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ON CHANGES IN THE FROBENIUS AND SYLVESTER NUMBERS

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Abstract

For any set of positive integers A with $\gcd A = 1$, let $\Gamma(A)$ denote the set of integers that are expressible as a linear combination of elements of A with non-negative integer coefficients. Then $\mathtt{g}(A)$, $\mathtt{n}(A)$, $\mathtt{s}(A)$ denote the *largest*, the *number* of, the *sum* of positive integer(s) not in $\Gamma(A)$, respectively. We investigate the change in $\mathtt{g}(A)$, $\mathtt{n}(A)$, and $\mathtt{s}(A)$ when A changes from a two-element set to a three-element set. In particular, we determine these numbers for certain families $A = \{a,b,c\}$. For the same families A, we also determine the set $\mathcal{S}^\star(A)$ which consists of positive integers n not in $\Gamma(A)$ for which $n + \big(\Gamma(A) \setminus \{0\}\big) \subset \Gamma(A) \setminus \{0\}$. The largest element in $\mathcal{S}^\star(A)$ is $\mathtt{g}(A)$.

1. Introduction

Consider a finite set $A = \{a_1, \ldots, a_k\}$ of positive integers with gcd $A := \gcd(a_1, \ldots, a_k) = 1$. Let $\Gamma(A) := \big\{a_1x_1 + \cdots + a_kx_k : x_i \in \mathbb{Z}_{\geq 0}\big\}$ and $\Gamma^\star(A) = \Gamma(A) \setminus \{0\}$. Then $\Gamma^c(A) := \mathbb{N} \setminus \Gamma(A)$ can be shown to be a *finite* set, and this allows us to define the *Frobenius* number $\mathtt{g}(A)$ and the *Sylvester* number $\mathtt{n}(A)$:

$$\mathtt{g}(A) := \max \Gamma^c(A), \quad \mathtt{n}(A) := \big|\Gamma^c(A)\big|.$$

The `Frobenius Problem` is to determine $\mathtt{g}(A)$ and $\mathtt{n}(A)$ in the general case.

For $A = \{a,b\}$, $\gcd(a,b) = 1$, Sylvester [10, 11] showed

$$\mathtt{g}(a,b) = ab - a - b, \quad \mathtt{n}(a,b) = \frac{1}{2}(a-1)(b-1).$$

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Exact values for $\mathtt{g}(A)$ have been known only for few cases when $|A| > 2$, in some cases when the elements of A satisfy a specific condition. For instance, $\mathtt{g}(ab, bc, ca) = 2abc - ab - bc - ca$ whenever $\gcd(a, b) = \gcd(b, c) = \gcd(c, a) = 1$. On the other hand, bounds and algorithms to compute $\mathtt{g}(A)$, especially in the case $|A| = 3$, have been a major source of research. Corresponding results for $\mathtt{n}(A)$ have been much rarer, even in special cases.

Brown and Shiue [1] introduced the related problem of determining the function

$$\mathtt{s}(A) := \sum_{n \in \Gamma^c(A)} n,$$

and found

$$\mathtt{s}(a, b) = \frac{1}{12}(a-1)(b-1)(2ab - a - b - 1)$$

when $\gcd(a, b) = 1$; see also [13].

The set $\Gamma(A)$ is closed under addition, and so $n + \Gamma(A) \subseteq \Gamma(A)$ whenever $n \in \Gamma(A)$. It is conceivable that $n \in \Gamma^c(A)$ satisfy a slightly modified condition, replacing $\Gamma(A)$ by $\Gamma(A) \setminus \{0\}$. In fact, $\mathtt{g}(A)$ is clearly the largest number satisfying such a condition. Thus we study the set given by

$$\mathcal{S}^\star(A) := \big\{n \in \Gamma^c(A) : n + \Gamma^\star(A) \subset \Gamma^\star(A)\big\},$$

where $\Gamma^\star(A) = \Gamma(A) \setminus \{0\}$. Members of $\mathcal{S}^\star(A)$ are called *pseudo-Frobenius* numbers, and the size of $S^\star(A)$ is called the *type* of A.

The main purpose of our paper is to investigate the change in the Frobenius number $\mathtt{g}(A)$ and the Sylvester number $\mathtt{n}(A)$ as we move from a 2-set $\{a, b\}$ to a 3-set $\{a, b, c\}$ for certain range of $c \in \Gamma^c(A)$ explictly formulated in the following paragraph. The fact that this forces $\gcd(a, b) = 1$ in the 3-set does not reduce the generality of our argument due to Proposition 2. We also investigate corresponding changes in $\mathtt{s}(A)$, and directly determine the set $\mathcal{S}^\star(\{a, b, c\})$. Since $\mathtt{g}(A)$, $\mathtt{n}(A)$, $\mathtt{s}(A)$, and $\mathcal{S}^\star(A)$ are well known when $|A| = 2$, determining changes in these functions would amount to the determination of these functions for the case $|A| = 3$. Explicit formulae for each of these functions is unknown, except for $\mathtt{g}(A)$.

