(z-1)^{k+1}}-\frac{zP_{a,b,1,p}(z)}{(z-1)^{p+1}}\right) \\
= \quad & z^{-m}\left(\prod_{k=0}^{m-1}\mathcal{Z}(\omega_k^n)\right)\mathcal{Z}\left((a(n+m)+b)^p-(an+b)^p\right). \qquad (132)
\end{aligned}$$

The product $\left(\prod_{k=0}^{m-1}\mathcal{Z}(\omega_k^n)\right)\mathcal{Z}\left((a(n+m)+b)^p-(an+b)^p\right)$ is the Z-transform of the convolution

$$\omega_{m-1}^n*\cdots*\omega_1^n*\omega_0^n*\left((a(n+m)+b)^p-(an+b)^p\right), \qquad (133)$$

and, according to (28), the factor z^{-m} in (132) produces an "m-delay" in the sequence (133). Thus we can write the term (B) in (127) as follows

$$\sum_{j_m=0}^{n-m}\cdots\sum_{j_2=0}^{j_3}\sum_{j_1=0}^{j_2}\omega_{m-1}^{n-j_m}\cdots\omega_1^{j_3-j_2}\omega_0^{j_2-j_1}\left((a(j_1+m)+b)^p-(aj_1+b)^p\right).$$

That is, formula (124) from Proposition 3 can be written as follows:

$$(an+b)^p = \sum_{k=0}^{m-1} \alpha_k \omega_k^n + \tag{134}$$
$$\sum_{j_m=0}^{n-m} \cdots \sum_{j_2=0}^{j_3} \sum_{j_1=0}^{j_2} \omega_{m-1}^{n-j_m} \cdots \omega_1^{j_3-j_2} \omega_0^{j_2-j_1} \left(\left(a\left(j_1+m \right) + b \right)^p - \left(aj_1+b \right)^p \right),$$

where α_k is as in Proposition 3.

In the case $m=1$, expression (128) is simply $\frac{z}{z-1}b^p = b^p \mathcal{Z}\left(1\right)$, and expression (131) is

$$\frac{1}{z-1}\sum_{k=0}^{p-1}\binom{p}{k}a^{p-k}\frac{zP_{a,b,1,k}\left(z\right)}{\left(z-1\right)^{k+1}} = \mathcal{Z}\left(\sum_{t=0}^{n-1}\left(\left(a\left(t+1\right)+b\right)^p - \left(at+b\right)^p\right)\right).$$

Thus, formula (124) is in this case

$$(an+b)^p = b^p + \sum_{t=0}^{n-1}\left(\left(a\left(t+1\right)+b\right)^p - \left(at+b\right)^p\right), \tag{135}$$

which is the standard telescoping sum of the sequence $(an+b)^p$.

When $m=2$, formula (124) can be written as follows:

$$\begin{aligned}(an+b)^p &= \frac{b^p+(a+b)^p}{2} + \frac{b^p-(a+b)^p}{2}\left(-1\right)^n \\ &+\frac{1}{2}\sum_{t=0}^{n-1}\left(1+\left(-1\right)^{n-t}\right)\left(\left(a\left(t+2\right)+b\right)^p - \left(at+b\right)^p\right),\end{aligned} \tag{136}$$

and the convolution version (134) can be written as follows:

$$\begin{aligned}(an+b)^p &= \frac{b^p+(a+b)^p}{2} + \frac{b^p-(a+b)^p}{2}\left(-1\right)^n \\ &+\sum_{j_2=0}^{n-2}\sum_{j_1=0}^{j_2}\left(-1\right)^{n-j_2}\left(\left(a\left(j_1+2\right)+b\right)^p - \left(aj_1+b\right)^p\right).\end{aligned} \tag{137}$$

