SOME NONLINEAR RADO NUMBERS

Kellen Myers
Departments of Mathematics and of Ecology & Evolutionary Biology, University of Tennessee, Knoxville, Tennessee
kellenmyers@gmail.com

Joseph Parrish
Kirkland, Washington

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Abstract

We discuss the computation of Rado numbers for several families of nonlinear equations, including equations comprised of sums of squares, in the spirit of the recently resolved conjecture of Erdős and Graham regarding the equation $x^2 + y^2 = z^2$. We provide (uniform) upper bounds for the 2- and 3-color Rado numbers of some such equations, as well as Rado numbers for a variety of other well-known nonlinear equations.

1. Introduction

Schur [42] shows that for any finite coloring of the positive integers (a function $\chi : \mathbb{Z}^+ \to \{1, 2, \ldots, r\}$), there will be a triple (x, y, z) such that $x + y = z$ and $\chi(x) = \chi(y) = \chi(z)$. This triple is said to be a monochromatic solution to the equation. The least N such that this statement holds for $\chi : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, r\}$ is called the r-color Schur number, which we will denote S_r.

It is easy to see that $S_2 = 5$, and one might note trivially that $S_1 = 2$. It is also straightforward to prove that $S_3 = 14$. The value $S_4 = 45$ is due to Baumert & Golomb [2], while Exoo [13] and Fredricksen [15] show that $161 \leq S_5 \leq 316$. Despite advances in computing, the difficulty of computing S_5 highlights the computational challenges in this area.

We can associate such a quantity to any equation $\mathcal{E}$, not just $x + y = z$, which we will denote $R_r(\mathcal{E})$. We make the definition formally as follows:

Definition 1 (Rado Number)**.** For a positive integer r and equation $\mathcal{E}$, $R_r(\mathcal{E})$ is the least number N such that any r-coloring of $\{1, 2, \ldots, N\}$ will yield a monochromatic solution to $\mathcal{E}$. We call this the *r-color Rado number of $\mathcal{E}$*.

In cases where no such N exists, we say $R_r(\mathcal{E}) = \infty$ (since the minimum of the empty set is ∞). An equation $\mathcal{E}$ is called r-regular if the quantity $R_r(\mathcal{E})$ is finite, and $\mathcal{E}$ is called regular if it is regular for all r. (In some literature, "regular" is "partition regular.")

Rado [34] provides necessary and sufficient conditions for homogeneous linear equations to be regular and 2-regular. For that reason, we call these quantities "Rado numbers." The conditions for 2-regularity are very lax (essentially just requiring that $\mathcal{E}$ is nontrivial).

Recently, there have been many results providing Rado numbers, mostly for the case $r = 2$ and almost exclusively for linear equations [3, 19, 24, 25, 26, 27, 33, 36, 37, 39, 41].

In this paper, we will stray from the safety of linear equations (and sometimes from 2 colors). We will provide Rado numbers for a few families of nonlinear equations, particularly those involving sums of squares. These results are related to a recently-proved conjecture of Erdős and Graham. We will discuss other families of Rado numbers inspired by previous work in Diophantine Ramsey Theory.

Our results will fall into two categories – some theorems, giving bounds and other characterizations of Rado numbers for various equation(s), will be presented with traditional proofs. Others will be computations, exhaustively and exactly computing Rado numbers for equation(s), often in cases where no other proof is yet known.

These computations were performed using standard algorithms in tree-searching and SAT-solving, and open-source implementation of such algorithms will be provided. Full certificates can be generated by these algorithms, but we often skip this step due to size restrictions in computing resources (depending on the method, hitting restrictions related to RAM and disk space or in the complexity of the computation). More details are provided below, and for greater discussion of these methods, see [1, 2, 21, 32].

2. Background

Consider the homogeneous linear equation with (nonzero) integer coefficients:

$$\sum_{i=1}^{n} a_i x_i = 0.$$

In [34], Rado gives the following two important theorems.

Theorem 2. *A linear equation (as above) is regular if and only if there is $J \subseteq \{1, 2, \ldots n\}$ such that $\sum_{i \in J} a_i = 0$.*

Theorem 3. *A linear equation (as above, with $n \geq 3$) is 2-regular if and only there are i_1, i_2 such that $a_{i_1} > 0$ and $a_{i_2} < 0$.*

Rado also gives an analogue of Theorem 2 with a criterion for systems of linear equations. The condition in Theorem 3 is quite weak; it only rules out equations that would have no solutions in the positive integers. Requiring $n \geq 3$ rules out equations like $3x = 2y$, and it is easy to see that $ax = by$ is 2-regular (indeed, regular) if and only if $a = b$.

Over time, a few theorems have been proved that give various Rado numbers; including the following.

Theorem 4 ([3])**.** *$R_2\left(x_1 + x_2 + \cdots + x_{m-1} = x_m\right) = m^2 - m - 1$ for $m \geq 2$.*

Theorem 5 ([20])**.** *$R_2\left(ax + ay = 2z\right) = \frac{a(a^2+1)}{2}$ for $a > 0$.*

Theorem 6 ([6])**.** *$R_2\left(x + y = kz\right) = \binom{k+1}{2}$ for $k \geq 4$.*

Theorem 7 ([40])**.** *$R_3\left(x + y + c = z\right) = 13c + 14$ for $c > 0$.*

The quantities in Theorem 4 are called "generalized Schur numbers" since the equation is similar to Schur's original $x + y = z$.

It is important to note that Theorem 7 is one of very few results proving or computing 3-color Rado numbers, which is a more formidable task considering there is no known necessary or sufficient condition(s) for 3-regularity.

Although there is much more to be done with Rado numbers for linear equations, we will turn our attention to nonlinear equations. We will first take a deeper look at equations comprised of sums of squares, followed by a few additional results regarding some other nonlinear equations that have appeared previously in the literature, mostly without quantitative results.

Before we proceed further, it is important to note that the work in this area spans many fields of mathematics and is a bit of a patchwork. For that reason, definitions, terminology, notation, and other conventions may vary. In particular, there are two significant conventions we will adopt that are not necessarily the same as those adopted by the authors of works cited here.

First, when considering Rado numbers like $R_3\left(x + y + 2z = y^2\right)$, we include solutions like $(x, y, z) = (2, 3, 2)$. So a coloring that assigns the integers 2 and 3 the same color includes a monochromatic solution. Some authors require distinct values for the variables in their equations (or other configurations). We do not normally make this restriction, although we can (and will) extend our definitions to this modified case where appropriate.

Second, we do not consider $R_2\left(x - y = n^2\right)$ to be the same as the least integer N such that coloring $\{1, 2, \ldots, N\}$ will yield two integers of the same color whose difference is a square (a result we will explore in Section 4). We would consider $R_2\left(x - y = n^2\right)$ to be the Rado number for an equation with three variables (x, y, and n), all of which would need to be monochromatic in the colorings we would expect to find. There are a few results we will discuss that do not prove the regularity of a particular equation due to variables like this n that are not required to be the same color as the values assigned to other variables like x and y.

2.1. Computational Methodology

Erdős and Graham [12] propose determining the 2-regularity of Pythagorean triples (solutions to $x^2 + y^2 = z^2$). Interest in this problem was renewed when Graham [18] stated that he believed it would be difficult to make this determination. However, Heule, Kullmann, & Marek [21] have computed $R_2\left(x^2 + y^2 = z^2\right) = 7825$, resolving a qualitative question by providing a quantitative answer, obtained with high-performance parallel computation.

The methods we will describe in this section were used to determine (independently and concurrently with [21]) that $R_2\left(x^2 + y^2 = z^2\right) > 7000$, as well as to determine many other Rado numbers of this type. We will focus on equations comprised of sums of squares, but also discuss several other nonlinear equations, once we have described our computational methods.

Our algorithms focus on computing $R_r(\mathcal{E})$ for a specific value of r and for an equation $\mathcal{E}$ with no unspecified coefficients or parameters. This narrowed focus allows us to make use of more precise, non-symbolic methods of computation. The result is, of course, limited by the specificity of the input, but for the Rado numbers we will compute in this paper, it is the appropriate trade-off to make.

We can proceed under a number of different paradigms. We will consider the problem in two ways: searching an r-ary tree and the satisfiability of Boolean clauses. These two methods are discussed at length in appendices, and at much greater length in the first author's Ph.D. thesis [32].

Both methods of computation focus on finding what we will call valid colorings, which themselves provide lower bounds – and the nonexistence of valid colorings, something harder perhaps to demonstrate, provides upper bounds.

Definition 8. For an integer n and some equation $\mathcal{E}$, a coloring of $\{1, 2, \ldots, n\}$ is said to be *valid* if it does not contain monochromatic solutions to $\mathcal{E}$.

We use a custom-built program named `RADO` to compute Rado numbers with an exhaustive search of the r-ary tree of all possible colorings using a backtracking depth-first search. However, many of the details in the implementation are significant in making feasible these large-scale computations.

Although `RADO` is a highly efficient and effective implementation of depth-first-search in the tree of all colorings, it is in the abstract nothing more than this. The true challenges are practical: memory management, parallelization, disk usage, and so forth. The details of implementing the depth-first search to make `RADO` are described at length in Appendix A and [32]. The basics of this type of depth-first search, termed "backtracking," are discussed in Baumert and Golomb [2]. Source code for an older non-parallel version of `RADO` is available at

`http://www.kellenmyers.org/papers/rado.zip`.

A depth-first search checks all possible colorings exhaustively, building upon each coloring whenever possible, by coloring the next largest integer, and if not, recoloring the largest integer, working through the tree of all possible colorings branch-by-branch. In the end, `RADO` provides (optionally) a certificate that is literally the entire tree of all valid colorings, in the form of a list of all maximal valid colorings, i.e., the branches of this tree.

We also use SAT-solvers to compute Rado numbers. Ahmed [1] describes how finding Ramsey-theoretic quantities can be translated into problems of logical satisfiability (or "SAT").

SAT-solving is a standard type of problem in computer science, both in theory and practice. In addition to the discussion in Appendix B and [1], there is a large body of literature regarding satisfiability problems. The source code for the parallel SAT-solver used (named ManySAT) can be found at

`http://www.cril.univ-artois.fr/∼jabbour/manysat.htm`.

SAT-solvers require us to translate our Ramsey-theoretic problem into the language of satisfying a formal logical statement. Most SAT-solvers can provide lower bounds in the form of one (but only one) maximum-length valid coloring. Depending on one's perspective, it may be good or bad, but SAT-solvers may be faster at proving upper bounds because they do not produce any concrete certificate that could prove this upper bound (at least, not without extra work).

These two approaches have different trade-offs. Most notably, SAT-solvers perform more reliably when the equation in question has many variables, while `RADO` performs more efficiently when the solution set is particularly dense.

3. Sums of Squares

Rather than exclusively considering the equation $x^2 + y^2 = z^2$, we will approach the more general problem, considering equations consisting of sums of squares. Let $\mathcal{E}_{a,b}$ be the equation:

$$\sum_{i=1}^{a} x_i^2 = \sum_{j=1}^{b} y_j^2.$$

We first revisit the groundbreaking result of Heule, Kullmann, & Marek.

Computation 9 ([21])**.** *The 2-color Rado number $R_2\left(x^2 + y^2 = z^2\right)$ is 7825.*

Heuristically speaking, there are more solutions to $\mathcal{E}_{a,b+1}$ than $\mathcal{E}_{a,b}$ in $[1, N]$, at least for N sufficiently large. That is not to say that solutions to $\mathcal{E}_{a,b+1}$ are a superset of those of $\mathcal{E}_{a,b}$, which would prove this heuristic. This is just a heuristic, a trend visible for some (but not all) values of $R_r(\mathcal{E}_{a,b})$ we have computed. This heuristic suggests $x^2 + y^2 = z^2$ is very likely to be the equation in this class with the largest Rado number. This also follows a more general heuristic that regardless of the class of equation being considered, having a greater number of variables is likely to bring down the Rado number (even though, in a practical sense, it makes the number harder to compute).