We list preliminary results that are key to this paper in Section 2, present our main results in Section 3, and conclude by listing the cases in which the problem remains open in Section 4. In moving from a 2-set $\{a, b\}$ to a 3-set $\{a, b, c\}$, we may assume that $c \notin \Gamma(\{a, b\})$, since $c \in \Gamma(\{a, b\})$ can be easily seen to imply $\Gamma(\{a, b, c\}) = \Gamma(\{a, b\})$. We may write any $c \notin \Gamma(\{a, b\})$ in the form $bs - ar$ with $s \in \{1, \ldots, a-1\}$ and $1 \leq r < \frac{bs}{a}$. Our main results in Section 3 are:

(i) Theorem 1, in which we determine the change in the least positive integer representable in each residue class modulo a, for the cases $1 \leq r \leq \left\lceil \frac{b(a-1)}{2a} \right\rceil$ and $\frac{a-1}{2} < s \leq a - 1$,

(ii) Theorem 2, in which we give precise results for the functions $\mathtt{g}(A)$, $\mathtt{n}(A)$, $\mathtt{s}(A)$, and determine the set $\mathcal{S}^\star(A)$ for the cases $1 \le r \le \left\lceil \frac{b(a-1)}{2a} \right\rceil$ and $\frac{a-1}{2} < s \le a-1$ using the result of Theorem 1, and

(iii) Theorem 3, in which we give a sharp lower bound for the differences $\mathtt{g}(\{a,b\}) - \mathtt{g}(\{a,b,c\})$ and $\mathtt{n}(\{a,b\}) - \mathtt{n}(\{a,b,c\})$. We also characterize c for which these lower bounds are attained.

Results corresponding to those in Theorem 1, and consequently corresponding to Theorem 2 for the remaining choices of r and s are much more difficult, and under ongoing investigation.

2. Preliminary Results

Suppose A is any set of positive integers with $\gcd A = 1$, and let $a \in A$. For each residue class $\mathbf{C}$ modulo a, let $\mathbf{m_C}$ denote the least integer in $\Gamma(A) \cap \mathbf{C}$. It is well known that the functions $\mathtt{g}$, $\mathtt{n}$ and $\mathtt{s}$ are easily determined from the values of $\mathbf{m_C}$. The following result, part (i) of which is due to Brauer and Shockley [2], part (ii) to Selmer [8], and part (iii) to Tripathi [13], is often a key step in this determination.

Proposition 1. ([2], [8], [13])
Let A be any set of positive integers with $\gcd(A) = 1$. For any $a \in A$,

(i) $\mathtt{g}(A) = \left(\max\limits_{\mathbf{C}} \mathbf{m_C}\right) - a.$

(ii) $\mathtt{n}(A) = \dfrac{1}{a}\sum\limits_{\mathbf{C}} \mathbf{m_C} - \dfrac{1}{2}(a-1).$

(iii) $\mathtt{s}(A) = \dfrac{1}{2a}\sum\limits_{\mathbf{C}} \mathbf{m_C}^2 - \dfrac{1}{2}\sum\limits_{\mathbf{C}} \mathbf{m_C} + \dfrac{1}{12}(a^2-1).$

In each case, the maximum and the sums are taken over all nonzero classes $\mathbf{C}$ modulo a.

The following reduction formulae for $\mathtt{g}(A)$, due to Johnson [5] for the three variable case and to Brauer and Shockley [2] for the general case, and for $\mathtt{n}(A)$ due to Rødseth [6], are useful in cases when all but one member of A share a common divisor greater than 1.

Proposition 2. ([2], [6])
Let A be any set of positive integers with $\gcd(A) = 1$. If $a \in A$ is such that $\gcd(A \setminus \{a\}) = d$, and $A' = \frac{1}{d}\big(A \setminus \{a\}\big)$, then

(i) $\mathtt{g}\big(A\big) = d \cdot \mathtt{g}\big(A' \cup \{a\}\big) + a(d-1).$

(ii) $\mathtt{n}\big(A\big) = d \cdot \mathtt{n}\big(A' \cup \{a\}\big) + \dfrac{1}{2}(a-1)(d-1)$.

The set $\mathcal{S}^\star(A)$ consists of positive integers n in $\Gamma^c(A)$ such that translating the set of positive integers in $\Gamma(A)$ by n results in a subset of $\Gamma(A)$. Since $\mathtt{g}(A) = \max \mathcal{S}^\star(A)$, determining $\mathcal{S}^\star(A)$ ensures that $\mathtt{g}(A)$ is also determined. The following result is due to Tripathi [12].