For $m=3$, formula (124) can be written as

$$\begin{aligned}(an+b)^p &= \\ &\frac{b^p+(a+b)^p+(2a+b)^p}{3} + \frac{2b^p-(a+b)^p-(2a+b)^p}{3}\cos\frac{2n\pi}{3} \\ &+\frac{(a+b)^p-(2a+b)^p}{\sqrt{3}}\sin\frac{2n\pi}{3} \\ &+\frac{1}{3}\sum_{t=0}^{n-1}\left(1-\cos\frac{2\left(n-1-t\right)\pi}{3} - \sqrt{3}\sin\frac{2\left(n-1-t\right)\pi}{3}\right)\times \\ &\times\left(\left(a\left(t+3\right)+b\right)^p - \left(at+b\right)^p\right).\end{aligned} \tag{138}$$

5.2. Sums of Powers of Binomial Coefficients

Plainly, expansion (3) together with (37), gives us at once that

$$\sum_{j=0}^{n}\binom{aj+b}{r}^{p}=\sum_{i=0}^{rp}A_{a,b,r}\left(p,i\right)\binom{n+1+rp-i}{rp+1}. \tag{139}$$

It turns out that, in some cases, the right-hand side of (139) can be written as the convolution of the odd positive integers sequence $2n+1$ with a polynomial. This is, for example, the case $a=1$, $b=0$, r odd, p even. In fact, formula (14) tells us that the GEP $P_{1,0,r,p}\left(z\right)=\sum_{i=0}^{rp}A_{1,0,r}\left(p,i\right)z^{rp-i}$ (which is a $r\left(p-1\right)$-th degree polynomial), with r odd and p even, is such that $P_{1,0,r,p}\left(-1\right)=0$. That is, we can write $P_{1,0,r,p}\left(z\right)=\left(z+1\right)\mathbb{Q}_{r(p-1)-1}\left(z\right)$, where $\mathbb{Q}_{r(p-1)-1}\left(z\right)$ is a $\left(r\left(p-1\right)-1\right)$-th degree polynomial. Indeed, it is not difficult to obtain the explicit factorization

$$P_{1,0,r,p}\left(z\right)=\left(z+1\right)\sum_{k=r}^{rp-1}\sum_{i=0}^{k}A_{1,0,r}\left(p,i\right)\left(-1\right)^{i+k}z^{rp-1-k}. \tag{140}$$

Thus, formula (99) gives us that

$$\begin{aligned}\mathcal{Z}\left(\sum_{j=0}^{n}\binom{j}{r}^{p}\right) &= \frac{z}{z-1}\frac{zP_{1,0,r,p}\left(z\right)}{\left(z-1\right)^{rp+1}}\\ &= \frac{z^{2}}{\left(z-1\right)^{rp+1}}\left(z+1\right)\sum_{k=r}^{rp-1}\sum_{i=0}^{k}A_{1,0,r}\left(p,i\right)\left(-1\right)^{i+k}z^{rp-1-k}\\ &= \frac{z\left(z+1\right)}{\left(z-1\right)^{2}}\sum_{k=r}^{rp-1}\sum_{i=0}^{k}A_{1,0,r}\left(p,i\right)\left(-1\right)^{i+k}\frac{z^{rp-k}}{\left(z-1\right)^{rp}}. \end{aligned} \tag{141}$$

From (141), together with (25), (33), and (35), we see that for r odd and p even, the sum $\sum_{j=0}^{n}\binom{j}{r}^{p}$ can be written as the convolution

$$\sum_{j=0}^{n}\binom{j}{r}^{p}=\left(2n+1\right)*\sum_{k=r}^{rp-1}\sum_{i=0}^{k}A_{1,0,r}\left(p,i\right)\left(-1\right)^{i+k}\binom{n+rp-1-k}{rp-1}. \tag{142}$$

Moreover, observe that

$$\sum_{k=r}^{rp-1}\sum_{i=0}^{k}A_{1,0,r}\left(p,i\right)\left(-1\right)^{i+k}\binom{n+rp-1-k}{rp-1}=0,$$

for $n=0,1,\ldots,r-1$. Thus, we can write

$$\sum_{k=r}^{rp-1}\sum_{i=0}^{k}A_{1,0,r}\left(p,i\right)\left(-1\right)^{i+k}\binom{n+rp-1-k}{rp-1}=\binom{n}{r}\mathbb{S}_{r(p-1)-1}\left(n\right),$$

where $\mathbb{S}_{r(p-1)-1}(n)$ is a $(r(p-1)-1)$-th degree n-polynomial.