So, to be clear, it is not entirely true that $R_2(\mathcal{E}_{a',b'}) \leq R_2(\mathcal{E}_{a,b})$ whenever $a' \geq a$ and $b' \geq b$, but this seems to carry some heuristic value. We will prove later that it is true in certain cases (for example, when $b' - a' = b - a$), but also demonstrate that it is not true in others (see the first row of Table 1).

Despite having previously formulated conjectures pessimistically, we are now optimistic that these Rado numbers should exist, even in general for r colors:

Conjecture 1. For fixed r and a, there is B sufficiently large such that for $b \geq B$, $R_r(\mathcal{E}_{a,b})$ is finite.

To begin, we offer the following two results:

Computation 10. *The 2-color Rado number $R_2(\mathcal{E}_{1,3})$ is 105.*

Computation 11. *The 2-color Rado number $R_2(\mathcal{E}_{2,3})$ is 19.*

Certificates for these two computations are given in Appendix C.1. We can take results of this type and prove the 2-regularity of a subset of all possible $\mathcal{E}_{a,b}$ according to the following theorem.

Theorem 12. *For some constant c there is a constant M such that for any a and b with $a \leq b \leq ca$, $R_2(\mathcal{E}_{a,b}) < M$.*

We will prove this theorem for $c = \frac{13}{12}$ and $M = 9$. However, we formulate the theorem in general because both the statement of the theorem and the proof are amenable to adaptation, producing better bounds for c (covering more (a, b) pairs) at the cost of a greater uniform bound M on those Rado numbers. (Skipping ahead to Table 1 might lead us to believe that $M = 9$ is the best possible.)

Proof. Consider the following family of subsets of [9]:

$$\begin{gathered}\{1,2,6\},\{1,3,4\},\{1,3,8\},\{2,3,4\},\{2,3,6\},\{3,4,5\},\\ \{3,5,8\},\{3,6,9\},\{4,6,8\},\{1,3,5,6\},\{1,4,5,6\},\\ \{2,5,8,9\},\{3,5,6,7\},\{1,2,4,5,9\},\{1,2,4,7,9\}.\end{gathered}$$

Each of these sets corresponds to a solution to an equation of the form $\mathcal{E}_{a',a'+1}$. It is trivial to check that no coloring of [9] avoids at least one of these sets being monochromatic.

In any case, given any coloring of [9], let S be the monochromatic set above (under this coloring). For any $a \leq b \leq \frac{a'+1}{a'}a$, S can be used to construct a solution to $\mathcal{E}_{a,b}$ by repeating solutions to $\mathcal{E}_{a',a'+1}$ until the number of unused variables on each side is the same. Any remaining variables can be assigned to any value in S. This process works precisely when $b \leq \frac{a'+1}{a'}a$. The inequality guarantees that this process will not exhaust the variables prematurely.

The worst case (a' greatest) is $a' = 12$, which means the theorem, with $M = 9$, holds for $c = \frac{13}{12}$. □

The proof presented here is meant to be a concise, simplified argument to prove the theorem (with these particular values of c and M). However, we present the argument at greater length, and with slight modification, in Appendix D. In that appendix, we will motivate the method of proof and explain how it can be reproduced for various values of c and M using the same methods.

We are able to compute many other values of $R_2(\mathcal{E}_{a,b})$ and offer a number of data for $R_2(\mathcal{E}_{a,b})$ in Table 1.

	b= 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a=1	1	7825	105	37	23	18	20	20	15	16	20	23	17	21	26	17	23	28	25	29
2		1	19	10	8	12	12	7	11	9	15	11	11	12	11	13	14	11	14	13
3			1	10	9	6	9	9	7	9	9	7	9	9	10	9	9	8	9	12
4				1	9	9	6	9	9	7	9	9	5	9	9	5	9	9	8	9
5					1	9	9	6	9	9	7	9	9	5	9	9	5	9	9	4
6						1	9	9	5	9	9	5	9	9	5	9	9	5	9	9
7							1	9	9	5	9	9	5	9	9	5	9	9	5	9
8								1	9	9	5	9	9	5	9	9	5	9	9	5
9									1	9	9	5	9	9	5	9	9	5	9	9
10										1	9	9	5	9	9	5	9	9	5	9
11											1	9	9	5	9	9	5	9	9	5
12												1	9	9	5	9	9	5	9	9
13													1	9	9	5	9	9	5	9
14														1	9	9	5	9	9	5
15															1	9	9	5	9	9
16																1	9	9	5	9
17																	1	9	9	5
18																		1	9	9
19																			1	9
20																				1

Table 1: Table of 2-color Rado numbers for sums of squares.

In this table, we see patterns along diagonals (for fixed values of $b - a$). The numbers along these diagonals appear to be (non-strictly) decreasing. This partially confirms our heuristic, and is also easily proved by applying the core idea of Theorem 12.

Lemma 1. *For $b - a = b' - a'$ and $a' > a$, $R_r(\mathcal{E}_{a,b}) \geq R_r(\mathcal{E}_{a',b'})$.*

Proof. Consider any r-coloring of $[1, R_r(E_{a,b})]$. There must be a monochromatic solution S to $\mathcal{E}_{a,b}$ therein. From S, form a solution S' to $\mathcal{E}_{a',b'}$ by assigning the values to all x_i and y_i that occur in both equations according to S, and assigning any value from S to the remaining x_i and y_i (which are equal in number, since $b - a = b' - a'$). □

Computation 13. *The 3-color Rado number* $R_3(\mathcal{E}_{3,4})$ *is 32.*

Like Computation 10, this was a significant computation to perform. The certificate for this computation is discussed in Appendix C.1. At this time, $R_3(\mathcal{E}_{2,3})$ and $R_3(\mathcal{E}_{1,2})$ are unknown, although we are optimistic that like $R_2(\mathcal{E}_{1,2})$, they will be found eventually. We can obtain from this any number of corollaries, either for specific known Rado numbers or a more general corollary true for any such quantity.

Corollary 1. *Theorem 12 holds with 3 colors for* $c = 4/3$ *and* $M = 32$.

Corollary 2. *Theorem 12 holds with* r *colors for* $c = (a+1)/a$ *and* $M = R_r(\mathcal{E}_{a,a+1})$.

We will now consider a subfamily of these equations, $\mathcal{E}_{1,k}$. We can provide the following table, which verifies exceptions to our heuristic: Increasing a or b will decrease $R_r(\mathcal{E}_{a,b})$ in many, but not all, cases. In fact, as k grows, we eventually see an up-tick, because fixing $a = 1$ and letting $b = k$ grow is not at all in the spirit of our heuristic. Indeed, we note that $R_2(\mathcal{E}_{1,k})$ is bounded below, trivially, by $\sqrt{k}$.

$k =$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$N =$	7825	105	37	23	18	20	20	15	16	20	23	17	21	26	17	23
$k =$	18	19	20	21	22	23	24	25	26	27	28	29	30			
$N =$	28	25	29	29	26	36	32	27	38	33	35	41	36			

Table 2: 2-color Rado numbers for $x_1^2 + \cdots + x_k^2 = z^2$

This is now entry `A250026` in the Online Encyclopedia of Integer Sequences (OEIS). These become difficult to compute using `RADO` for larger values of k due to the increasing number of variables.

Note that the family $\mathcal{E}_{1,k}$ provides the worst possible case for Theorem 12 (in the sense that it requires the value $c = k$), even though for k sufficiently large it seems clear that we should be able to prove that $R_2(\mathcal{E}_{1,k})$ is finite (but with no bound uniform with respect to k). The authors expect that the methods of additive combinatorics or ergodic theory will be effective and hope that a proof is forthcoming.

3.1. Primitive Pythagorean Triples

In this section, we consider the set of primitive Pythagorean triples.

Definition 14. A *primitive* Pythagorean triple is a solution (x, y, z) to the equation $x^2 + y^2 = z^2$ where $\gcd(x, y, z) = 1$.

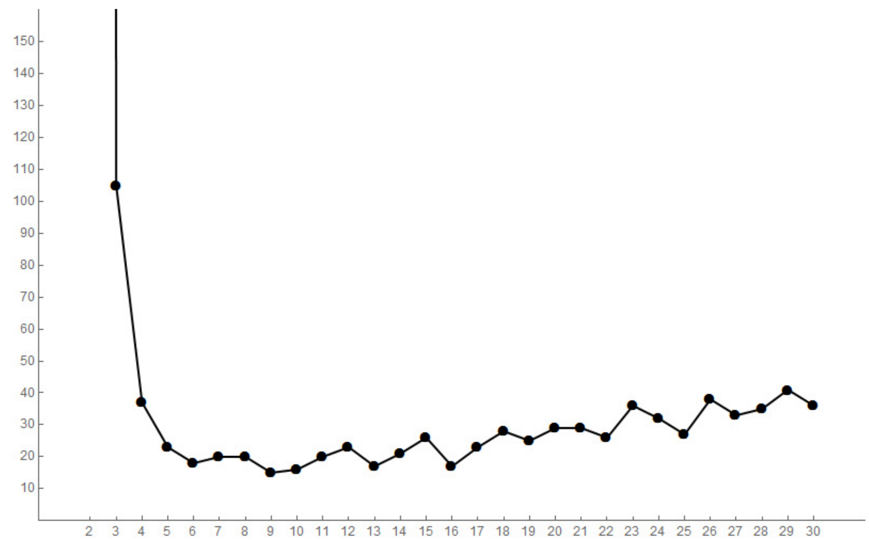

Figure 1: 2-color Rado numbers for $x_1^2 + \cdots + x_k^2 = z^2$, with $k > 2$

Initially, we considered whether there may be a relationship between the regularity of the set of primitive triples and that of the set of all triples. However, while it has been proved that $R_2\left(x^2 + y^2 = z^2\right)$ is finite, it is easy to prove the following.

Theorem 15. *The set of primitive Pythagorean triples is not 2-regular.*

Although we have not explicitly defined what this means, since this set is not strictly the set of solutions to an equation, we believe the definition is clear. To remove any confusion, this theorem asserts that there is a 2-coloring of the positive integers that includes no monochromatic primitive Pythagorean triple. (We provide such a coloring as proof.)

This theorem is offered mostly as an aside that we hope is of interest to those who may have wondered whether $R_2\left(x^2 + y^2 = z^2\right)$ could be related to some other quantity associated to primitive triples.

Proof. Consider a coloring where any integer that occurs as z in a primitive triple $x^2 + y^2 = z^2$ is colored red, and all other integers are blue. There are clearly no blue primitive triples because the z value of any such triple is red.

On the other hand, if we have a red triple, all three must occur as z values, say that $z_1^2 + z_2^2 = z_3^2$. These z_i must be parametrized as:

$$\begin{aligned} z_1 &= m_1^2 + n_1^2, \\ z_2 &= m_2^2 + n_2^2, \\ z_3 &= m_3^2 + n_3^2, \\ z_1 &= 2m_3n_3, \\ z_2 &= m_3^2 - n_3^2, \end{aligned}$$

where each (m_i, n_i) pair is coprime and not both odd. (This is the standard parametrization of primitive triples.) Modulo 4, this gives:

$$\begin{aligned} m_1^2 + n_1^2 &\equiv 2m_3n_3 \pmod 4, \\ m_2^2 + n_2^2 &\equiv m_3^2 - n_3^2 \pmod 4, \end{aligned}$$

which is easily verified as unsatisfiable. One need only check 8 possibilities for each pair (a total of $8^3 = 512$ possibilities). Moreover, the first of the two congruences is itself unsatisfiable (one need only check 64 possibilities to confirm this). More directly, we note that since one of m_3 and n_3 must be even, $2m_3n_3 \equiv 0 \pmod 4$. Because 0 and 1 are the only squares modulo 4, this would require that any solution would have $m_1^2 \equiv n_2^2 \equiv 0 \pmod 4$. This implies m_1 and n_1 are both even. □

3.2. Closely Related Equations

Before closing this section, we consider a few equations within the family $ax^2+by^2 = cz^2$, for positive constants a, b, c. Besides cases where $a + b = c$, we presume that computing Rado numbers for $ax^2 + by^2 = cz^2$ would be as hard computing that of $x^2 + y^2 = z^2$. We can offer the following lower bounds.

