IDENTITIES FOR THE GENERALIZED FIBONACCI POLYNOMIAL

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Abstract

A second order polynomial sequence is of Fibonacci type (Lucas type) if its Binet formula is similar in structure to the Binet formula for the Fibonacci (Lucas) numbers. In this paper we generalize identities from Fibonacci numbers and Lucas numbers to Fibonacci type and Lucas type polynomials. A Fibonacci type polynomial is equivalent to a Lucas type polynomial if they both satisfy the same recurrence relations. Most of the identities provide relationships between two equivalent polynomials. In particular, each type of identities in this paper relate the following polynomial sequences: Fibonacci with Lucas, Pell with Pell-Lucas, Fermat with Fermat-Lucas, both types of Chebyshev polynomials, Jacobsthal with Jacobsthal-Lucas and both types of Morgan-Voyce.

1. Introduction

A second order polynomial sequence is of *Fibonacci type* (*Lucas type*) if its Binet formula has a structure similar to that for Fibonacci (Lucas) numbers. In the

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literature, these types of sequences are known as *Generalized Fibonacci Polynomial* (GFP). They are actually a natural generalization of the sequence $F_n(x) = xF_{n-1}(x) + 1F_{n-2}(x)$ where $F_0(x) = 0$, $F_1(x) = 1$, called the *Fibonacci polynomial sequence*. However, there is no unique generalization of this sequence, one can refer to articles by several authors like André-Jeannin [1, 2], Bergum et al. [4], and Flórez et al. [6, 7], to see this. In this paper we use the definition of the GFP given in [6, 7, 9]. Flórez et al. [6], proved the strong divisibility property for the GFP, but while working on it the authors needed some identities involving the polynomials. When searching through the existing literature they realized that there were a limited number of those identities, even just for Fibonacci polynomials.

The study of identities for Fibonacci polynomials and Lucas polynomials have received less attention than their counterparts for numerical sequences, even if many of these identities can be proved easily. A natural question to ask is: under what conditions is it possible to extend identities that already exist for Fibonacci and Lucas numbers to the GFP? We observe here that the identities involving Fibonacci and Lucas numbers extend naturally to the GFP that satisfy closed formulas similar to the Binet formulas satisfied by Fibonacci and Lucas numbers.

While this paper does not intend to prove new numerical identities, we have concluded with a substantial list of known numerical identities extended to the GFP. We also adapt some known identities given for Fibonacci or Lucas polynomials to the GFP.

Bergum and Hoggatt [4] gave a list of more than twenty identities for their definition of generalized Fibonacci polynomials. Koshy [13] has also a big collection of identities for Fibonacci polynomials and Lucas polynomials. We generalize or adapt many of the identities given by either Bergum et al. [4], Wu et al. [20], or Koshy [13] to our definition of the GFP. Most of the numerical identities for Fibonacci and Lucas numbers that we extend to GFP can be found in the books, articles or webpages on Fibonacci and Lucas numbers and their applications, [8, 14, 16, 17, 12, 19].

We note that once we have an identity –from literature– for numerical sequences it is not too complicated to extend this to GFPs. However, the numerical identities do not extend automatically to GFPs; they need some adjustments (some of the identities in the paper were balanced using *Mathematica*®). Therefore, the aim of this paper is to give a collection of identities for the GFP. In particular, the identities that we give here apply to the following familiar polynomial sequences: Fibonacci polynomials, Lucas polynomials, Pell polynomials, Pell-Lucas polynomials, Fermat polynomials, Fermat-Lucas polynomials, Chebyshev first kind polynomials, Chebyshev second kind polynomials, Jacobsthal polynomials, Jacobsthal-Lucas polynomials, and Morgan-Voyce polynomials.

2. Generalized Fibonacci Polynomials

In this section we introduce the generalized Fibonacci polynomial sequences. This definition gives rise to some known polynomial sequences which are mentioned below. The definition matches the definitions of polynomial sequences given in other papers on this topic, for example the definition given by Flórez et al., Hoggatt et al., and Koshy [6, 9, 14], respectively.

For the remaining part of this section we reproduce the definitions by Flórez et al. [5, 6, 7], for generalized Fibonacci polynomials. We now give the two second order polynomial recurrence relations in which we divide the GFP.

$$\mathcal{F}_0(x) = 0,\ \mathcal{F}_1(x) = 1,\ \text{and}\ \mathcal{F}_n(x) = d(x)\mathcal{F}_{n-1}(x) + g(x)\mathcal{F}_{n-2}(x) \text{ for } n \geq 2 \quad (1)$$

where $d(x)$, and $g(x)$ are fixed non-zero polynomials in $\mathbb{Q}[x]$.

We say a polynomial recurrence relation is of *Fibonacci type* if it satisfies the relation given in (1), and of *Lucas type* if:

$$\mathcal{L}_0(x) = p_0,\ \mathcal{L}_1(x) = p_1(x),\ \text{and}\ \mathcal{L}_n(x) = d(x)\mathcal{L}_{n-1}(x) + g(x)\mathcal{L}_{n-2}(x) \text{ for } n \geq 2 \quad (2)$$

where $|p_0| = 1$ or 2 and $p_1(x)$, $d(x) = \alpha p_1(x)$, and $g(x)$ are fixed non-zero polynomials in $\mathbb{Q}[x]$ with α an integer of the form $2/p_0$.

To use similar notation for (1) and (2) on certain occasions we write $p_0 = 0$, $p_1(x) = 1$ to indicate the initial conditions of Fibonacci type polynomials. Some known examples of Fibonacci type polynomials and of Lucas type polynomials are in Table 2 (see also [6, 7, 18, 11, 14]).