Summarizing, for r odd and p even, we have

$$\sum_{j=0}^{n}\binom{j}{r}^{p}=(2n+1)*\binom{n}{r}\mathbb{S}_{r(p-1)-1}(n),$$

where the polynomial $\mathbb{S}_{r(p-1)-1}(n)$ is given by

$$\mathbb{S}_{r(p-1)-1}(n)=\binom{n}{r}^{-1}\sum_{k=r}^{rp-1}\sum_{i=0}^{k}A_{1,0,r}(p,i)(-1)^{i+k}\binom{n+rp-1-k}{rp-1}.$$

As we will see in the following examples, the polynomial $\mathbb{S}_{r(p-1)-1}(n)$ has a nice form when we write it in powers of $n-s$, where $r=2s+1$. When $r=1$, we have the following formula for sums of even powers of integers

$$\sum_{j=0}^{n}j^{2p}=(2n+1)*n\mathbb{S}_{2p-2}(n),$$

where $p\in\mathbb{N}$ and $\mathbb{S}_{2p-2}(n)=n^{-1}\sum_{k=1}^{2p-1}\sum_{i=0}^{k}A_{1,0,1}(2p,i)(-1)^{i+k}\binom{n+2p-1-k}{2p-1}$. Concrete examples are

$$\begin{aligned}\sum_{k=0}^{n}k^{2}&=(2n+1)*n,\\ \sum_{k=0}^{n}k^{4}&=(2n+1)*n\left(2n^{2}-1\right),\\ \sum_{k=0}^{n}k^{6}&=(2n+1)*n\left(3n^{4}-5n^{2}+3\right).\end{aligned}$$

Some examples of (142) with $r=3$ are

$$\begin{aligned}\sum_{k=0}^{n}\binom{k}{3}^{2}&=\frac{1}{2}(2n+1)*\binom{n}{3}\left((n-1)^{2}-2\right),\\ \sum_{k=0}^{n}\binom{k}{3}^{4}&=\frac{1}{72}(2n+1)*\binom{n}{3}\times\\ &\times\left(2(n-1)^{8}-23(n-1)^{6}+157(n-1)^{4}-604(n-1)^{2}+936\right),\end{aligned}$$

and with $r=5$

$$\begin{aligned}
\sum_{k=0}^{n}\binom{k}{5}^{2} &= \frac{1}{4!}\left(2n+1\right)*\binom{n}{5}\left(\left(n-2\right)^{4}-9\left(n-2\right)^{2}+24\right),\\
\sum_{k=0}^{n}\binom{k}{5}^{4} &= \frac{1}{345600}\left(2n+1\right)*\binom{n}{5}\times\\
&\quad\times\left(\begin{array}{c} 2\left(n-2\right)^{14}-83\left(n-2\right)^{12}+2209\left(n-2\right)^{10}\\ -44389\left(n-2\right)^{8}+648681\left(n-2\right)^{6}-6375828\left(n-2\right)^{4}\\ +36988608\left(n-2\right)^{2}-93657600\end{array}\right).
\end{aligned}$$

Let us now consider alternating sums of powers of binomial coefficients. Observe that, by using (3), (21) and (37), we can write

$$\mathcal{Z}\left(\sum_{k=0}^{n}\left(-1\right)^{k}\binom{k}{r}^{p}\right)=\frac{z}{z-1}\frac{-zP_{1,0,r,p}\left(-z\right)}{\left(-z-1\right)^{rp+1}}=\left(-1\right)^{rp}\frac{z^{2}P_{1,0,r,p}\left(-z\right)}{\left(z-1\right)\left(z+1\right)^{rp+1}}.$$