Computation 16. *For* $3 \leq k \leq 20$, $R_2\left(x^2 + y^2 = kz^2\right) > 5000$.

These lower bounds can almost certainly be improved, but in order to compute them quickly and state the theorem succinctly, we offer this as a starting point. We might expect that as k increases, the Rado number generally increases, so one may want to start with $k = 3$ somewhere higher than 7825.

We now consider the equation $\mathcal{E}_k$ to be $x^2 + y^2 + kz^2 = w^2$, inspired in part by our desire to understand the previous families, and by the approach taken to understand Rado numbers for linear equations, where this form (one new coefficient, one variable on the right-hand side) seems to be the most tractable.

We first present the following computations, which were relatively intensive but very much tractable using the `RADO` package.

Computation 17. *The 2-color Rado numbers for* $x^2 + y^2 + kz^2 = w^2$ *are as presented in Table 3.*

k	1	2	3	4	5	6	7	8	9	10	11	12
$R_2(\mathcal{E}_k)$	105	37	40	41	55	85	43	68	77	84	70	77

Table 3: 2-color Rado numbers for $\mathcal{E}_k$, $R_2\left(x^2+y^2+kz^2=w^2\right)$

This table could be extended using the `RADO` package, except that $k=13$ is extremely difficult to compute (but not $k=14$). This phenomenon was discussed in [32]. These data are interesting because it is not at all clear how k affects $R_2(\mathcal{E}_k)$ or the amount of computing power required to compute it.

For some variety, we also offer the following:

Computation 18. *The 2-color Rado number $R_2\left(x^2+y^2=z^2+2w^2\right)$ is 33.*

The certificate for this computation is discussed in Appendix C.1.

We will discuss a variety of other nonlinear equations the next section, but we will note that Luperi Baglini & Di Nasso [8] prove the regularity of the equation $x_1x_2+y_1y_2=z^2$ (for any number of colors), which we believe represents the current result closest to proving the regularity of $x^2+y^2=z^2$ – which could, by extension, help us prove that many other equations $\mathcal{E}_{a,b}$ are regular as well (using the methods of Theorem 12 or otherwise).

4. Other Nonlinear Equations

Despite being mostly unexplored, there are a few results that speak to the regularity of nonlinear equations. We start with a seminal result proved independently by Furstenberg [16] and Sárközy [38]:

Theorem 19. *For any r-coloring of $\mathbb{Z}^+$ there is $n\in\mathbb{Z}^+$ such that the equation $x-y=n^2$ has a monochromatic solution (x,y).*

This type of property is frequently also called regularity or partition regularity, but it has a slightly different meaning (due to the introduction of a parameter n that may depend on the coloring). This is sometimes (e.g. Luperi Baglini [31]) called "partial (partition) regularity," which we agree fits well.

There are other results of this type, including Frantzikinakis & Host [14], where a similar result is proved for $9x^2+16y^2=n^2$ and certain other quadratic equations, using the machinery of Fourier analysis.

We can verify the theorem for $r=2$ by simply computing $R_2\left(x-y=z^2\right)=9$, which would be larger than what we could call the 2-color Furstenberg-Sarközy number, FS_2, since the latter relaxes the monochromatic conditions to only x and y. We offer several of these Furstenberg-Sarközy numbers and some of the corresponding Rado numbers. We have only been able to verify that the 5-color number is greater than 181.

r	1	2	3	4	5
FS_r	2	5	29	58	181+
$R_r\left(x-y=z^2\right)$	2	9	204	800+	

Table 4: Furstenberg-Sarközy numbers

The certificates for the third column (3 colors) are discussed in Appendix C.2. It is important to note that in Csikvári, K. Gyarmati, and A. Sárközy [7], it is also shown that $x - y = z^2$ is not regular, so the bottom row will eventually fail to yield finite values.

Erdős, Sárközy, and Sós [11], and later Khalfalah and Szémeredi [28], provide a number of generalizations of this sort of result, where they examine $x - y = f(n)$ or $x + y = f(n)$ and look for monochromatic x and y that satisfy the equation for one or even infinitely many values of n (but without regard to the color of n).

4.1. Reciprocal Schur Numbers

In 1991, Brown and Rödl [5] and Lefmann [29] proved independently that the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$ is regular. (See also Graham [17].)

Brown and Rödl prove a stronger result, that the equation is regular even if we ignore solutions in which some of the variables are the same, e.g. $(x, y, z) = (4, 4, 2)$. We will discuss this distinction more in the next section.

It is useful to note that $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$ is, equivalently, a quadratic polynomial $xz + yz = xy$. In addition to the proofs by Brown & Rödl and Lefmann, Luperi Baglini proves the following theorem.

Theorem 20 ([31]). *If $P(x, y, z)$ is a degree d homogeneous polynomial and $P(x, y, z) = 0$ is regular, then so is $Q(x, y, z) = x^d y^d z^d P\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) = 0$.*

(This is proved for any number of variables, but we present it with three.) Luperi Baglini notes that this is another way of proving the regularity of this reciprocal equation (and generalizations originally included in both of the papers from 1991). Setting $P(x, y, z) = x+y-z$ yields $Q(x, y, z) = xz+yz-xy$, and likewise Theorem 4 gives a generalization with more variables.

We have been able to quantify this result by computing the following.

Computation 21. *The 2-color Rado number $R_2\left(yz + xz = xy\right) = R_2\left(\frac{1}{x} + \frac{1}{y} = \frac{1}{z}\right)$ is 60. Furthermore, $R_2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{w}\right)$ is 40. With 5 variables it is 48, and with 6 variables it is 39.*

The certificate for this computation with three variables is discussed in Appendix C.3.

In the past, e.g. [32], we have called these Rödl numbers, although they might better be called Brown-Rödl-Lefmann numbers. For the time being, we will call them reciprocal Schur numbers instead, especially in light of Luperi Baglini linking them to the generalized Schur numbers in Theorem 4 via Theorem 20.

We also offer the following with a greater number of colors. The certificate for this computation is discussed in Appendix C.3.

Computation 22. *The 3-color reciprocal Schur number* $R_3\left(\frac{1}{x}+\frac{1}{y}=\frac{1}{z}\right)$ *is 3276.*

We are only able to bound the 4-color number from below:

$$R_4\left(\frac{1}{x}+\frac{1}{y}=\frac{1}{z}\right) > 87,000.$$

4.2. Hindman Numbers

Hindman [22, 23] discusses the question of finding sets whose sums and products are monochromatic. Taking this idea and applying it to any number of more specific statements provides us several equations (and systems of equations) that are regular. For example, we have confirmed the following.

Theorem 23. *For any* 2*-coloring of* $\mathbb{Z}^+$*, there are monochromatic* x, y, z, w *such that* $x + y = z$*and* $xy = w$*.*

The full weight of Hindman's results is much greater, and slightly different, proving for instance this much stronger result.

Theorem 24 ([22])**.** *For any* r*-coloring of* $\mathbb{Z}^+$*, there is an infinite set* A *such that all nonempty finite sums of* A *are monochromatic. Moreover, there is an infinite* B *such that its nonempty finite products are monochromatic in the same color as the sums of* A*.*

It is useful to note that in order to compute Rado numbers for systems of non-linear equations, particularly polynomials, it is no different than computing a single Rado number. If we want the r-coloring of $[N]$ to satisfy the equations $\mathcal{E}_1$ and $\mathcal{E}_2$ we can write $\mathcal{E}_i$ as $f_{\mathcal{E}_i} = 0$ and find $R_r\left((f_{\mathcal{E}_1})^2 + (f_{\mathcal{E}_2})^2 = 0\right)$.

So in Theorem 23, we might be interested in $R_2\left((x + y - z)^2 + (xy - w)^2 = 0\right)$.

Although it is not explicit in previous statements of Theorem 23, we should note carefully that it is fairly clear in the existing literature that this theorem usually includes the stronger condition that all of x, y, z, w are distinct.

Hindman's theorem has been strengthened in certain cases so that one may take $A = B$, with some restrictions to the coloring, and we proceed with some quantitative exploration regarding a proposition that is an unproven, stronger version of Hindman's theorem.

Proposition 1. *For any r-coloring of $\mathbb{Z}^+$ there is an infinite set A such that all nonempty finite sums and products of A are monochromatic.*

Please note carefully, although similar to Theorem 23, we will proceed *as if* this proposition is true. We do not conjecture that it is true (or false), nor do we offer proof. Instead, it is the model – an infinitary statement (like Schur's theorem on the integers $\mathbb{Z}^+$) that we will investigate quantitatively. The proposition is amenable to a computational approach (despite being not proved or even conjectured) and we will proceed from there.

We can define H_r, for a fixed number of colors r, as the least N for which Theorem 23 (but with r colors) holds on $\{1, 2, \ldots, N\}$ instead of $\mathbb{Z}^+$. Hindman [22] communicates that Graham has proved a result about these numbers (by computer). However, the lower bound for this example is demonstrated by two color classes:

$$\begin{aligned} \mathit{Red} &= \{1, 3, 5, 8, 12, 14, 16, 18, 20, 22, 24, \ldots, 250\}, \\ \mathit{Blue} &= \{2, 4, 6, 7, 9, 10, 11, 13, 15, 17, 19, \ldots, 251\}. \end{aligned}$$

Without our careful note above, we might immediately balk at 2 and 4 being blue, since $(x, y, z, w) = (2, 2, 4, 4)$ is a solution! But, as we noted, Graham and others implicitly assume that this is not a solution we are considering – that none of the variables can be equal. This discrepancy is confirmed by computing $R_2\left((x + y - z)^2 + (xy - w)^2 = 0\right)$, which is 39, not 252.

The certificate for $H_2 = 39$ is discussed in Appendix C.4.

So we can instead define H_r' to be the injective Hindman number, a sort of Rado number, defined informally as:

$$H_r' = R_r\left(x + y = z \text{ and } xy = w \text{ and } x \neq y\right).$$

Graham's proposition should, indeed, be a statement about H_2':

Computation 25 ([22])**.** *The 2-color Hindman number is $H_2' = 252$.*

It is trivial to note that $H_r \le H_r'$, so we know $H_2 \le 252$ based on Graham's result (indeed, it is 39). For three colors, we have only been able to determine that:

$$100{,}000 < H_3 \le H_3'.$$

For this reason, we mostly focus on two colors for further computation. We can define $H_r(m)$ and $H_r'(m)$ to be the corresponding quantities for the following system:

$$\begin{aligned} x_1 + \cdots + x_{m-1} &= x_m \\ x_1 \cdot \cdots \cdot x_{m-1} &= z \end{aligned}$$

This simplifies significantly what Hindman's results could inspire, since [22] discusses sets of integers and *all* of their (nonempty) sums and products. However, we are able to compute some other such quantities.

Computation 26. *The 2-color 4-variable generalized Hindman number* $H_2(4)$ *is* 450. *Furthermore,* $H_2(5) = 11,000$*,* $H_2(6) = 62,500$*, and* $H_2'(4) = 23,100$*.*

We summarize our results in Table 5.

m	3	4	5	6
$H_2(m)$	39	450	11,000	62,500
$H_2'(m)$	252	23,100		

Table 5: Generalized Hindman numbers

We believe that these computations demonstrate that Hindman numbers are interesting. We offer the following conjecture in hopes that these numbers are indeed well defined, in the spirit of Hindman's original theorem.

Conjecture 2. For any r-coloring of $\mathbb{Z}^+$ and any integer m, there are monochromatic $x_1, x_2, \ldots, x_m, z$ such that $x_1 + \cdots + x_{m-1} = x_m$ and $x_1 \cdot \cdots \cdot x_{m-1} = z$.

Note that this does not follow from Theorem 24, and indeed, even Theorem 23 does not follow from Theorem 24. The trouble is that Theorem 24 does not necessarily have $A = B$ (as stated). While we can prove or confirm Theorem 23 by computing the value H_2, and we could likewise prove special cases of the conjecture for each value of m, none of these follow from Theorem 24. It is our hope that this computation is encouraging, and that perhaps by focusing only on the total sum/product of A and B (respectively), one can prove a weaker version of 24 with $A = B$ (as Conjecture 2 or something similar).