Suppose that G_n is either $\mathcal{F}_n(x)$ or $\mathcal{L}_n(x)$ for all $n \geq 0$ and $d^2(x) + 4g(x) > 0$, then the explicit formula for the recurrence relations in (1) and (2) is given by

$$G_n(x) = t_1 a^n(x) + t_2 b^n(x)$$

where $a(x)$ and $b(x)$ are the solutions of the quadratic equation associated with the second order recurrence relation $G_n(x)$. That is, $a(x)$ and $b(x)$ are the solutions of $z^2 - d(x)z - g(x) = 0$. If $\alpha = 2/p_0$, then the Binet formula for Fibonacci type polynomials is stated in (3) and the Binet formula for Lucas type polynomials is stated in (4) (for details on the construction of the two Binet formulas see [6]).

$$\mathcal{F}_n(x) = \frac{a^n(x) - b^n(x)}{a(x) - b(x)} \quad (3)$$

and

$$\mathcal{L}_n(x) = \frac{a^n(x) + b^n(x)}{\alpha}. \quad (4)$$

Note that for both types of sequences:

$$a(x)+b(x)=d(x), \quad a(x)b(x)=-g(x), \quad \text{and} \quad a(x)-b(x)=\sqrt{d^2(x)+4g(x)}$$

where $d(x)$ and $g(x)$ are the polynomials defined in (1) and (2).

A sequence of Lucas type (Fibonacci type) is *equivalent* or *conjugate* to a sequence of Fibonacci type (Lucas type), if their recursive sequences are determined by the same polynomials $d(x)$ and $g(x)$. Notice that two equivalent polynomials have the same $a(x)$ and $b(x)$ in their Binet representations. Examples of equivalent polynomials are given in Table 2. Note that the leftmost polynomials in Table 2 are of Lucas type and their equivalent Fibonacci type polynomials are in the second column on the same line. Table 1 shows some familiar examples of those types of polynomial sequences.

Polynomial	Initial value $G_0(x)=p_0(x)$	Initial value $G_1(x)=p_1(x)$	Recursive Formula $G_n(x)=d(x)G_{n-1}(x)+g(x)G_{n-2}(x)$
Fibonacci	0	1	$F_n(x)=xF_{n-1}(x)+F_{n-2}(x)$
Lucas	2	x	$D_n(x)=xD_{n-1}(x)+D_{n-2}(x)$
Pell	0	1	$P_n(x)=2xP_{n-1}(x)+P_{n-2}(x)$
Pell-Lucas	2	$2x$	$Q_n(x)=2xQ_{n-1}(x)+Q_{n-2}(x)$
Pell-Lucas-prime	1	x	$Q'_n(x)=2xQ'_{n-1}(x)+Q'_{n-2}(x)$
Fermat	0	1	$\Phi_n(x)=3x\Phi_{n-1}(x)-2\Phi_{n-2}(x)$
Fermat-Lucas	2	$3x$	$\vartheta_n(x)=3x\vartheta_{n-1}(x)-2\vartheta_{n-2}(x)$
Chebyshev second kind	0	1	$U_n(x)=2xU_{n-1}(x)-U_{n-2}(x)$
Chebyshev first kind	1	x	$T_n(x)=2xT_{n-1}(x)-T_{n-2}(x)$
Jacobsthal	0	1	$J_n(x)=J_{n-1}(x)+2xJ_{n-2}(x)$
Jacobsthal-Lucas	2	1	$j_n(x)=j_{n-1}(x)+2xj_{n-2}(x)$
Morgan-Voyce	0	1	$B_n(x)=(x+2)B_{n-1}(x)-B_{n-2}(x)$
Morgan-Voyce	2	$x+2$	$C_n(x)=(x+2)C_{n-1}(x)-C_{n-2}(x)$
Vieta	0	1	$V_n(x)=xV_{n-1}(x)-V_{n-2}(x)$
Vieta-Lucas	2	x	$v_n(x)=xv_{n-1}(x)-v_{n-2}(x)$

Table 1: Recurrence relation of some GFP.

Note. The definition of the Generalized Fibonacci Polynomial by Flórez et al. [7], differs from the definition in this paper due to the initial conditions of the Fibonacci type polynomials. Thus, the initial conditions for the Fibonacci type polynomials given by Flórez, Higuita, and Mukherjee [7] are $G_0(x)=p_0(x)=1$ and so implicitly $G_{-1}(x)=0$. However, our definition for the Lucas type polynomial is identical to that found in the same article.

3. Identities

In this section we give a collection of identities for the GFP. These identities apply, in particular, to our familiar set of GFP. Thus, the identities apply to: Fibonacci

Polynomial of First type $\mathcal{L}_n(x)$	Polynomial of Second type $\mathcal{F}_n(x)$	α	$d(x)$	$g(x)$	$a(x)$	$b(x)$
$D_n(x)$	$F_n(x)$	1	x	1	$(x+\sqrt{x^2+4})/2$	$(x-\sqrt{x^2+4})/2$
$Q'_n(x)$	$P_n(x)$	1	$2x$	1	$x+\sqrt{x^2+1}$	$x-\sqrt{x^2+1}$
$\vartheta_n(x)$	$\Phi_n(x)$	1	$3x$	-2	$(3x+\sqrt{9x^2-8})/2$	$(3x-\sqrt{9x^2-8})/2$
$T_n(x)$	$U_n(x)$	2	$2x$	-1	$x+\sqrt{x^2-1}$	$x-\sqrt{x^2-1}$
$j_n(x)$	$J_n(x)$	1	1	$2x$	$(1+\sqrt{1+8x})/2$	$(1-\sqrt{1+8x})/2$
$C_n(x)$	$B_n(x)$	1	$x+2$	-1	$(x+2+\sqrt{x^2+4x})/2$	$(x+2-\sqrt{x^2+4x})/2$
$v_n(x)$	$V_n(x)$	1	x	-1	$(x+\sqrt{x^2-4})/2$	$(x-\sqrt{x^2-4})/2$