If r is odd and p is even, we have, using (140), that

$$\begin{aligned}
&\mathcal{Z}\left(\sum_{k=0}^{n}\left(-1\right)^{k}\binom{k}{r}^{p}\right)\\
&\qquad=\left(-1\right)^{rp}\frac{z^{2}\left(-z+1\right)\sum_{k=r}^{rp-1}\sum_{i=0}^{k}A_{1,0,r}\left(p,i\right)\left(-1\right)^{i+k}\left(-z\right)^{rp-1-k}}{\left(z-1\right)\left(z+1\right)^{rp+1}}\\
&\qquad=\sum_{k=r}^{rp-1}\sum_{i=0}^{k}A_{1,0,r}\left(p,i\right)\left(-1\right)^{i+k}\frac{\left(-z\right)^{rp+1-k}}{\left(-z-1\right)^{rp+1}},
\end{aligned}$$

from which we conclude that

$$\sum_{j=0}^{n}\left(-1\right)^{k+n}\binom{k}{r}^{p}=\sum_{k=r}^{rp-1}\sum_{i=0}^{k}A_{1,0,r}\left(p,i\right)\left(-1\right)^{i+k}\binom{n+rp-k}{rp}. \tag{143}$$

Some examples are

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^{k+n}k^{2} &= \binom{n+1}{2}, \\
\sum_{k=0}^{n}(-1)^{k+n}k^{4} &= \binom{n+3}{4}+10\binom{n+2}{4}+\binom{n+1}{4}, \\
\sum_{k=0}^{n}(-1)^{k+n}\binom{k}{3}^{2} &= \binom{n+3}{6}+8\binom{n+2}{6}+\binom{n+1}{6}, \\
\sum_{k=0}^{n}(-1)^{k+n}\binom{k}{5}^{2} &= \binom{n+5}{10}+24\binom{n+4}{10}+76\binom{n+3}{10} \\
&\quad +24\binom{n+2}{10}+\binom{n+1}{10}.
\end{aligned}$$

Similar discussions, beginning with the alternating sums (15) or (16) replacing (14), yield the corresponding formulas for sums of powers of binomial coefficients.

Related to (15), for r odd and p even, we have

$$\sum_{k=0}^{n}\binom{k+r}{r}^{p}=(2n+1)*\sum_{k=0}^{r(p-1)-1}\sum_{i=0}^{k}A_{1,r,r}(p,i)(-1)^{i+k}\binom{n+rp-1-k}{rp-1}.$$

Related to (16), for $r\equiv 1 \mod 4$ and p odd, or $r\equiv 3 \mod 4$ and p even, we have

$$\sum_{k=r}^{n}\binom{2k+1}{r}^{p}=$$

$$(2n+1)*\sum_{k=0}^{rp-1-\lfloor r/2\rfloor}\sum_{i=0}^{k+(r-1)/2}A_{2,1,r}(p,i)(-1)^{i+k+(r-1)/2}\binom{n+rp-1-k-\lfloor r/2\rfloor}{rp-1}.$$

A different approach to the alternating sums of powers, including more general weighted sums, begins with the advance-shifting property of the Z-transform (20) (see (66) and (67)). Using (20) and (99), we can write

$$\begin{aligned}
\frac{zP_{a,am+b,r,p}(z)}{(z-1)^{rp+1}} &= \mathcal{Z}\left(\binom{a(n+m)+b}{r}^{p}\right) \\
&= z^{m}\left(\mathcal{Z}\left(\binom{an+b}{r}^{p}\right)-\sum_{j=0}^{m-1}\frac{\binom{aj+b}{r}^{p}}{z^{j}}\right) \\
&= z^{m}\left(\frac{zP_{a,b,r,p}(z)}{(z-1)^{rp+1}}-\sum_{j=0}^{m-1}\frac{\binom{aj+b}{r}^{p}}{z^{j}}\right),
\end{aligned}$$

from where we get

$$\sum_{k=0}^{m-1}\frac{\binom{ak+b}{r}^p}{z^k}=\frac{zP_{a,b,r,p}\left(z\right)}{\left(z-1\right)^{rp+1}}-z^{-m}\frac{zP_{a,am+b,r,p}\left(z\right)}{\left(z-1\right)^{rp+1}},$$

or, replacing z by z^{-1},

$$\sum_{k=0}^{m-1}z^k\binom{ak+b}{r}^p=\frac{z^{rp}P_{a,b,r,p}\left(z^{-1}\right)}{\left(1-z\right)^{rp+1}}-z^m\frac{z^{rp}P_{a,am+b,r,p}\left(z^{-1}\right)}{\left(1-z\right)^{rp+1}}. \tag{144}$$