We also propose that one could expand this definition to include all finite sums and products of some $x_1, \ldots, x_m$, which we would call "strong"' Hindman numbers, with or without the condition that the x_i be distinct. We would denote these $H_r^*(m)$ and $H_r^{*\prime}(m)$, but unfortunately we can say very little beyond the fact that these might be interesting, particularly since $m > 3$ is required (or else the distinction between H_r and H_r^* is meaningless). All we have determined at this time is a lower bound: the smallest of these, $H_2^*(4)$, is at least 300,000. It could be that $H_2^*(4) = \infty$, since Theorem 24 does not guarantee this quantity is finite.

4.3. Exponential Rado Numbers

Sisto [43] proves that the equation $z = x^y$ is regular (with $x, y, z > 1$). Brown [4] extends this to systems of exponential equations, while Sahasrabudhe [35] shows that the system $\{xy = w,\ x^y = z\}$ is regular, which we could call "geometric Hindman numbers," perhaps. We can offer the following quantitative result to complement these advances, with a slight modification of our usual idea of Rado number to allow $x, y, z > 1$, i.e., we color only $\{2, 3, 4, 5, \ldots\}$.

Computation 27. *The 2-color Rado number for colorings of integers greater than 2 for the exponential equation $z = x^y$ is 65,536.*

That is 2^{16}, which is of course no coincidence, since this is a result about exponential quantities. We have also been able to determine that the corresponding quantity for the system from [35] is over 2^{19}.

At this time, we are unable to compute corresponding results for three colors, but we are optimistic that further investigation into this thread will yield additional results of this type.

Finally, we propose something new. Consider the exponential equation:

$$x_1 + \cdots + x_{m-1} = kn^{x_m}.$$

Fixing $k = 1$ and $n = 2$, we can determine the following Rado numbers.

Computation 28. *The 2-color Rado numbers $R_2\left(x_1 + \cdots + x_{m-1} = 2^{x_m}\right)$ are 1, 6, 3, 9, 11, 5, 5 for $m = 3$, 4, 5, 6, 7, 8, 9 (respectively).*

The certificate for this computation with $m = 7$ is discussed in Appendix C.4. We hope in the future to learn more about Rado numbers for non-polynomial equations like these.

4.4. Miscellany

Finally, we will discuss a few other Rado numbers for nonlinear equations. We should acknowledge the important work of Doss, Saracino, and Vestal [10], in which the following famlies of Rado numbers are determined.

Theorem 29. *For any positive integer n, $R_2\left(x + y^n = z\right) = 1 + 2^{n+1}$.*

Theorem 30. *For any integer $c \geq 2$, $R_2\left(x + y^2 + c = z\right) = c^2 + 7c + 7$.*

Theorem 31. *For any integer $a \geq 2$, $R_2\left(x + y^2 = az\right) = a - 1$.*

To our knowledge, these are the only parametrized families of nonlinear equations for which the Rado number has been determined exactly (in the style of previous results on linear equations, e.g. Theorems 4 through 7). Two of these equations are quadratic, but one is of arbitrary degree n (where n is the parameter in this family of equations).

We have computed Rado numbers for several other nonlinear equations that have been mentioned elsewhere, often having been proved regular in the course of other work. Most of these come from Luperi Baglini [30]. We give these miscellaneous results in Table 6, including the reference in which this equation can be found, the equation, and the Rado number for at least $r = 2$ colors and others as computational limits permit. We hope to add many more rows to this table in the future.

Ref.	Equation	$r = 2$	3	4	5
[8]	$x_1x_2 + y_1y_2 = z^2$	5	13	41	60+
[30]	$x_1y_1 + x_2y_2 = x_3$	13	128+		
[30]	$2x_1 + 3x_2y_1y_2 + x_4y_2y_3 = 5x_3y_1$	5	20	146+	
[30]	$t_1t_2x^2 + t_3t_4y^2 = t_5t_6z^2$	5	13		
[30]	$x_1y_1y_2 + 4x_2y_1y_2y_3 + x_5 = 3x_3y_3 + 2x_4y_1$	9	53+		

Table 6: Miscellaneous Nonlinear Rado Numbers

5. Future Work

In the future, we hope to be able to more efficiently compute Rado numbers for more diverse classes of linear and nonlinear equations, and that this study will continue to inform (and be informed by) the ongoing theoretical work towards proving the existence of these numbers. In particular, Di Nasso & Luperi Baglini [8] give us hope that we will soon know whether equations comprised of sums of squares are regular.

We hope that progress in computing 3-color Rado numbers (even just for linear equations) will also provide further insight into the conditions for an equation to be 3-regular. (We do not know this, even for linear equations.)

Concretely, we expect improvements and new insights into our computational approaches. The use of SAT-solvers and backtracking algorithms can be tweaked and improved incrementally, allowing for incremental progress – sometimes tedious, but sometimes extraordinary (as in Heule, Kullmann, & Marek [21]). Likewise, physical computer resources continue to expand. We also expect innovative and hybrid approaches to these algorithms to allow for more rapid and robust exploration of these problems.

Our tree-searching program `RADO` is a highly-specialized, sophisticated search algorithm developed over several years. It makes a number of very significant gains in practical computational speed by capitalizing on some nuances of these problems. Yet, like any computational tool, it is not perfectly optimized or ideal in some way. There may be ways of combining the methods of `RADO` with other backtracking algorithms or using hardware-facilitated improvements to the existing algorithm to substantially increase its computational power.

In addition to the general open question of determining Rado numbers for many unknown classes of linear and nonlinear equations, there are also variations of the idea of Rado numbers of great interest.

We might consider modified Rado numbers that require monochromatic solutions where each integer is different, as discussed above. Not only does this pose a slightly different problem, but it makes more natural questions about equations like $x^2+y^2 = z^2 + w^2$ or even $x + y = z + w$ (which would otherwise have trivial Rado numbers). Again, Luperi Baglini [31] and elsewhere has used terminology we suggest is wellsuited to this: calling such an equation injectively partition regular. Computing injective Rado numbers seems like an obvious avenue for future exploration.

We may also be interested in *rainbow* Rado numbers, which require a solution set have one element of each color. Generally, the number of variables is equal to the number of colors, by design, and consequently each part of the solution must be be distinct. Investigations into rainbow Rado numbers may provide new computational challenges, since colorings must be restricted to provide nearly (or exactly) equally sized color classes. Adapting the methods of SAT-solving (see Appendix B) to such a problem is not trivial, since there is no simple clause that would enforce a requirement like "1/3 of this 3-coloring of $\{1, 2, \ldots, 90\}$ is red."

There are many other variations on the theme of Rado numbers, but we will leave these two as examples and invite the reader to explore the literature for more such variations – or to invent his/her own.

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Appendices

A. Tree-Searching Methods

We use a custom-built program named `RADO` to compute Rado numbers with an exhaustive search of the r-ary tree using a backtracking depth-first search. However, many of the details in the implementation are significant in making feasible these large-scale computations.

The tree search is parallelized by splitting an initial fixed-depth search into a queue of valid colorings. This search factors out any symmetry in the tree by only using the i^{th} color if the $(i-1)^{\text{th}}$ color is already in use. (In other words, the branch beginning red-red-blue is the equivalent to that of red-red-green, so we only do one of those two). Each worker is assigned a branch (an initial valid coloring) and explores this branch depth-first. When a worker has explored the entire branch, it is assigned a new branch to explore.

The parallelized algorithm is controlled by a master process that directs the workers in their search. This master process keeps track of any idle workers and the queue of branches not yet explored.

If a full certificate (the entire search tree) is being created, there are other manager tasks that handle the output (a list of all maximal valid colorings) from the workers, in order to avoid overwhelming the master process with this task. These manager processes write to disk asynchronously in order to prevent thrashing the hard disk and other I/O issues. The managers also compress the output using LZ4 compression, which allows for compression in real time, at a relatively good compression ratio. A partial certificate, containing only summary information, can also be produced (in which case, there are no manager processes).

In order to compute these Rado numbers, the numbers of colorings checked (i.e., number of nodes in the search) is extremely large, and for each coloring checked, the number of times the equation $\mathcal{E}$ is evaluated is likewise very large. For example, proving that $R_3\left(x^2+y^2+z^2=w^2\right)=105$ requires checking over 50 million colorings, makes billions of tests for satisfying the equation $\mathcal{E}$, and generates a full certificate of roughly 250 megabytes. (That is, by no means, the worst case – some certificates are in the terabyte scale, and some problems require even greater numbers of computations.)

In order to evaluate $\mathcal{E}$ this many times, within a feasible time-frame, the arithmetic operations that define $\mathcal{E}$ are translated into a function $f_{\mathcal{E}}$ (subtracting all terms to one side), and this function evaluated repeatedly (and $\mathcal{E}$ is satisfied if and only if $f_{\mathcal{E}}=0$). This function is not hard-coded, nor is it a complex method in C++ or some other language – instead, a user-input string representing $f_{\mathcal{E}}$ is parsed and compiled. This just-in-time (JIT) compilation allows the method to be comprised of a set of low-level operations corresponding to the arithmetical operations in $f_{\mathcal{E}}$.

For a fixed coloring, a worker node will have to check whether that coloring is valid. Because this is a depth first search, each such coloring is an extension (by one element) of a valid coloring, so the integer at the end of the coloring would have to be a part of any monochromatic zero of $f_{\mathcal{E}}$. In order to check for monochromatic solutions, we check all possible evaluations of $f_{\mathcal{E}}$ (where one or more of the variables must be the element from the end of the coloring) using what we have termed the value-iterator (VI).

The VI checks quickly, in a predetermined order, for any possible monochromatic solution. The VI is optimized in the case of certain symmetry in the equations, like in the case of $x_1^2+\cdots+x_k^2=z^2$, assuming $x_1 \le x_2 \le \cdots \le x_k \le z$ (and so we know z must be the largest integer in the coloring as well). At this time, the VI will not optimize for equations like $x+y+3z=10w$, where the only assumption we can make is that $x \le y$.

It is important to note that having more variables in $\mathcal{E}$ will (generally speaking) make the Rado number smaller. However, it increases the complexity of the VI by a factor of the current depth of the search (which is not a favorable trade-off). There is significant room for improvement in the VI, especially when handling nonlinear equations like $x^2+y^2=z^2$, where solutions to the equation are sparse.

B. Satisfiability (SAT)

Ahmed [1] describes how the problem of computing van der Waerden numbers can be translated into a problem of logical satisfiability (or "SAT"). Like many problems in computer science, satisfiability has a yes or no answer: "Is a certain logical statement satisfiable, for some assignment of the variables?" For example, the statement $x \wedge y$ is satisfiable, with the assignment $x = y =$ True. However, the statement $x \wedge (\neg x)$ is not satisfiable, because any assignment of x will make this statement false. Here $\wedge$ represents "and," $\vee$ represents "or," and $\neg$ represents "not." Satisfiability could also be rephrased in terms of Boolean algebra, but we will continue to use the logical notation $\wedge$, $\vee$, $\neg$.

We will reserve x and y for variables in our Diophantine equation $\mathcal{E}$ from now on, and we will enumerate our logical variables in this section using v instead.

For our purposes, we have an equation $\mathcal{E}$ and a specific N and r, and we want to know whether there is any r-coloring of $\{1, 2, \ldots, N\}$ such that no monochromatic solutions to $\mathcal{E}$ exist, i.e., a valid coloring. First, we will formulate our method in the case of $r = 2$. We designate the variable v_i to indicate that i is colored blue (and so if v_i is false, i must be colored red).

SAT solving algorithms generally accept input in one of several normal forms. In this case, we will use the DIMACS format [9] for SAT problems, which uses conjunctive normal form (CNF). The statement to be satisfied must be the conjunction ("and") of a number of clauses. These clauses are themselves all disjunctions ("or") of literals (variables or their negations). It is important to note that all logical formulas can be formulated in CNF, but we will find that this form is quite natural for our problem.