Table 2: Binet formulas for Lucas type $L_n(x)$ and its equivalent Fibonacci type $R_n(x)$.

polynomials, Lucas polynomials, Pell polynomials, Pell-Lucas polynomials, Fermat polynomials, Fermat-Lucas polynomials, Chebyshev first kind polynomials, Chebyshev second kind polynomials, Jacobsthal polynomials, Jacobsthal-Lucas polynomials and Morgan-Voyce polynomials. The identities below and their proofs are expressed in terms of α, $d(x)$, $g(x)$, $a(x)$, and $b(x)$. Table 2 gives those values for the polynomials mentioned above. If one is interested in identities for the special case of Fibonacci polynomials and Lucas polynomials, they may see a larger collection in Koshy [13].

For example, suppose we want to apply the identity in Proposition 3 part (1) which is $(a(x)-b(x))^2\mathcal{F}_n(x) = \alpha(\mathcal{L}_{n+1}(x) + g\mathcal{L}_{n-1}(x))$, to Chebyshev polynomials of the first and second kind, namely $T_n(x)$ and $U_n(x)$ respectively. We begin by getting the appropriated information for $U_n(x)$ and $T_n(x)$ given in Table 2 line 4. Thus, $a(x) = x+\sqrt{x^2-1}$, $b(x) = x-\sqrt{x^2-1}$, $\alpha = 2$, and $g(x) = -1$. Next, we observe that $U_n(x)$ is Fibonacci type, so $\mathcal{F}_n(x)$ equals $U_n(x)$ and from Table 2 line 4, we get that its equivalent Lucas type polynomial $\mathcal{L}_n(x)$ equals $T_n(x)$. Note that $(a(x)-b(x))^2 = 4x^2-4$. Now substituting all this information into the identity given in Proposition 3 part (1) we obtain,

$$(4x^2-4)U_n(x) = 2\left(T_{n+1}(x) + (-1)T_{n-1}(x)\right), \text{ for all } n.$$

If we want to apply the same identity for Jacobsthal $J_n(x)$, then we substitute the information given in Table 2 line 5 into Proposition 3 part (1). So, we have

$$(1+8x)J_n(x) = j_{n+1}(x) + (2x)j_{n-1}(x), \text{ for all } n.$$

The identities (1)–(3), (5)–(15), (42), (49)–(51), (56), (57), (63), (65), (67)–(69) are generalizations of numerical identities by Vajda [19]. The identities (16)–(39), (54), (55), (64), (82)–(94) are generalizations of numerical identities provided by Koshy [14]. The identities (4), (40), (41), (44)–(46), (52), (53), (58)–(62), (66), (70)–(81) are generalizations of numerical identities found in the webpage [12]. The identities (43), (47), (48) are generalizations of numerical identities by Benjamin and Quinn [3]. The identities (95)–(100) are generalizations of numerical identities present in [15].

Note. For the sake of simplicity throughout the rest of this paper, we use $\mathcal{F}_0$, $\mathcal{L}_0$, d, g, a, and b instead of $\mathcal{F}_0(x)$, $\mathcal{L}_0(x)$, $d(x)$, $g(x)$, $a(x)$, and $b(x)$, respectively.

The expression $(a-b)^2$ appears very often in the following list of identities. It is easy to see that

$$(a-b)^2 = \alpha\mathcal{L}_2 + 2g, \quad (a-b)^2 = \mathcal{F}_3 + 3g \quad \text{and} \quad (a-b)^2 = d^2 + 4g.$$

However, in the following list of identities we keep the factor $(a-b)^2$ as is.

Proposition 2 parts (1) and (2) are generalizations of Cassini and Catalan identities respectively. One can refer to the book by Koshy [14] to learn more about them. We have found that all of the identities in this paper can be proved by substituting the Binet formula on each side of the proposed identity. Since the reader can check the proofs without major difficulty, we prove some identities and leave the others as exercises.

Proposition 1 ([6]). *If $\{\mathcal{L}_n\}$ and $\{\mathcal{F}_n\}$ are equivalent generalized Fibonacci polynomial sequences, then*

(1) $\mathcal{F}_{m+n+1} = \mathcal{F}_{m+1}\mathcal{F}_{n+1} + g\mathcal{F}_m\mathcal{F}_n$

(2) if $n \geq m$, then $\mathcal{F}_{n+m} = \alpha\mathcal{F}_n\mathcal{L}_m - (-g)^m\mathcal{F}_{n-m}$

(3) if $n \geq m$, then $\mathcal{F}_{n+m} = \alpha\mathcal{F}_m\mathcal{L}_n + (-g)^m\mathcal{F}_{n-m}$

(4) $(a-b)^2\mathcal{F}_{m+n+1} = \alpha^2\mathcal{L}_{m+1}\mathcal{L}_{n+1} + \alpha^2 g\mathcal{L}_m\mathcal{L}_n$.

Proposition 2. *If $\{\mathcal{L}_n\}$ and $\{\mathcal{F}_n\}$ are equivalent generalized Fibonacci polynomial sequences, then*

(1) $\mathcal{F}_{n+1}\mathcal{F}_{n-1} - \mathcal{F}_n^2 = (-1)^n g^{n-1}$

(2) if $n \geq m$, then $\mathcal{F}_n^2 - (-g)^{n-m}\mathcal{F}_m^2 = \mathcal{F}_{n+m}\mathcal{F}_{n-m}$.