Further, by using (100) we can write (144) as follows

$$\sum_{k=0}^{m-1}z^k\binom{ak+b}{r}^p=\frac{P_{a,a-b+r-1,r,p}\left(z\right)}{\left(1-z\right)^{rp+1}}-z^m\frac{P_{a,a(1-m)-b+r-1,r,p}\left(z\right)}{\left(1-z\right)^{rp+1}},$$

and finally, replacing b by $a-b+r-1$ we get

$$\sum_{k=0}^{m-1}z^k\binom{ak+a-b+r-1}{r}^p=\frac{P_{a,b,r,p}\left(z\right)}{\left(1-z\right)^{rp+1}}-z^m\frac{P_{a,b-am,r,p}\left(z\right)}{\left(1-z\right)^{rp+1}}. \tag{145}$$

Formula (145) gives us the value of the weighted sum of the left-hand side, where $z\in\mathbb{C}$, $z\neq 0,1$ is the weight, in terms of the GEP $P_{a,b,r,p}\left(z\right)$ and $P_{a,b-am,r,p}\left(z\right)$.

If we set $a=1,b=0,z=-1$, we get from (144) and (145) that

$$\sum_{k=0}^{m-1}\left(-1\right)^k\binom{k}{r}^p=\frac{\left(-1\right)^{rp}P_{1,0,r,p}\left(-1\right)}{2^{rp+1}}-\left(-1\right)^m\frac{\left(-1\right)^{rp}P_{1,m,r,p}\left(-1\right)}{2^{rp+1}},$$

and

$$\sum_{k=0}^{m-1}\left(-1\right)^k\binom{k+r}{r}^p=\frac{P_{1,0,r,p}\left(-1\right)}{2^{rp+1}}-\left(-1\right)^m\frac{P_{1,-m,r,p}\left(-1\right)}{2^{rp+1}},$$

respectively. For r odd and p even we have $P_{1,0,r,p}\left(-1\right)=0$, and then we obtain the following alternating sums of powers of binomial coefficients

$$\sum_{k=0}^{m-1}\left(-1\right)^{k+m}\binom{k}{r}^p=-\frac{1}{2^{rp+1}}P_{1,m,r,p}\left(-1\right),$$

and

$$\sum_{k=0}^{m-1}\left(-1\right)^{k+m}\binom{k+r}{r}^p=-\frac{1}{2^{rp+1}}P_{1,-m,r,p}\left(-1\right),$$

in terms of the value of the GEP $P_{1,m,r,p}\left(z\right)$ or $P_{1,m,r,p}\left(z\right)$ at $z=-1$. Explicitly, we have, for r odd and p even:

$$\begin{aligned}\sum_{k=0}^{m-1}\left(-1\right)^{k+m}\binom{k}{r}^p &= -\frac{1}{2^{rp+1}}\sum_{i=0}^{rp}A_{1,m,r}\left(p,i\right)\left(-1\right)^i\\ &= -\frac{1}{2^{rp+1}}\sum_{i=0}^{rp}\sum_{j=0}^{i}\left(-1\right)^{j+i}\binom{rp+1}{j}\binom{i-j+m}{r}^p,\end{aligned}$$

(compare with (143)), and

$$\begin{aligned}\sum_{k=0}^{m-1}(-1)^{k+m}\binom{k+r}{r}^{p} &= -\frac{1}{2^{rp+1}}\sum_{i=0}^{rp}A_{1,-m,r}\left(p,i\right)\left(-1\right)^{i}\\ &= -\frac{1}{2^{rp+1}}\sum_{i=0}^{rp}\sum_{j=0}^{i}(-1)^{j+i}\binom{rp+1}{j}\binom{i-j-m}{r}^{p}.\end{aligned}$$

Acknowledgements. I thank the referee for his/her careful reading of the first version of the paper, which produced suggestions and corrections improving this final version. I also thank him/her for drawing my attention to some combinatorial developments involving generalized Eulerian polynomials, as colored Eulerian polynomials, and multiset Eulerian polynomials. He/she also pointed out some questions for future work: combinatorial interpretations of some particular cases, and the question on whether some of the sequences of generalized Eulerian polynomials are unimodal. Some final words for Dr. Bruce Landman: thank you very much for your help and patience with this paper.