The conditions we must impose are relatively simple: We know any satisfying assignment will result in a valid coloring (and if not, there is no such coloring) because each v_i will be either true or false. We take every solution $(x_1, x_2, \ldots, x_m)$ and form two clauses:

$$\big(v_{x_1} \vee v_{x_2} \vee \cdots \vee v_{x_m}\big) \wedge \big(\neg v_{x_1} \vee \neg v_{x_2} \vee \cdots \vee \neg v_{x_m}\big).$$

These two clauses force at least one of the x_i to be red, and at least one to be blue, i.e., not monochromatic. As an example, if we wanted to prove the 2-color Schur number $S(2) > 4$, we would need to gather up all solutions to $x + y = z$ (we may assume $x \leq y$ to eliminate some redundant clauses). They are:

$$\big\{(1,1,2), (1,2,3), (1,3,4), (2,2,4)\big\}.$$

This gives us the following formula:

$$\begin{aligned} F(v_1, v_2, v_3, v_4) \quad = \quad & \big(v_1 \vee v_2\big) \wedge \big(\neg v_1 \vee \neg v_2\big) \wedge \\ & \big(v_1 \vee v_2 \vee v_3\big) \wedge \big(\neg v_1 \vee \neg v_2 \vee \neg v_3\big) \wedge \\ & \big(v_1 \vee v_3 \vee v_4\big) \wedge \big(\neg v_1 \vee \neg v_3 \vee \neg v_4\big) \wedge \\ & \big(v_2 \vee v_4\big) \wedge \big(\neg v_2 \vee \neg v_4\big). \end{aligned}$$

We can use a SAT solver to produce a satisfying assignment to $F(v_1, v_2, v_3, v_4)$, e.g. $v_1 = v_4 = \text{False}$, $v_2 = v_3 = \text{True}$, which is red-blue-blue-red.

We could then consider two additional solutions, $(1, 4, 5)$ and $(2, 3, 5)$, obtaining:

$$\begin{aligned} G(v_1, v_2, v_3, v_4, v_5) \quad = \quad & F(v_1, v_2, v_3, v_4) \wedge \big(v_1 \vee v_4 \vee v_5\big) \wedge \big(\neg v_1 \vee \neg v_4 \vee \neg v_5\big) \wedge \\ & \big(v_2 \vee v_3 \vee v_5\big) \wedge \big(\neg v_2 \vee \neg v_3 \vee \neg v_5\big). \end{aligned}$$

A SAT solver would tell us that G is not satisfiable, proving $S(2) \leq 5$.

Because our usual equations include three variables (or more), most of our clauses will too. This means our problems fall into the framework of 3-SAT, the problem of deciding the satisfiability of a general formula in CNF where each clause has 3 literals. This problem is well-known to be NP-hard.

Our approach to these problems is complicated by the need to have more than two colors, in which case we we will introduce additional clauses. For each color j and each solution $(x_1, \ldots, x_m)$ to $\mathcal{E}$, we have the clause:

$$\big(v_{x_1,j} \vee v_{x_2,j} \vee \cdots \vee v_{x_m,j}\big).$$

We have now doubly indexed our variables as $v_{i,j}$, representing that integer i is colored the color j. In practice, $v_{i,j}$ is translated to v_{Nj+i}, since SAT solvers do not usually allow a double index. We no longer require the corresponding clause with negative literals, since that was to accommodate the second color of two. But there is more to consider.

We must explicitly enforce that a color is assigned to each i, so we must include for each i the clause:

$$\big(v_{i,1} \vee v_{i,2} \vee \cdots \vee v_{i,r}\big).$$

We must also insist that no i is assigned to multiple colors. For each i, we include the following clauses, which are a bit more complicated because we need one for each $1 \le j < j' \le r$, i.e., for $\binom{r}{2}$ pairs:

$$\neg v_{i,j} \vee \neg v_{i,j'}.$$

The DIMACS format for CNF formulas requires us to count the numbers of variables and clauses. This is actually not simple, at least not in most cases, since we would need to enumerate the number of solutions to $\mathcal{E}$ in $[N]$. Since we have already listed them, the best way is to simply count the length of that list – rather than try to exploit some special cases (e.g. if $\mathcal{E}$ is $x + y = z$, it is not hard). Each solution to $\mathcal{E}$ contributes r clauses, and we have to add on $(1 + \binom{r}{2})N$ clauses to enforce the well-definedness of our coloring. If the solution set is dense (i.e., $\mathcal{E}$ has roughly $o(N^{m-1})$ solutions in $[N]$), we will have $o(N^{m-1} + Nr^2)$ clauses.

Fortunately, it is not so difficult to count the number of variables. Even in cases where some i is never used in a solution to $\mathcal{E}$, it is used in the clauses that enforce the well-definedness of the coloring. So, we know there are rN variables.

In some respects, using a SAT solver is a generalization or restatement of methods used in previous work of this type that did not utilize computers. For more on this, see [32].

SAT solvers are not tailor-made for these types of systems, but SAT solvers are important tools and many different SAT solvers exist to tackle these tough problems like computing Rado numbers, including `WalkSAT`[1], `GRASP`[2], and `MiniSAT`[3]. These SAT solvers rely on high performance computing, written for efficiency and effectiveness in lower-level languages than computer algebra systems or the like, and are highly effective in many cases.

SAT-solving and `RADO` may be similar in some ways, but they are very different. `RADO` is robust and effective in many cases, and provides exhaustive output, while SAT solvers have their own advantages. In particular, when the solution set is sparse, the `RADO` program does not adapt to this and will consider a significant

[1]available at `http://www.cs.rochester.edu/u/kautz/walksat/`
[2]available at `http://vlsicad.eecs.umich.edu/BK/Slots/cache/sat.inesc.pt/∼jpms/grasp/`
[3]available at `http://minisat.se/`

number of redundant cases. A SAT solver does not have this issue and will instead reap the benefits of the sparsity by having far fewer clauses to satisfy.

C. Selected Certificates of Computations

In this appendix, we present various certificates of computation.

A full certificate, proving a particular Rado number, is comprised of the list of all maximal valid colorings (which, if arranged as such, would constitute the tree of all valid colorings). Each coloring listed is valid, meaning it has no monochromatic solutions to the equation in question. Each is maximal, meaning it cannot be extended by one of the colors and remain valid. The list is exhaustive, covering all possible colorings – any coloring is either a prefix of one of these colorings (and therefore valid), on the list (also valid), or an extension of one of the colorings on the list (therefore not valid).

Note there is some nuance: a maximal valid coloring, for the purposes of our certificates, means there is at least one color that cannot be appended and remain valid – some colors may be possible to append, but importantly, such extensions will occur as prefixes for maximal colorings later in the certificate. For those few colorings of maximum length, which we will discuss in detail below, indeed it cannot be extended by any color. We might limit ourselves to only these colorings, but in order to check such a certificate, we would have to check extensions of *any prefix* of such a coloring.

Generally it is impractical to present such certificates on paper. This is due to the size, as certificates range from 127 KB (which is possible, just impractical, totaling at least 30 pages) to 1.5 GB (perhaps filling an infeasible quarter million pages). While we might present a short certificate entirely, we must provide large full certificates on the Internet.

These four examples and all other examples will include at least one valid coloring of maximum length. This will prove the lower bound for the computation in question. We will discuss each such example at length, and as we examine the maximum-length valid colorings (those that prove the lower bound), we will show why they (in particular) cannot be extended by any color.

Certificates can be given in a few different ways. A full certificate is a list of colorings, while a partial certificate may consist only of a single coloring of maximum length (maximum, not just maximal). Each coloring can be given as r sets, each representing a color class. However, a more compact representation is using binary (ternary, quaternary, etc.) strings for 2 (3, 4, etc.) colors. This more easily provides a single lengthy coloring or a long list of colorings.

These colorings are listed using r digits, so that a string of digits represents an r-coloring. The integer i is colored with the j^{th} color, where j is the digit in position i of the coloring. For example, 01012211 represents the coloring $R = \{1, 3\}$, $B = \{2, 4, 7, 8\}$, $G = \{5, 6\}$ (where R, B, G are the color classes).

In order to reduce the size of each certificate, we assume 1 is in the first color class ("red" or "0"), 2 is in the second ("blue" or "1") if not red, and so forth. This eliminates redundancy one could obtain by permuting the colors.

We will use these two methods (color classes given as explicit sets and r-ary strings) to represent colorings. There are several other ways to represent lengthy colorings and large sets of colorings, including color-coded diagrams and other pictorial encodings, but we will not use them here.

C.1. Sums of Squares

Recall that $\mathcal{E}_{a,b}$ is our notation for:

$$\sum_{i=1}^{a} x_i^2 = \sum_{j=1}^{b} y_j^2.$$

For four such equations, we provide a full certificate. However, these certificates are too lengthy to print in most cases. We will provide links to electronic versions of these certificates instead.

The Equation $\mathcal{E}_{2,3}$ With 2 Colors

In this first example, we can provide a full certificate, spanning only a few pages. The following set of binary strings is a certificate for $R_2\left(\mathcal{E}_{2,3}\right) = 19$.