Proof of part (1). We prove this part using Binet formula (3) from Section 2. Therefore, substituting formula (3) in $\mathcal{F}_{n+1}\mathcal{F}_{n-1} - (\mathcal{F}_n)^2$, we obtain,

$$\left[(a^{n+1}-b^{n+1})(a^{n-1}-b^{n-1}) - (a^n-b^n)^2\right]/(a-b)^2.$$

Simplifying this expression, we obtain

$$\frac{2(ab)^n - (a^{n+1}b^{n-1} + b^{n+1}a^{n-1})}{(a-b)^2}.$$

Factoring and simplifying we see that the last expression reduces to $-(ab)^{n-1}$, and this is equal to $(-1)^n g^{n-1}$. □

Proof of part (2). Using Binet formula (3) in $\mathcal{F}_n^2 - (-g)^{n-m}\mathcal{F}_m^2$, we have,

$$\left[(a^n - b^n)^2 - (ab)^{n-m}(a^m - b^m)^2\right]/(a-b)^2.$$

Simplifying the right hand side of the last expression we obtain,

$$\left[a^{2n} + b^{2n} - a^{n+m}b^{n-m} - a^{n-m}b^{n+m}\right]/(a-b)^2.$$

Factoring the last expression, we get $\left[(a^{n+m} - b^{n+m})(a^{n-m} - b^{n-m})\right]/(a-b)^2$, and this is equal to $\mathcal{F}_{n+m}\mathcal{F}_{n-m}$. □

Proposition 3. *Let $\{\mathcal{L}_n\}$ and $\{\mathcal{F}_n\}$ be equivalent generalized Fibonacci polynomial sequences. If m and n are positive integers, then*

(1) $(a-b)^2\mathcal{F}_n = \alpha(\mathcal{L}_{n+1} + g\mathcal{L}_{n-1})$

(2) $g\mathcal{F}_{n-1} + \mathcal{F}_{n+1} = \alpha\mathcal{L}_n$

(3) $\mathcal{F}_{n+2} - g^2\mathcal{F}_{n-2} = \alpha^2\mathcal{L}_1\mathcal{L}_n$

(4) $\mathcal{F}_{n+2} + g^2\mathcal{F}_{n-2} = (d^2 + 2g)\mathcal{F}_n$

(5) $\mathcal{F}_{n+2} + g^2\mathcal{F}_{n-2} = \alpha\mathcal{L}_2\mathcal{F}_n$

(6) $\alpha\mathcal{L}_n\mathcal{F}_n = \mathcal{F}_{2n}$

(7) $\alpha(\mathcal{F}_{n+1}\mathcal{L}_{n+1} - \mathcal{F}_n\mathcal{L}_n) = \mathcal{F}_{2n+2} - \mathcal{F}_{2n}$

(8) $\alpha(\mathcal{L}_m\mathcal{F}_n + \mathcal{L}_n\mathcal{F}_m) = 2\mathcal{F}_{n+m}$

(9) *if* $n \geq m$, *then* $\alpha(\mathcal{L}_m\mathcal{F}_n - \mathcal{L}_n\mathcal{F}_m) = 2(-g)^m\mathcal{F}_{n-m}$

(10) *if* $\alpha = 1$, *then* $2\mathcal{L}_{2n} - \mathcal{L}_n^2 = (a-b)^2\mathcal{F}_n^2$

(11) $\mathcal{L}_{2n} - 2(-g)^n = \left(((a-b)\mathcal{F}_n)^2 - 2(\alpha-1)(-g)^n\right)/\alpha$

(12) *if* $\alpha = 1$, *then* $(a-b)^2\mathcal{F}_n^2 - \mathcal{L}_n^2 = 4(-1)^{n+1}g^n$

(13) $\alpha^2\left(g{\mathcal{L}_n}^2 + \mathcal{L}_{n+1}^2\right) = (a-b)^2\left(g\mathcal{F}_n^2 + \mathcal{F}_{n+1}^2\right) = (a-b)^2\mathcal{F}_{2n+1}$

(14) $\mathcal{L}_2\mathcal{F}_n + \alpha\mathcal{L}_1\mathcal{L}_n = 2\mathcal{F}_{n+2}/\alpha$

(15) $(a-b)^2\mathcal{L}_1\mathcal{F}_n + \alpha\mathcal{L}_2\mathcal{L}_n = 2\mathcal{L}_{n+2}$

(16) $\alpha\mathcal{F}_{n+1}\mathcal{L}_n = \mathcal{F}_{2n+1} + (-g)^n$

(17) *if* $n \neq m$, *then* $\mathcal{L}_m\mathcal{F}_{n-m+1} + g\mathcal{L}_{m-1}\mathcal{F}_{n-m} = \mathcal{L}_n$

(18) *if* $n \geq m$, *then* $\alpha^2\mathcal{L}_{m+n}^2 + g^{2n}(a-b)^2\mathcal{F}_{m-n}^2 = \alpha^2\mathcal{L}_{2n}\mathcal{L}_{2m}$

(19) *if* $m \geq n$, *then* $\alpha^2 g^{2n} \mathcal{L}_{m-n}^2 + (a-b)^2 \mathcal{F}_{m+n}^2 = \alpha^2 \mathcal{L}_{2n} \mathcal{L}_{2m}$

(20) *if* $m \geq n$, *then* $\alpha^2 \left(\mathcal{L}_{2m+2n}^2 - g^{4n} \mathcal{L}_{2m-2n}^2 \right) = (a-b)^2 \mathcal{F}_{4m} \mathcal{F}_{4n}$