References

[1] G. F. C. de Bruyn, Formulas for $a + a^2 2^p + a^3 3^p + \cdots + a^n n^p$, *Fibonacci Quart.* **33** (1995), 98–103.

[2] L. Carlitz, Note on a paper of Shanks, *Amer. Math. Monthly* **59** (4) (1952), 239–241.

[3] L. Carlitz, A note on q-Eulerian numbers, *J. Combin. Theory Ser. A* **25** (1978), 90–94.

[4] L. Carlitz, Eulerian numbers and polynomials, *Math. Mag.* **32** (1959), 247–260.

[5] L. Carlitz, q-Bernoulli and Eulerian numbers, *Trans. Amer. Math. Soc.* **76** (1954), 332–350.

[6] L. Carlitz, A combinatorial property of q-Eulerian numbers, *Amer. Math. Monthly* **82** (1975), 51–54.

[7] L. Carlitz, Eulerian numbers and polynomials of higher order, *Duke Math. J.* **27** (1960), 401–423.

[8] L. Carlitz, Extended Bernoulli and Eulerian numbers, *Duke Math. J.* **31** (1964), 667–689.

[9] L. Carlitz and J. Riordan, Congruences for Eulerian numbers, *Duke Math. J.* **20** (1953), 339–343.

[10] L. Euler, *Institutiones Calculi Differentialis,* Academiae Imperialis Scientiarum Petropolitanae, 1755.

[11] D. Foata, *Eulerian Polynomials: from Euler's Time to the Present*, in "The Legacy of Alladi Ramakrishnan in the Mathematical Sciences", Springer, 2010, 253–273.

[12] D. Foata and M. P. Schützenberger, *Théorie géométrique des polynômes Eulériens,* Lecture Notes in Mathematics **138**, Springer, 1970.

[13] H. W. Gould, *Combinatorial Identities*, Morgantown, W. Va. 1972.

[14] U. Graf, *Applied Laplace Transforms and z-Transforms for Scientists and Engineers: A Computational Approach Using a* Mathematica *Package*, Birkhäuser, 2004.

[15] D. H. Lehmer, Generalized Eulerian numbers, *J. Combin. Theory Ser. A* **32** (1982), 195–215.

[16] Z. Lin, On some generalized q-Eulerian polynomials, DMTCS proc. **AS.** 2013, 439–450.

[17] T. K. Petersen, *Eulerian Numbers,* Birkhäuser, 2015.

[18] C. Pita-Ruiz, On a generalization of eulerian numbers, Submitted.

[19] C. D. Savage and G. Viswanathan, The $1/k$-Eulerian polynomials, Available at: `http://www4.ncsu.edu/~savage/PAPERS/The_1_over_k_Eulerian_Polynomials.pdf`

[20] H. S. Vandiver, An arithmetical theory of the Bernoulli numbers, *Trans. Amer. Math. Soc.* **51** (1942), 502–531.

[21] R. Vilch, *Z Transform. Theory and Applications,* D. Reidel Publishing Company, 1987.

[22] H. S. Wilf, *Generatingfunctionology,* Academic Press Inc., 1994.

[23] J. Worpitzky, Studien über die Bernoullischen und Eulerschen zahlen, *J. Reine Angew. Math.* **94** (1883), 203–232.

[24] T. Xiong, H. P. Tsao, and J. I. Hall, General Eulerian numbers and Eulerian polynomials, *Journal of Mathematics* **2013** (2013), ID 629132.