{ 000, 0001000, 00010001000, 000100010001000, 000100010001001, 000100010001010, 000100010001011, 000100010001100, 000100010001101, 000100010001110, 000100010001111, 00010001001, 000100010011000, 000100010011001, 000100010011010, 000100010011011, 000100010011100, 000100010011101, 000100010011110, 000100010011111, 00010001010, 000100010101000, 000100010101001, 0001000101010, 000100010101100, 00010001010110, 0001000101011, 00010001011, 000100010111000, 000100010111001, 0001000101110, 000100010111100, 00010001011110, 0001000101111, 00010001100, 000100011001000, 00010001100100, 000100011001010, 00010001100101, 000100011001100, 00010001100110, 000100011001110, 00010001100111, 00010001101, 000100011011000, 00010001101100, 000100011011010, 00010001101101, 000100011011100, 00010001101110, 000100011011110, 00010001101111, 00010001110, 000100011101000, 00010001110100, 0001000111010, 000100011101100, 00010001110110, 0001000111011, 000100011110, 0001000111101, 00010001111011, 000100011111000, 00010001111100, 0001000111110, 000100011111100, 00010001111110, 0001000111111, 0001001, 00010011000, 000100110001000, 000100110001001, 000100110001010, 000100110001011, 000100110001100, 00010011000110, 000100110001110, 00010011000111, 00010011001, 00010011010, 000100110101000, 000100110101001, 0001001101010, 000100110101100, 00010011010110, 0001001101011, 00010011011, 00010011, 0001010, 00010101000, 000101010001000, 000101010001001, 000101010001010, 000101010001011, 000101010001100, 00010101000110, 000101010001110, 00010101000111, 00010101001, 000101010011000, 000101010011001, 000101010011010, 000101010011011, 000101010011100, 00010101001110, 000101010011110, 00010101001111, 000101010, 00010101100, 000101011001000, 00010101100100, 000101011001010, 00010101100101, 000101011001100, 00010101100110, 0001010110011, 00010101101, 000101011011000, 00010101101100, 000101011011010, 00010101101101, 000101011011100, 00010101101110, 0001010110111, 000101011, 00010110, 00010110100, 000101101001, 0001011010011000, 0001011010011001, 00010110100110011, 000101101001100, 00010110100110, 0001011010011, 00010110101, 00010110110, 000101101101, 00010110111, 00010111000, 000101110001000, 000101110001001, 000101110001010, 00010111000101, 000101110001100, 00010111000110, 000101110001110, 00010111000111, 00010111001, 000101110, 00010111, 0001100, 00011001000, 000110010001000, 000110010001001, 000110010001010, 000110010001011, 000110010001, 00011001001, 00011001010, 000110010101000, 000110010101001, 0001100101010, 000110010101, 00011001011, 00011001100, 000110011001000, 00011001100100, 000110011001010, 00011001100101, 000110011001, 0001100110, 00011001110, 0001100111, 0001101, 00011011000, 000110110001000, 000110110001001, 000110110001010, 000110110001011, 000110110001, 00011011001, 00011011010, 000110110101000, 000110110101001, 0001101101010, 000110110101, 00011011011, 00011011, 00011100, 00011101000, 000111010001000, 000111010001001, 000111010001010, 000111010001011, 000111010001, 00011101001, 000111010, 00011101, 00011110, 00011111000, 000111110001000, 000111110001001, 000111110001010, 00011111000101, 000111110001, 00011111001, 000111110, 00011111, 00100, 00100100, 00100100100, 00100100101, 00100100110, 001001001110, 0010010011101, 00100100111011, 001001001110111, 00100100111, 00100101, 00100101100, 00100101101, 001001011, 00100110, 00100110100, 00100110101, 00100110110, 001001101110, 0010011011101, 00100110111011, 001001101110111, 00100110111, 001001110, 00100111, 001010, 0010101, 00101100, 00101100100, 00101100101, 00101100110, 0010110011100, 00101100111001, 001011001110011, 0010110011101, 00101100111011, 001011001110111, 00101100111, 00101101, 00101101100, 00101101101, 001011011, 001011100, 0010111001000, 00101110010001, 001011100100011, 0010111001001, 00101110010011, 001011100100111, 00101110010100, 001011100101001, 0010111001010, 001011100101, 0010111001100, 00101110011001, 001011100110011, 0010111001101, 00101110011011, 001011100110111, 00101110011, 00101110100, 00101110101, 0010111011000, 00101110110001, 001011101100011, 0010111011001, 00101110110011, 001011101100111, 00101110110, 0010111011100, 00101110111001, 001011101110011, 0010111011101, 00101110111011, 001011101110111, 00101110111, 0010111, 0011000, 00110001000, 001100010001000, 001100010001001, 001100010001010, 001100010001011, 001100010001100, 001100010001101, 001100010001110, 001100010001111, 0011000100, 00110001010, 001100010101000, 001100010101001, 0011000101010, 001100010101100, 00110001010110, 0011000101011, 0011000101, 00110001100, 0011000110, 00110001110, 0011000111, 0011001, 00110011000, 001100110001000, 001100110001001, 001100110001010, 001100110001011, 001100110001100, 00110011000110, 001100110001110, 00110011000111, 0011001100, 00110011010, 001100110101000, 001100110101001, 0011001101010, 001100110101100, 00110011010110, 0011001101011, 0011001101, 00110011, 00110100, 00110100100, 00110100101, 00110100110, 001101001110, 00110100111, 00110101000, 001101010001000, 001101010001001, 001101010001010, 00110101000101, 001101010001100, 00110101000110, 001101010001110, 00110101000111, 0011010100, 001101010, 00110101100, 0011010110, 001101011, 00110110, 00110110100, 001101101010, 00110110101, 00110110110, 001101101110, 00110110111, 00110111000, 001101110001000, 001101110001001, 001101110001010, 00110111000101, 001101110001100, 00110111000110, 001101110001110, 00110111000111, 0011011100, 001101110, 00110111, 0011, 010, 0101000, 01010001000, 010100010001000, 010100010001001, 0101000100010, 010100010001100, 01010001000110, 0101000100011, 01010001001, 010100010011000, 010100010011001, 0101000100110, 010100010011100, 01010001001110, 0101000100111, 01010001010, 01010001011, 01010001100, 010100011001000, 01010001100100, 0101000110010, 010100011001100, 01010001100110, 0101000110011, 01010001101, 010100011011000, 01010001101100, 0101000110110, 010100011011100, 01010001101110, 0101000110111, 01010001110, 0101000111, 0101001, 01010011000, 010100110001000, 010100110001001, 0101001100010, 010100110001100, 01010011000110, 0101001100011, 01010011001, 01010011010, 01010011011, 01010011, 0101010, 01010110, 01010110100, 010101101001, 0101011010011000, 01010110100110001, 010101101001100011, 010101101001100, 01010110100110, 0101011010011, 01010110101, 01010110110, 010101101101, 01010110111, 0101011, 0101100, 01011001000, 010110010001000, 010110010001001, 0101100100010, 010110010001, 01011001001, 01011001010, 01011001011, 01011001, 0101101, 01011011000, 010110110001000, 010110110001001, 0101101100010, 010110110001, 01011011001, 01011011010, 01011011011, 01011011, 01011100, 0101110, 010111, 01100000, 011000001, 01100000110, 011000001110, 01100000111010, 011000001110101, 01100000111011, 011000001110111, 01100000111, 01100001, 011000011, 01100001110, 0110000111, 01100010, 011000101, 01100010110, 011000101110, 01100010111010, 011000101110101, 01100010111011, 011000101110111, 01100010111, 01100011, 01100100, 01100100100, 011001001010, 0110010010101, 01100100101011, 011001001010111, 01100100101, 01100100110, 011001001110, 01100100111010, 011001001110101, 01100100111011, 011001001110111, 01100100111, 0110010, 0110011000, 01100110001, 0110011001, 011001100110, 01100110011010, 011001100110101, 01100110011011, 011001100110111, 01100110011, 01100110100, 011001101010, 0110011010101, 01100110101011, 011001101010111, 01100110101, 01100110110, 011001101110, 01100110111010, 011001101110101, 01100110111011, 011001101110111, 01100110111, 0110011, 01101000, 011010001, 01101000110, 0110100011100, 01101000111001, 011010001110011, 01101000111010, 011010001110101, 01101000111011, 011010001110111, 01101000111, 01101001, 011010100, 0110101001000, 01101010010001, 011010100100011, 01101010010010, 011010100100101, 01101010010011, 011010100100111, 01101010010, 0110101001100, 01101010011001, 011010100110011, 01101010011010, 011010100110101, 01101010011011, 011010100110111, 01101010011, 011010101, 0110101011000, 01101010110001, 011010101100011, 01101010110010, 011010101100101, 01101010110011, 011010101100111, 01101010110, 0110101011100, 01101010111001, 011010101110011, 01101010111010, 011010101110101, 01101010111011, 011010101110111, 01101010111, 0110101, 01101100, 01101100100, 0110110010100, 01101100101001, 011011001010011, 0110110010101, 01101100101011, 011011001010111, 01101100101, 01101100110, 01101100111000, 011011001110001, 01101100111001, 011011001110011, 01101100111010, 011011001110101, 01101100111011, 011011001110111, 01101100111, 0110110, 0110111000000, 01101110000001, 011011100000011, 0110111000001, 01101110000011, 011011100000111, 0110111000010, 011011100001, 0110111000100, 01101110001001, 011011100010011, 0110111000101, 01101110001011, 011011100010111, 01101110001, 01101110010000, 011011100100001, 01101110010001, 011011100100011, 01101110010010, 011011100100101, 01101110010011, 011011100100111, 01101110010, 01101110011000, 011011100110001, 01101110011001, 011011100110011, 01101110011010, 011011100110101, 01101110011011, 011011100110111, 01101110011, 0110111010000, 01101110100001, 011011101000011, 0110111010001, 01101110100011, 011011101000111, 01101110100, 0110111010100, 01101110101001, 011011101010011, 0110111010101, 01101110101011, 011011101010111, 01101110101, 01101110110000, 011011101100001, 01101110110001, 011011101100011, 01101110110010, 011011101100101, 01101110110011, 011011101100111, 01101110110, 01101110111000, 011011101110001, 01101110111001, 011011101110011, 01101110111010, 011011101110101, 01101110111011, 011011101110111, 01101110111, 0110111, 01110000, 011100001, 01110000110, 0111000011, 01110001000, 011100010001000, 011100010001001, 0111000100010, 011100010001100, 01110001000110, 0111000100011, 0111000100, 01110001010, 0111000101, 01110001100, 0111000110, 01110001110, 0111000111, 011100100, 0111001001, 011100101, 01110010110, 0111001011, 01110011000, 011100110001000, 011100110001001, 0111001100010, 011100110001100, 01110011000110, 0111001100011, 0111001100, 011100110, 01110011, 01110100, 01110100100, 011101001010, 01110100101, 01110100110, 0111010011, 0111010, 0111011000, 01110110001, 0111011001, 01110110100, 011101101010, 01110110101, 01110110110, 0111011011, 0111011, 0111 }

This is a set of 696 maximal valid colorings, varying in length from 3 to 18. There are two colorings of maximum length:

$$\{000101101001100011, 010101101001100011\}.$$

The first of these can be written as sets as:

$$\begin{aligned} R &= \{1, 2, 3, 5, 8, 10, 11, 14, 15, 16\}, \\ B &= \{4, 6, 7, 9, 12, 13, 17, 18\}. \end{aligned}$$

The second maximum length coloring is the same except for the coloring of 2. In either case, one can verify relatively quickly no solutions exist in a single color class to $\mathcal{E}_{2,3}$. Neither color class (in either coloring) can be extended to include 19 due to the following potential monochromatic solutions (red and blue, respectively):

$$\begin{aligned} 1^2 + 15^2 + 16^2 &= 11^2 + 19^2, \\ 4^2 + 9^2 + 19^2 &= 13^2 + 17^2. \end{aligned}$$

(These do not represent all such triples – coloring 19 red or blue in any case introduces a number of these solutions. In most cases, including the rest of these examples, it is conceivable to have many such potential monochromatic solutions, although one of each color will suffice to quash the possibility of extension.)

The Equation $\mathcal{E}_{1,3}$ With 2 Colors

It would be impossible to present a certificate for $R_2(\mathcal{E}_{1,3}) = 105$ in print. This certificate consists of 19,471,702 maximal valid colorings, 188,160 of which are maximum length (104, that is). We will examine only one such maximum length coloring:

$$\begin{aligned} R &= \{1,4,6,9,10,11,14,16,17,21,24,25,26,29,30,31,34,36,\\ &\quad 37,39,41,43,44,45,46,49,53,54,56,57,60,63,64,65,66,\\ &\quad 67,70,73,74,76,77,80,82,83,84,87,88,92,93,94,95,96,\\ &\quad 97,98,100,102,103,104\},\\ B &= \{2,3,5,7,8,12,13,15,18,19,20,22,23,27,28,32,33,35,\\ &\quad 38,40,42,47,48,50,51,52,55,58,59,61,62,68,69,71,72,\\ &\quad 75,78,79,81,85,86,89,90,91,99,101\}. \end{aligned}$$

One can check that this coloring has no monochromatic solutions to $\mathcal{E}_{1,3}$, but it cannot be extended and remain valid, due to the following potential monochromatic triples (in red and blue, respectively):

$$\begin{aligned} 14^2+70^2+77^2 &= 105^2,\\ 18^2+51^2+90^2 &= 105^2. \end{aligned}$$

A full certificate for this Rado number is available at
`http://www.kellenmyers.org/rado/`.
It is 35 MB compressed and can be uncompressed to 1.04 GB.

The Equation $\mathcal{E}_{3,4}$ With 3 Colors

A certificate for $R_3(\mathcal{E}_{3,4}) = 32$ is much smaller than that for $R_2(\mathcal{E}_{1,3}) = 105$, but is still much too long to present in print. It consists of 3,612,110 maximal valid colorings. However, only six of them are maximum length (31) and they are as follows:

$$\begin{aligned} \{&0112010021010212000212011222000, 0112010021010212000212011222010,\\ &0112010021010212000212011222100, 0112010021010212000212011222110,\\ &0112201200212001022210010102220, 0112201200212001022210010102221\}. \end{aligned}$$

We can examine the first of these, which can be expressed as:

$$\begin{aligned} R &= \{1,5,7,8,11,13,17,18,19,23,29,30,31\},\\ B &= \{2,3,6,10,12,15,21,24,25\},\\ G &= \{4,9,14,16,20,22,26,27,28\}. \end{aligned}$$

One can verify that there are no solutions to $\mathcal{E}_{3,4}$ in any one of these color classes, so this coloring is valid, but it cannot be extended to include 32 and remain valid. This is demonstrated by the following potential monochromatic solutions, in red, blue, and green respectively:

$$
\begin{aligned}
7^2+18^2+30^2 &= 8^2+8^2+11^2+32^2,\\
3^2+15^2+32^2 &= 2^2+2^2+25^2+25^2,\\
14^2+20^2+32^2 &= 9^2+9^2+27^2+27^2.
\end{aligned}
$$

A full certificate for this Rado number is available at

`http://www.kellenmyers.org/rado/`.