(21) $(a-b)^2 \mathcal{F}_{2n}^2 + 2g^{2n} = \alpha \mathcal{L}_{4n}$

(22) $(a-b)^2 \mathcal{F}_{2n+1}^2 - 2(g)^{2n+1} = \alpha \mathcal{L}_{4n+2}$

(23) $\alpha \mathcal{L}_n \mathcal{F}_{2n} - (-g)^n \mathcal{F}_n = \mathcal{F}_{3n}$

(24) $\mathcal{F}_n \left(\alpha \mathcal{L}_{2n} + (-g)^n \right) = \mathcal{F}_{3n}$

(25) $\mathcal{F}_{4n+1} - g^{2n} = \alpha \mathcal{L}_{2n+1} \mathcal{F}_{2n}$

(26) $\mathcal{F}_{4n+3} + (-g)^{2n+1} = \alpha \mathcal{L}_{2n+1} \mathcal{F}_{2n+2}$

(27) $\alpha \mathcal{L}_{n+1} - (a-b)^2 \mathcal{F}_n = \alpha(-g) \mathcal{L}_{n-1}$

(28) $\alpha \left(\mathcal{F}_{n+1} \mathcal{L}_{n+2} - d \mathcal{F}_{n+2} \mathcal{L}_n \right) = g \mathcal{F}_{2n+1} - (-g)^n (d^2 - g)$

(29) *if* $n \geq m$, *then* $(a-b)^2 \left(\mathcal{F}_{n+m}^2 + g^{2m} \mathcal{F}_{n-m}^2 \right) = \alpha^2 \mathcal{L}_{2n} \mathcal{L}_{2m} - 4(-g)^{n+m}$

(30) *if* $n \geq m$, *then*

$$(a-b)^2 \left(\mathcal{F}_{n+m} \mathcal{F}_{n+m+1} + g^{2m} \mathcal{F}_{n-m} \mathcal{F}_{n-m+1} \right) = \alpha^2 \mathcal{L}_{2n+1} \mathcal{L}_{2m} - 2(-g)^{n+m} d$$

(31) $\mathcal{F}_{n+4} \mathcal{F}_{n+1}^2 - \mathcal{F}_n \mathcal{F}_{n+3}^2 = \alpha d (-g)^n \mathcal{L}_{n+2}$

(32) $\alpha \left(\mathcal{F}_{n+4} \mathcal{L}_{n+1}^2 - \mathcal{F}_n \mathcal{L}_{n+3}^2 \right) = d^3 (-g)^n \mathcal{L}_{n+2}$

(33) $2\alpha \mathcal{L}_{m+n} = \alpha^2 \mathcal{L}_m \mathcal{L}_n + (a-b)^2 \mathcal{F}_m \mathcal{F}_n$

(34) $\alpha(\mathcal{L}_{m+n} - (-g)^n \mathcal{L}_{m-n}) = (a-b)^2 \mathcal{F}_m \mathcal{F}_n$

(35) $\alpha^2 \mathcal{L}_{2m+1} \mathcal{L}_{2n+1} = \alpha^2 \mathcal{L}_{m+n+1}^2 - (a-b)^2 (g)^{2n+1} \mathcal{F}_{m-n}^2$

(36) *if* $n \geq m$, *then*

$$\alpha(a-b) \mathcal{F}_n \mathcal{L}_{n+m} - \alpha^2 \mathcal{L}_n \mathcal{L}_{n-m}$$

equals

$$(a-b)(\mathcal{F}_{2n+m} - (-g)^n \mathcal{F}_m) - \alpha \left(\mathcal{L}_{2n-m} - (-g)^{n-m} \mathcal{L}_n \right)$$

(37) $\alpha^2 \mathcal{L}_{n-1} \mathcal{L}_{n+1} - (a-b)^2 (\mathcal{F}_n)^2 = (-g)^{n-1} \left(\alpha \mathcal{L}_2 - 2g \right)$

(38) $(a-b)^2 \mathcal{F}_{2n+3} \mathcal{F}_{2n-3} = \alpha \left(\mathcal{L}_{4n} - (-g)^{2n-3} \mathcal{L}_6 \right)$

(39) $\mathcal{L}_{5n} = \mathcal{L}_n \left[(\alpha \mathcal{L}_{2n} - (-g)^n)^2 + (a-b)^2 (-g)^n \mathcal{F}_n^2 \right]$

(40) $\mathcal{F}_{n+5} - g^2 \mathcal{F}_{n+1} = d \alpha \mathcal{L}_{n+3}$

(41) $\mathcal{F}_{n+5} + g^2 \mathcal{F}_{n+1} = \alpha \mathcal{L}_2 \mathcal{F}_{n+3}$

(42) $d\mathcal{F}_n^2 + 2g\mathcal{F}_{n-1}\mathcal{F}_{n+1} = \mathcal{F}_{2n}$

(43) $\mathcal{F}_{n+1}^2 - g^2\mathcal{F}_{n-1}^2 = d\mathcal{F}_{2n}$

(44) $\mathcal{F}_{n+3}^2 + g^3\mathcal{F}_n^2 = \mathcal{F}_{2n+3}\mathcal{F}_3$

(45) *if* $n \geq m$, *then* $\mathcal{F}_{n+m+1}^2 + g^{2m+1}\mathcal{F}_{n-m}^2 = \mathcal{F}_{2n+1}\mathcal{F}_{2m+1}$

(46) *if* $n \geq m$, *then* $\mathcal{F}_{n+m}^2 - g^{2m}\mathcal{F}_{n-m}^2 = \mathcal{F}_{2n}\mathcal{F}_{2m}$

(47) *if* $n \geq m$, *then* $\mathcal{F}_n\mathcal{F}_{m+1} - \mathcal{F}_m\mathcal{F}_{n+1} = (-g)^m\mathcal{F}_{n-m}$