It is 6.4 MB compressed and can be uncompressed to 84 MB.

The Equation $x^2 + y^2 = z^2 + 2w^2$ With 2 Colors

We are able to generate a full certificate for this Rado number, but again it is too lengthy to reproduce in print. For this variation on the theme of $\mathcal{E}_{a,b}$, we find that there are 3845 maximal valid colorings, of which 20 are length 32. They are given below:

$$
\begin{aligned}
\{&00010110110111001001000100010100, 00010110110111001001000100010110,\\
&00010110110111001001000100011100, 00010110110111001001000100011110,\\
&00010110110111001001000100110100, 00010110110111001001000100110110,\\
&00010110110111001001001100010100, 00010110110111001001001100010110,\\
&00010110110111001001001100011100, 00010110110111001001001100011110,\\
&01010110110111001001000100010100, 01010110110111001001000100010110,\\
&01010110110111001001000100011100, 01010110110111001001000100011110,\\
&01010110110111001001000100110100, 01010110110111001001000100110110,\\
&01010110110111001001001100010100, 01010110110111001001001100010110,\\
&01010110110111001001001100011100, 01010110110111001001001100011110\}.
\end{aligned}
$$

The first of these can be split into the following color classes:

$$
\begin{aligned}
R &= \{1,2,3,5,8,11,15,16,18,19,21,22,23,25,26,27,29,31,32\},\\
B &= \{4,6,7,9,10,12,13,14,17,20,24,28,30\}.
\end{aligned}
$$

One may verify that there are no solutions to $x^2 + y^2 = z^2 + 2w^2$ among either color class in relatively short order. But this coloring (like the other nineteen others in the list above) cannot be extended and remain valid, considering for example the following potentially monochromatic solutions involving 33 (in R and B respectively):

$$
\begin{aligned}
15^2+33^2 &= 8^2+25^2,\\
12^2+33^2 &= 9^2+24^2.
\end{aligned}
$$

A full certificate for this Rado number is available at

`http://www.kellenmyers.org/rado/`.

It is 7 KB compressed and can be uncompressed to 80 KB.

C.2. Square Gaps

The Furstenberg-Sárközy Number FS_3

Recall that FS_r is the least integer such that any r coloring of $[1, FS_r]$ must contain monochromatic x and y such that $x - y$ is a square. To prove that $FS_3 \geq 29$, we must provide a coloring of $[1, 28]$

$$
\begin{aligned}
R &= \{2, 5, 7, 10, 15, 17, 20, 25, 28\},\\
B &= \{1, 4, 9, 12, 14, 19, 22, 24, 27\},\\
G &= \{3, 6, 8, 11, 13, 16, 18, 21, 23, 26\}.
\end{aligned}
$$

Although it does not prove $FS_3 \leq 29$ exhaustively, we remark that this particular valid coloring cannot be extended. If 29 were added to R, $29 - 28 = 1$ is a square; if B, $29 - 4 = 25$ is a square; and if G, $29 - 13 = 16$ is a square.

The Corresponding Rado Number $R_3\left(x - y = z^2\right)$

Likewise, we considered Rado numbers for the equation $x - y = z^2$ (bounded below by FS_r) and obtained $R_3\left(x - y = z^2\right) = 204$. The following 3-coloring of $[1, 203]$ provides the lower-bound:

$$
\begin{aligned}
R = \{&3, 5, 7, 10, 13, 20, 23, 27, 34, 37, 40, 44, 47, 50, 57, 60, 64, 67,\\
&70, 77, 80, 84, 87, 94, 97, 101, 104, 111, 114, 117, 125, 128, 130,\\
&131, 138, 141, 148, 151, 154, 158, 161, 165, 168, 171, 175, 178, 181,\\
&185, 188, 191, 195, 198, 199, 202\},\\
B = \{&2, 8, 11, 14, 16, 17, 19, 22, 25, 28, 30, 31, 33, 36, 39, 42, 45,\\
&48, 51, 53, 54, 56, 59, 62, 65, 68, 71, 73, 74, 76, 79, 82, 85, 88,\\
&90, 91, 93, 96, 99, 102, 105, 107, 108, 110, 113, 116, 119, 121, 122,\\
&124, 127, 133, 134, 136, 139, 142, 144, 145, 147, 150, 153, 156, 159,\\
&162, 164, 167, 170, 173, 176, 179, 182, 184, 187, 190, 193, 196, 201\},\\
G = \{&1, 4, 6, 9, 12, 15, 18, 21, 24, 26, 29, 32, 35, 38, 41, 43, 46, 49,\\
&52, 55, 58, 61, 63, 66, 69, 72, 75, 78, 81, 83, 86, 89, 92, 95, 98,\\
&100, 103, 106, 109, 112, 115, 118, 120, 123, 126, 129, 132, 135, 137,\\
&140, 143, 146, 149, 152, 155, 157, 160, 163, 166, 169, 172, 174, 177,\\
&180, 183, 186, 189, 192, 194, 197, 200, 203\}.
\end{aligned}
$$

Again, we can check (perhaps in more computing time than the previous, but with no more complex an algorithm to do the checking) that this is a valid coloring for this problem: None of these three sets has monochromatic solutions to $x-y=z^2$. However, this coloring can not be extended, because:

$$\begin{aligned} 204 - 104 &= 10^2 \\ 204 - 8 &= 14^2 \\ 204 - 203 &= 1^2, \end{aligned}$$

where we can observe that 10 and 104 are red, that 8 and 13 are blue, and that 1 and 203 are green. (These are not necessarily the only examples that exclude 204 from each color class.)

C.3. Reciprocal Schur Numbers

We produced two reciprocal Schur numbers with three variables. The first, with two colors, was $R_2\left(\frac{1}{x}+\frac{1}{y}=\frac{1}{z}\right)=60$. The following coloring has no monochromatic solutions to the equation:

$$\begin{aligned} R &= \{2,3,5,8,11,12,13,14,15,18,20,31,32,33,34,35,37,38,41,42,43, \\ &\quad 44,45,46,47,48,49,50,51,52,53,54,55,56,57,58,59\}, \\ B &= \{1,4,6,7,9,10,16,17,19,21,22,23,24,25,26,27,28,29,30,36,39,40\}. \end{aligned}$$

This valid coloring cannot be extended because we have the following potential red and blue (respectively) solutions:

$$\frac{1}{60}+\frac{1}{15}=\frac{1}{12},$$

$$\frac{1}{60}+\frac{1}{40}=\frac{1}{24}.$$

Likewise, with three colors, we have $R_3\left(\frac{1}{x}+\frac{1}{y}=\frac{1}{z}\right)=3276$. The lower bound is demonstrated by the following 3-coloring. For the sake of brevity, this is presented in the ternary-string format as described above.

100202001120021210221021010002011022201022021022010001210110
121010210001102222010001111000012020012222210102010001020212
002102210001000201020120000201012101011222122120202210221221
220201121002120021201020100001100200112011222202222112100011
111100021200001012021212121112110110211112220111210101101100
010020120200112210002120000000102001100210000002100000000022

1011011001120200001101000010022200020022000100011002011101 22
1202111011120200000120020001000002002111212212212211012112 00
1220200202210100002011221101020120101101000000000000010100221
0022210002010000000002000000011000002000001100112001102100211
0110111200020010110012021000011000010222001011110201010101 02
0001110111020001011010111111211210100010201110101111210010
1011101011000022110020000101020102210002222202010202020222 01
0121222202202221010121122220220101000012220022220200112220 00
0000020000000002200020221122022222222221221212220122202201202
2222222021122120202020200000002202020112012222120201212200201
2200202200222201222121222222212122222002222211012220111001 01
1110202220220220022011110111021121021111111200110011110101 00
0101000000020202112002020111011111121111001121121121111102 12
1101122212002111112201121011222202011111122012102212121000 12
1100101110101010110101002111021210000101001001111001120122 20
0001010101001101110111010001001002202200220001020102212201 01
0201121010000100122022100201012201120202222122222200221210 11
1220200201201211010221212112121022100110101022122102111212 02
2202222020221121100002121212121202010010120022221212221120 201
0200012111120202012000101110021011022101011011011101111001 22
1220001010110021111110111001100011001100001010011011001111 11
1110011011021111101111111111100101120022122212022212120222 212
2202020222212222220212222202012012022211210000000001000200 21
0000002111110000100000000012101111010001110011111012001001 11
0000110000010011000000011110111111111011111110011101111111 11
1011111110121010111100100000000000111100110000000011111111 112
1111111111111011010101100001110000000000000000010000100011 012
0000000000010001010000000000002220200011001010010000111000 102
0000101111111111111111111111111010101002200000000010010022 221
0000000001010010000010000000001010010110102001000010022222 121
2222212021212102220212012220220022220022022022202212220222 22
2202222202212002222222221222221222200022220022222222022220 00221
2222222222202101022122220022020202220021020212200222202200 001
2000012022222002022202202022221000220020222002202222222220 010
0202222122020002020220222022202022200222022022102002020020 200
2201202220222020022022220220210122022200222202022122222020 2021
2222222222022222220022220222212022202222222022222221122122 2212
2221022021220002022222021020222021220212222220000200210102 0220
2220220220210022222122220222222220202022220202022220022020 222
2000220222000220210120202002200020212000222220200220002222 22
0020200200022220022222022220002222222022220202020220020200 02200

2202002022020220000222002000211200200202022000202200000002121
0000222020000200221001120220222002222202002001000022220000000
2001202220221222222000021022001002000202220202002002102202200
2200022202222022202200000220222001222021220202020020222020120
2202222022212022211001022222202222000222222022220000222202021
2200202222212202200222020122212200222202222020220020202220022
0222000000221202022012222022200222220222120000202002000002010

Again, this valid coloring cannot be extended, due to the potential red, blue, and green (respectively) solutions below:

$$\frac{1}{3276} + \frac{1}{273} = \frac{1}{252},$$

$$\frac{1}{3276} + \frac{1}{546} = \frac{1}{468},$$

$$\frac{1}{3276} + \frac{1}{2808} = \frac{1}{1512}.$$

C.4. Other

Recall that $H_r(m)$ is the r-color Hindman number, such that for any $N \geq H_r(m)$ there is a monochromatic solution to the system:

$$\begin{aligned} x_1 + x_2 + \cdots + x_{m-1} &= x_m, \\ x_1 \cdot x_2 \cdot \cdots \cdot x_{m-1} &= z. \end{aligned}$$

We have computed $H_2(2) = 39$, and we can demonstrate the lower bound with the following coloring:

$$\begin{aligned} R &= \{2, 3, 5, 7, 9, 11, 13, 15, 16, 17, 19, 20, 21, 23, 24, 25, 27, \\ &\quad 28, 29, 31, 32, 33, 35, 36, 37\} \\ B &= \{1, 4, 6, 8, 10, 12, 14, 18, 22, 26, 30, 34, 38\} \end{aligned}$$

One may check that for each pair in R, either the sum, product, or both is absent from R, and likewise for B. However, this coloring cannot be extended, because $3 + 13 = 16$ and $3 \cdot 13 = 39$ would be monochromatic (red), or else $1 + 38 = 39$ and $1 \cdot 38 = 38$ (blue).

As a final example, consider $R_2\left(x_1 + x_2 + \cdots + x_6 = 2^{x_7}\right) = 11$. The lower-bound for this Rado number can be proved by the following coloring:

$$\begin{aligned} R &= \{3, 4, 6, 7, 8, 9, 10\} \\ B &= \{1, 2, 5\} \end{aligned}$$

One can check that R and B contain no monochromatic solutions to the equation in question, a task simplified by the fact that most 6-fold sums from R or B will not be powers of 2. However, this valid coloring cannot be extended due to the potential red and blue (respectively) solutions below:

$$\begin{aligned} 9+11+11+11+11+11 &= 2^6 \\ 1+2+2+5+11+11 &= 2^5 \end{aligned}$$

D. Extended Proof of Theorem 12

We will first restate the theorem:

Theorem. *For some constant c there is a constant M such that for any a and b with $a \le b \le ca$, $R_2(\mathcal{E}_{a,b}) < M$.*

The idea behind our proof is straightforward, and it follows a depth-first search, much in the spirit of our computational methods. We will begin without a clear understanding of which values of c or M we can use, but instead, considering how we could prove such a theorem by searching the tree of colorings.

First, assume there is a valid coloring in which 1, 2, and 3 are all red. We want to construct a monochromatic solution that only uses 1, 2, and 3. We observe such a solution exists: $3^2 = 1^2 + 2^2 + 2^2$. This works along this branch of the tree not just for $\mathcal{E}_{1,3}$. We can also add sort of dummy variable to each side, like $1^2 + 3^2 = 1^2 + 1^2 + 3^2 + 2^2$, covering $\mathcal{E}_{2,4}$or indeed any $\mathcal{E}_{1+k,3+k}$. We could also double-up on the solution $(3, 1, 2, 2)$ to cover $\mathcal{E}_{2,6}$ and then any $\mathcal{E}_{2+k,6+k}$.

In summary, if $b-a$ is even, say $b-a=2k$, then consider the solution:

$$\underbrace{1^2+\cdots+1^2}_{a-k}+\underbrace{3^2+\cdots+3^2}_{k}=\underbrace{1^2+\cdots+1^2}_{a-k=b-3k}+\underbrace{1^2+2^2+2^2+\cdots+1^2+2^2+2^2}_{3k}.$$

This solution is feasible because each $x_i = 3$ for $i > a-k$ corresponds to three y terms 1,1,2, and with k of them, that gives us the correct number of x and y terms (and of course, it is important to note $3^2 = 1^2 + 2^2 + 2^2$). This requires $b \le 3a$ (equivalent to $a - k \ge 0$) in order to have enough variables on each side to make this work. So for a certain set of $\mathcal{E}_{a,b}$, we have used a solution to a smaller equation $\mathcal{E}_{a',b'} = \mathcal{E}_{1,3}$, yielding a bound $c \le \frac{b'}{a'}$.

Likewise, if $b-a=3k$, we can similarly obtain monochromatic solutions using the identity $1^2+1^2+1^2+1^2 = 2^2$, requiring $b \le 4a$.

However, we would rather have $b-a=1$ and use the identity $2^2+2^2+2^2 = 1^2+1^2+1^2+3^2$, with no divisibility restriction, so that we can find solutions of the form:

$$\underbrace{\cdots}_{a-3k}+\underbrace{2^2+\cdots+2^2}_{3k}=\underbrace{\cdots}_{b-4k}+\underbrace{1^2+1^2+1^2+3^2+\cdots+1^2+1^2+1^2+3^2}_{4k}.$$

This requires $a - 3k \geq 0$, i.e., $b \leq \frac{4}{3}a$, so $c = \frac{4}{3}$ will suffice.

For other initial colorings, like 1 red, 2 red, 3 blue, we must repeat this argument and obtain a similar bound. But we have our start – this is how we traverse the tree of colorings. Depth-first, using smaller solutions of the form $\mathcal{E}_{a',a'+1}$ to cover solutions to any $\mathcal{E}_{a,b}$ with the approriate restriction that $a \leq b \leq ca$ with $c = \frac{a'+1}{a'}$.

After tabulating the outcome, we obtain the following proof:

Proof. In Figure 2 we illustrate the tree of all colorings, truncated according to Table 7. In this table, the monochromatic elements of each branch are given as well as the corresponding solutions to some $\mathcal{E}_{a',a'+1}$.

The worst case (greatest number of squares) is $12 \cdot 2^2 = 12 \cdot 1^2 + 6^2$, which proves that this theorem holds for $c = \frac{13}{12}$ and $M = 9$. □

Monoch. Set	Sums of squares		
$\{1,2,3\}$	$2^2+2^2+2^2$	$=$	$1^2+1^2+1^2+3^2$
$\{1,2,6\}$	12×2^2	$=$	$12\times 1^2+6^2$
$\{1,3,4\}$	3^2+3^2	$=$	$1^2+1^2+4^2$
$\{1,3,8\}$	8×3^2	$=$	$8\times 1^2+9^2$
$\{2,3,4\}$	$2^2+4^2+4^2$	$=$	$3^2+3^2+3^2+3^2$
$\{2,3,5\}$	$3^2+3^2+3^2+3^2+3^2$	$=$	$2^2+2^2+2^2+2^2+2^2+5^2$
$\{2,3,6\}$	$9\times 2^2+6^2+6^2$	$=$	12×3^2
$\{2,4,6\}$	$4^2+4^2+4^2$	$=$	$2^2+2^2+2^2+6^2$
$\{3,4,5\}$	5^2	$=$	3^2+4^2
$\{3,4,7\}$	7×4^2	$=$	$7\times 3^2+7^2$
$\{3,5,8\}$	$5^2+5^2+5^2+5^2$	$=$	$3^2+3^2+3^2+3^2+8^2$
$\{3,6,9\}$	$6^2+6^2+6^2$	$=$	$3^2+3^2+3^2+9^2$
$\{4,6,8\}$	4^2+8^2+8	$=$	$6^2+6^2+6^2+6^2$
$\{1,3,5,6\}$	$1^2+6^2+6^2+6^2$	$=$	$3^2+5^2+5^2+5^2+5^2$
$\{1,3,6,7\}$	$6^2+6^2+6^2$	$=$	$1^2+3^2+7^2+7^2$
$\{1,4,5,6\}$	$4^2+4^2+4^2+4^2$	$=$	$1^2+1^2+1^2+5^2+6^2$
$\{2,5,8,9\}$	5^2+8^2	$=$	$2^2+2^2+9^2$
$\{3,5,6,7\}$	$6^2+6^2+6^2$	$=$	$3^2+5^2+5^2+7^2$
$\{1,2,4,5,9\}$	$2^2+4^2+4^2+5^2+5^2$	$=$	$1^2+1^2+1^2+1^2+1^2+9^2$
$\{1,2,4,7,9\}$	$2^2+4^2+4^2+7^2$	$=$	$1^2+1^2+1^2+1^2+9^2$
$\{1,2,4,8,9\}$	$2^2+4^2+8^2$	$=$	$1^2+1^2+1^2+9^2$
$\{1,2,5,7,9\}$	$2^2+2^2+2^2+5^2+7^2$	$=$	$1^2+1^2+1^2+1^2+1^2+9^2$

Table 7: Table for proof of Theorem 12

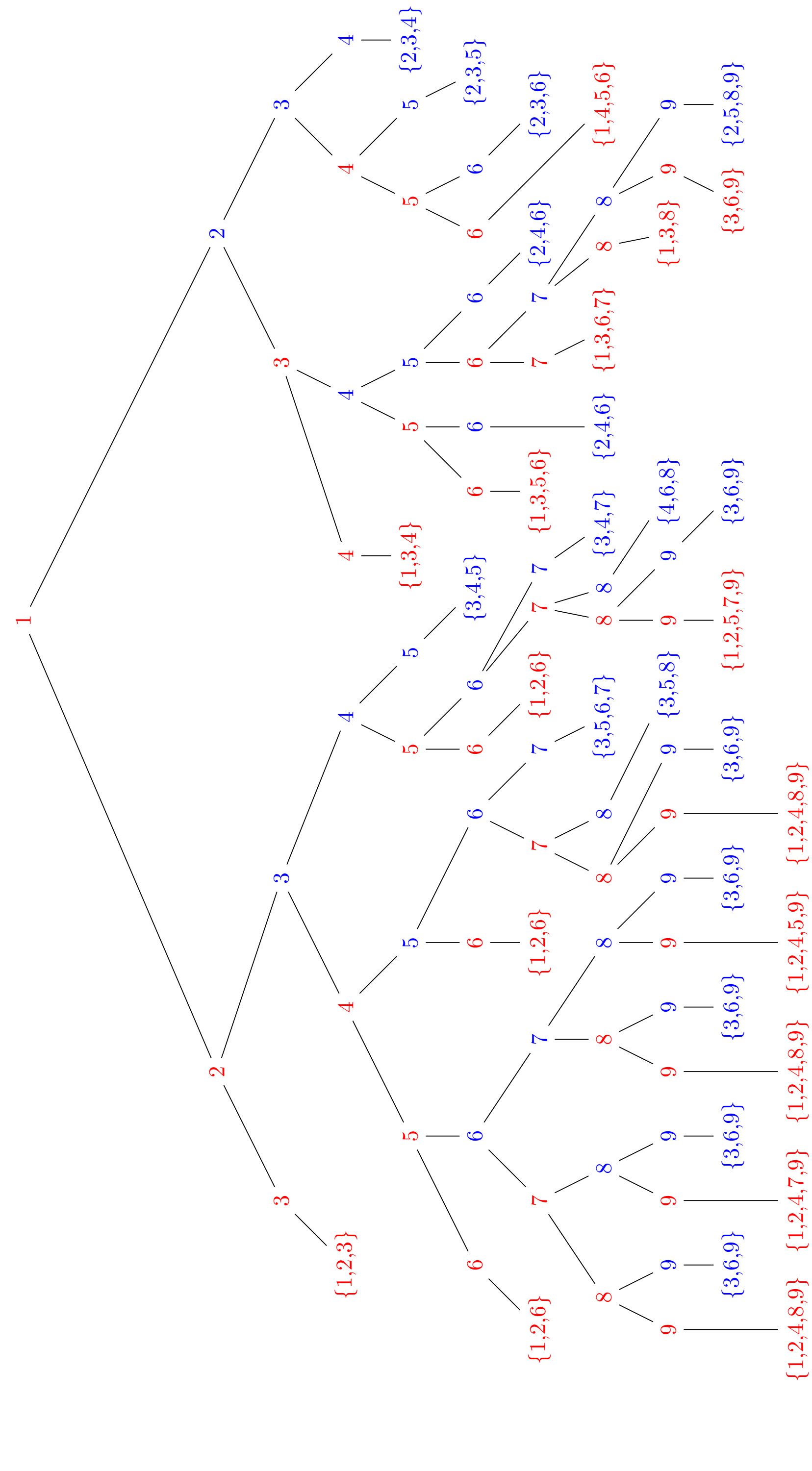

Figure 2: Figure for proof of Theorem 12

This proof was constructed in a way that prioritized three things: first, to terminate every branch of the tree as soon as feasible; second, to minimize value of M; third, to minimize a'. The proof could be reconstructed to give better (higher) values of c, allowing instead poorer (larger) values of M. Regardless, whatever value of c and M one may obtain, it is important to note that this value M is a uniform bound, confirming our intuition that in some cases, increasing a and b will not increase the Rado number of $\mathcal{E}_{a,b}$ (in general, not increasing one of a or b significantly without increasing the other).

This conveniently excludes the counter-example to our intuition discussed previously, the first row of Table 1, since no value of c gives a region of $a \leq b \leq ca$ that includes a whole row of these values. Indeed, it could not, since our observation that the first row ($\mathcal{E}_{1,k}$) eventually grows without bound (bounded below by $\sqrt{k}$) should apply to any row of the table (with analogous lower bound $\sqrt{b/a}$, likewise unbounded with a fixed and b increasing).

After obtaining the set of solutions listed in Table 7, it is possible to delete some of them (as we have done for our previous, more concise proof). Some of them may end up being unnecessary, once this set is assembled. This corresponds to exploring certain branches of the tree longer but reusing solution-sets more efficiently.

A more streamlined reconstruction of the proof could also refer to any particular value of $R_2\left(\mathcal{E}_{a',a'+1}\right)$, which would itself provide a bound M for $c = \frac{a'+1}{a'}$. (The tree in this version of the proof would be the same as the tree of valid colorings that would prove $R_2\left(\mathcal{E}_{a',a'+1}\right)$ in the first place.).

That means that the result of Heule, Kullmann, & Marek [21] proves the theorem for $c = 2$ (which covers, in some sense, half of all possible $\mathcal{E}_{a,b}$) with a very large value of $M = 7825$, while our value of $R_2\left(\mathcal{E}_{2,3}\right)$ proves this theorem with $c = 3/2$ with a tigheter bound $M = 19$.