(48) *if* $n \geq m$, *then* $\mathcal{F}_{n+1}\mathcal{F}_{m+1} - g^2\mathcal{F}_{n-1}\mathcal{F}_{m-1} = d\mathcal{F}_{n+m}$

(49) *if* $n \geq m$, *then* $\mathcal{F}_{n+1}\mathcal{F}_m + g\mathcal{F}_n\mathcal{F}_{m-1} = \mathcal{F}_{n+m}$

(50) *if* $n \geq m$, *then* $\mathcal{F}_{n-m+1}\mathcal{F}_m + g\mathcal{F}_{n-m}\mathcal{F}_{m-1} = \mathcal{F}_n$

(51) $\mathcal{F}_n\mathcal{F}_{n+1} - \mathcal{F}_{n-1}\mathcal{F}_{n+2} = d(-g)^{n-1}$

(52) *if* $i \geq 0$, *then* $\mathcal{F}_{n+i}\mathcal{F}_{n+m} - \mathcal{F}_n\mathcal{F}_{n+m+i} = (-g)^n\mathcal{F}_i\mathcal{F}_m$

(53) $\sqrt{(a-b)^2\mathcal{F}_n^2 + 4(-g)^n} - g\mathcal{F}_{n-1} = \mathcal{F}_{n+1}$

(54) *if* $r \leq \min\{m, n, s, t\}$ *and* $m + n = s + t$, *then*

$$\mathcal{F}_m\mathcal{F}_n - \mathcal{F}_s\mathcal{F}_t = (-g)^r\left(\mathcal{F}_{m-r}\mathcal{F}_{n-r} - \mathcal{F}_{s-r}\mathcal{F}_{t-r}\right)$$

(55) $\mathcal{F}_{n+2}\mathcal{F}_{m+1} - g^2\mathcal{F}_n\mathcal{F}_{m-1} = d\mathcal{F}_{n+m+1}$

(56) $\mathcal{L}_{2n}\left[\alpha^2(\mathcal{L}_{4n} - g^{2n}/\alpha)^2 + g^{2n}(a-b)^2\mathcal{F}_{2n}^2\right] = \mathcal{L}_{10n}$

(57) $\mathcal{L}_{2n} + 2(-g)^n/\alpha = \alpha\mathcal{L}_n^2$

(58) *if* $n \geq m$, *then* $\mathcal{L}_{n+m} + (-g)^m\mathcal{L}_{n-m} = \alpha\mathcal{L}_m\mathcal{L}_n$

(59) $(a-b)\sqrt{\mathcal{L}_n^2 - 4(-g)^n/\alpha^2} - g\mathcal{L}_{n-1} = \mathcal{L}_{n+1}$

(60) $(a-b)^2\mathcal{F}_{n+2}^2 + g^2\alpha^2\mathcal{L}_n^2 = \alpha^2\mathcal{L}_2\mathcal{L}_{2n+2}$

(61) $\alpha^2\left(\mathcal{L}_{n+1}^2 - g^2\mathcal{L}_{n-1}^2\right) = (a-b)^2\mathcal{F}_2\mathcal{F}_{2n}$

(62) $\alpha^2\mathcal{L}_{n+1}^2 - (a-b)^2 g\mathcal{F}_n^2 = \alpha^2 d\mathcal{L}_{2n+1}$

(63) *if* $n \geq m$, *then* $\mathcal{L}_{n+m} + (-g)^m\mathcal{L}_{n-m} = \alpha\mathcal{L}_n\mathcal{L}_m$

(64) $\mathcal{F}_{n+1}\mathcal{L}_{n+1} + g\mathcal{F}_n\mathcal{L}_n = \mathcal{L}_{2n+1}$

(65) *if* $m \geq n$, *then* $\mathcal{F}_m\mathcal{F}_{2m}\mathcal{F}_{3n} = \mathcal{F}_{m+n}^3 - (-g)^{3n}\mathcal{F}_{m-n}^3 - \alpha(-g)^m\mathcal{F}_n^3\mathcal{L}_m$

(66) $(a-b)^2\left[\mathcal{F}_n^2 + \mathcal{F}_{n+1}^2\right] = \alpha^2\left[\mathcal{L}_n^2 + \mathcal{L}_{n+1}^2\right] + 4(-g)^n(g-1)$

(67) $\mathcal{F}_m\mathcal{L}_n + g\mathcal{F}_{m-1}\mathcal{L}_{n-1} = \mathcal{L}_{m+n-1}$

(68) $\mathcal{F}_{n+m+i}\mathcal{L}_n - \mathcal{F}_{n+m}\mathcal{L}_{n+i} = (-g)^n\mathcal{L}_m\mathcal{F}_i$

(69) $\mathcal{F}_{n+i}\mathcal{L}_{n+m} - \mathcal{F}_n\mathcal{L}_{n+m+i} = (-g)^n\mathcal{L}_m\mathcal{F}_i$

(70) $\alpha^2(\mathcal{L}_{n+m+i}\mathcal{L}_n - \mathcal{L}_{n+m}\mathcal{L}_{n+i}) = (-g)^n(a-b)^2\mathcal{F}_m\mathcal{F}_i$

(71) $(-g)^k\mathcal{F}_n\mathcal{F}_{m-k} + (-g)^m\mathcal{F}_{n-m}\mathcal{F}_k + (-g)^n\mathcal{F}_m\mathcal{F}_{k-n} = 0$

(72) $(-g)^k\mathcal{L}_n\mathcal{F}_{m-k} + (-g)^m\mathcal{L}_k\mathcal{F}_{n-m} + (-g)^n\mathcal{L}_m\mathcal{F}_{k-n} = 0$

(73) $(a-b)^2\mathcal{F}_{jk+r}\mathcal{F}_{ju+v} = \alpha\left[\mathcal{L}_{j(k+u)+r+v} - (-g)^{ju+v}\mathcal{L}_{j(k-u)+r-v}\right]$

(74) $\alpha\mathcal{F}_{jk+r}\mathcal{L}_{ju+v} = \left[\mathcal{F}_{j(k+u)+r+v} + (-g)^{ju+v}\mathcal{F}_{j(k-u)+r-v}\right]$

(75) $\alpha\mathcal{L}_{jk+r}\mathcal{L}_{ju+v} = \left[\mathcal{L}_{j(k+u)+r+v} + (-g)^{ju+v}\mathcal{L}_{j(k-u)+r-v}\right]$

(76) *if* $m+n = s+t$, *then* $\mathcal{F}_m\mathcal{L}_n - \mathcal{F}_s\mathcal{L}_t = (-g)^r\left[\mathcal{F}_{m-r}\mathcal{L}_{n-r} - \mathcal{F}_{s-r}\mathcal{L}_{t-r}\right]$

(77) *if* $m+n = s+t$, *then*

$$(a-b)^2\mathcal{F}_m\mathcal{F}_n - \alpha^2\mathcal{F}_s\mathcal{L}_t = (-g)^r\left[(a-b)^2\mathcal{F}_{m-r}\mathcal{F}_{n-r} - \alpha^2\mathcal{L}_{s-r}\mathcal{L}_{t-r}\right]$$

(78) $\mathcal{F}_{n+1}^3 + gd\mathcal{F}_n^3 - g^3\mathcal{F}_{n-1}^3 = d\mathcal{F}_{3n}$

(79) $\alpha^2\left[\mathcal{L}_{n+1}^3 + gd\mathcal{L}_n^3 - g^3\mathcal{L}_{n-1}^3\right] = d(a-b)^2\mathcal{L}_{3n}$

(80) $\alpha^2\left[\mathcal{F}_{n+1}\mathcal{L}_{n+1}^2 + gd\mathcal{F}_n\mathcal{L}_n^2 - g^3\mathcal{F}_{n-1}\mathcal{L}_{n-1}^2\right] = d(a-b)^2\mathcal{F}_{3n}$

(81) $\mathcal{L}_{n+1}\mathcal{F}_{n+1}^2 + gd\mathcal{L}_n\mathcal{F}_n^2 - g^3\mathcal{L}_{n-1}\mathcal{F}_{n-1}^2 = d\mathcal{L}_{3n}$

(82) $\mathcal{F}_{3n} = \alpha\mathcal{L}_3\mathcal{F}_{3n-3} + g^3\mathcal{F}_{3n-6}$

(83) $\mathcal{F}_{3n} = (a-b)^2\mathcal{F}_n^3 + 3(-g)^n\mathcal{F}_n$

(84) $\mathcal{F}_{r+s+t} = \mathcal{F}_{r+1}\mathcal{F}_{s+1}\mathcal{F}_{t+1} + gd\mathcal{F}_r\mathcal{F}_s\mathcal{F}_t - g^3\mathcal{F}_{r-1}\mathcal{F}_{s-1}\mathcal{F}_{t-1}$

(85) *if* $k < s < m$ *are non-negative integers, then*

$$\mathcal{F}_{m+k}\mathcal{F}_{m-k} - \mathcal{F}_{m+s}\mathcal{F}_{m-s} = (-g)^{m-s}\mathcal{F}_{s-k}\mathcal{F}_{s+k}$$

(86) *if* $r < n$ *is a positive integer, then* $\mathcal{F}_r\mathcal{F}_{m+n} = \mathcal{F}_{m+r}\mathcal{F}_n - (-g)^r\mathcal{F}_{n-r}\mathcal{F}_m$

(87) *if* $r < m$ *is a positive integer, then*

$$d\mathcal{F}_{2r}\left[(a-b)^2\mathcal{F}_m^2 + 2(-g)^m\right]$$

equals

$$\mathcal{F}_{m+r+1}^2 - g^2\mathcal{F}_{m+r-1}^2 - g^{2r}\mathcal{F}_{m-r+1}^2 + g^{2r+2}\mathcal{F}_{m-r-1}^2$$

(88) *if* $r < m$ *is a positive integer, then*

$$\alpha^2 \mathcal{L}_1 \mathcal{L}_{2m} \mathcal{F}_{2r} = \mathcal{F}^2_{m+r+1} - g^2 \mathcal{F}^2_{m+r-1} - g^{2r} \mathcal{F}^2_{m-r+1} + g^{2r+2} \mathcal{F}^2_{m-r-1}$$

(89) $\left(d^2 \mathcal{F}^2_{n+1} + \mathcal{F}^2_{n+2}\right)^2 = \mathcal{F}^2_n \left(\mathcal{F}_{n+4} - d^2 \mathcal{F}_{n+2}\right)^2 + (2d\mathcal{F}_{n+1}\mathcal{F}_{n+2})^2$
(This is an adaptation of the Pythagorean Theorem for the Fibonacci type polynomials)

(90) $\left(d^2 \mathcal{L}^2_{n+1} + \mathcal{L}^2_{n+2}\right)^2 = \mathcal{L}^2_n \left(\mathcal{L}_{n+4} - d^2 \mathcal{L}_{n+2}\right)^2 + (2d\mathcal{L}_{n+1}\mathcal{L}_{n+2})^2$
(This is an adaptation of the Pythagorean Theorem for the Lucas type polynomials)

(91) $\mathcal{F}^2_{n+2m+1} + g^{2m+1} \mathcal{F}^2_n = \mathcal{F}_{2m+1} \mathcal{F}_{2n+2m+1}$

(92) $\alpha^2 \left(\mathcal{F}^2_{n+m} \mathcal{L}^2_{n+m} - g^{2n} \mathcal{F}^2_m \mathcal{L}^2_m\right) = \mathcal{F}_{2n} \mathcal{F}_{4m+2n}$

(93) *if* $r < m$ *is a positive integer, then*

$$\alpha^2 \mathcal{L}_{2m-2} \mathcal{L}_{2r} - 2(-g)^{m+r-2}(2g + d^2)$$

equals

$$(a-b)^2 \left(\mathcal{F}_{m+r} \mathcal{F}_{m+r-2} + g^{2r} \mathcal{F}_{m-r} \mathcal{F}_{m-r-2}\right)$$

(94) $\left[\left(\alpha\mathcal{L}_n + (a-b)\mathcal{F}_n\right)/2\right]^m = \left(\alpha\mathcal{L}_{mn} + (a-b)\mathcal{F}_{mn}\right)/2$

(95) $\mathcal{F}_{3^{n+1}} = (a-b)^2 \mathcal{F}^3_{3^n} + 3(-g)^{3^n} \mathcal{F}_{3^n}$

(96) $\mathcal{F}_{5^{n+1}} = (a-b)^4 \mathcal{F}^5_{5^n} + 5(a-b)^2(-g)^{5^n} \mathcal{F}^3_{5^n} + 5(-g)^{2(5^n)} \mathcal{F}_{5^n}$

(97) $\mathcal{F}_{7^{n+1}} = A^6 \mathcal{F}^7_{7^n} + 7A^4(-g)^{7^n} \mathcal{F}^5_{7^n} + 14A^2(-g)^{2(7^n)} \mathcal{F}^3_{7^n} + 7(-g)^{3(7^n)} \mathcal{F}_{7^n}$, *where* $A = (a-b)$.

(98) $\mathcal{L}_{2^{n+1}} = \alpha \mathcal{L}^2_{2^n} - 2\alpha^{-1}(-g)^{2^n}$

(99) $\mathcal{L}_{4^{n+1}} = \alpha^3 \mathcal{L}^4_{4^n} - 4\alpha(-g)^{4^n} \mathcal{L}^2_{4^n} + 2\alpha^{-1}(-g)^{2(4^n)}$

(100) $\mathcal{L}_{6^{n+1}} = \alpha^5 \mathcal{L}^6_{6^n} - 6\alpha^3(-g)^{6^n} \mathcal{L}^4_{6^n} + 9\alpha(-g)^{2(6^n)} \mathcal{L}^2_{6^n} - 2\alpha^{-1}(-g)^{3(6^n)}$.

Proof of part (1). We know $\mathcal{F}_n$ is a Fibonacci type polynomial, so we can represent it using Binet formula (3). Therefore, $(a-b)^2\mathcal{F}_n$ can be written as $(a-b)^2[(a^n - b^n)/(a-b)]$. Simplifying, we see the above expression equals $(a^{n+1}+b^{n+1}-ab^n-ba^n)$. This reduces to $a^{n+1} + b^{n+1} + g(a^{n-1} + b^{n-1})$, since we know that $ab = -g$. It is easy to see that

$$a^{n+1} + b^{n+1} + g(a^{n-1} + b^{n-1}) = \alpha \left(\frac{a^{n+1} + b^{n+1}}{\alpha} + g\frac{a^{n-1} + b^{n-1}}{\alpha}\right).$$

This and Binet formula (4) give us $\alpha(\mathcal{L}_{n+1} + g\mathcal{L}_{n-1})$ and the proof is complete. □

Proof of part (2). Similar to part (1), we know $\mathcal{F}_n$ is a Fibonacci type polynomial, so from Binet formula (3) and $g = -ab$, we have

$$g\mathcal{F}_{n-1} + \mathcal{F}_{n+1} = \left(-ab(a^{n-1} - b^{n-1}) + a^{n+1} - b^{n+1}\right)/(a-b).$$

Simplifying the numerator we have $(a^n(a-b) + b^n(a-b))/(a-b)$. This and Binet formula (4) imply that $a^n + b^n = \alpha\mathcal{L}_n$. □

Proof of part (3). Substituting $\mathcal{F}_n$ with its Binet formula (3) and $-ab$ with g we obtain

$$\mathcal{F}_{n+2} - g^2\mathcal{F}_{n-2} = \left(a^{n+2} - b^{n+2} - a^2b^2(a^{n-2} - b^{n+2})\right)/(a-b).$$

Factoring the numerator we have $\left((a^2-b^2)(a^n+b^n)\right)/(a-b)$. This implies that,

$$\mathcal{F}_{n+2} - g^2\mathcal{F}_{n-2} = (a+b)(a^n+b^n).$$

Rewriting the right hand side we obtain

$$\mathcal{F}_{n+2} - g^2\mathcal{F}_{n-2} = \alpha^2\left(\frac{a+b}{\alpha}\right)\left(\frac{a^n+b^n}{\alpha}\right).$$

This and Binet formula (4) for Lucas type polynomials complete the proof. □

Proof of part (4). We use the Binet formula (3), $d = (a+b)$, and $g = -ab$ in the expression $[(d^2+2g)\mathcal{F}_n]$, to obtain $[(a^2+b^2)(a^n-b^n)]/(a-b)$. Expanding and simplifying we get $[(a^{n+2}-b^{n+2}) + (ab)^2(a^{n-2}-b^{n-2})]/(a-b)$ which is the same as $[\mathcal{F}_{n+2} + g^2\mathcal{F}_{n-2}]$. □

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