Proposition 3. ([12])
Let A be any set of positive integers with $\gcd(A) = 1$. Let $a \in A$, and let $\mathbf{m}_x$ denote the least integer in $\Gamma(A)$ congruent to x modulo a, $1 \le x \le a-1$. Then

$$\mathcal{S}^\star(A) = \big\{\mathbf{m}_x - a : \mathbf{m}_x + \mathbf{m}_y \ge \mathbf{m}_{x+y} + a \text{ for } 1 \le y \le a-1\big\}.$$

For the case where $|A| = 3$, Rosales and García-Sánchez [7] provide analogues of the results in Proposition 1 for the functions $\mathtt{g}(A)$ and $\mathtt{n}(A)$ and of the result in Proposition 3 for the set $\mathcal{S}^\star(A)$. Tripathi and Vijay [14] have also provided results similiar to these, but only for the function $\mathtt{g}(A)$ and for the set $\mathcal{S}^\star(A)$.

Proposition 4. ([7])
Let $A = \{a, b, c\}$ be a set of positive integers, with $\gcd(a, b, c) = 1$. Define c_1, c_2, c_3 by

$$\begin{aligned} c_1 &:= \min\big\{x \in \mathbb{N} : xa \in \Gamma(\{b,c\})\big\}, \\ c_2 &:= \min\big\{x \in \mathbb{N} : xb \in \Gamma(\{a,c\})\big\}, \\ c_3 &:= \min\big\{x \in \mathbb{N} : xc \in \Gamma(\{a,b\})\big\}. \end{aligned}$$

Then there exist nonnegative integers $r_{12}, r_{13}, r_{21}, r_{23}, r_{31}, r_{32}$ such that

$$c_1 a = r_{12} b + r_{13} c, \quad c_2 b = r_{21} a + r_{23} c, \quad c_3 c = r_{31} a + r_{32} b.$$

Then

(i) $\mathtt{g}(A) = \max\big\{(c_3-1)c + (r_{12}-1)b - a, (c_2-1)b + (r_{13}-1)c - a\big\}$.

(ii) $\mathtt{n}(A) = \dfrac{1}{2}\big((c_1-1)a + (c_2-1)b + (c_3-1)c - c_1c_2c_3 + 1\big)$.

(iii) $\mathcal{S}^\star(A) = \big\{(c_3-1)c + (r_{12}-1)b - a, (c_2-1)b + (r_{13}-1)c - a\big\}$.

3. Main Results

Let A be any set of positive integers that are relatively prime. Unless otherwise specified, we use Proposition 1 to compute $\mathtt{g}(A)$. We consider the congruence classes modulo $\min A$, and denote the least integer in $\Gamma(A)$ congruent to i modulo $\min A$ by $\mathbf{m}_i$. It is trivial that $\Gamma^c(A) = \emptyset$ if $1 \in A$, and consequently that $\mathtt{g}(A) = -1$ in this case.

Proposition 5. *Let a, b be positive integers, with* $\gcd(a, b) = 1$*.*

(a) *Every $c \in \mathbb{Z}$ is expressible in the form $ax + by$ with $0 \le y \le a - 1$ and $x \in \mathbb{Z}$.*

(b) *Every $c \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is expressible in the form $ax + by$ with $0 \le y \le a - 1$ and $x \ge -\frac{by}{a}$.*

(c) *Every $c \in \Gamma(\{a, b\})$ is expressible in the form $ax + by$ with $0 \le y \le a - 1$ and $x \ge 0$.*

(d) *Every $c \in \Gamma^c(\{a, b\}) = \mathbb{N} \setminus \Gamma(\{a, b\})$ is expressible in the form $ax + by$ with $1 \le y \le a - 1$ and $-\frac{by}{a} < x < 0$.*

Each expression is unique.

Proof. (a) Every integer is expressible in the form $ax + by$ with $x, y \in \mathbb{Z}$ since $\gcd(a, b) = 1$. Moreover if $c = ax_0 + by_0$, then all solution to $ax + by = c$ are given by $x = x_0 + bt$, $y = y_0 - at$ with $t \in \mathbb{Z}$. Therefore there is a unique representation of c by the form $ax + by$ with $0 \le y \le a - 1$.

(b) By part (a), there is a unique representation of c by the form $ax + by$ with $0 \le y \le a - 1$. Now $c \in \mathbb{N}_0$ if and only if $ax + by \ge 0$, which is the same as $x \ge -\frac{by}{a}$.

(c) By part (a), there is a unique representation of c by the form $ax + by$ with $0 \le y \le a - 1$. If $c = ax_0 + by_0$ with $x_0 \ge 0$ and $0 \le y_0 \le a - 1$, then $c \in \Gamma(\{a, b\})$. Suppose $x_0 < 0$ and $c = ax + by$ with $x \ge 0$. Since $x = x_0 + bt$, $t \ge 1$, so that $y = y_0 - at \le y_0 - a < 0$. Hence there is no representation of c by the form $ax + by$ with both $x, y \ge 0$ when $x_0 < 0$.

(d) This follows from parts (b) and (c) since $\Gamma^c(\{a, b\}) = \mathbb{N} \setminus \Gamma(\{a, b\}) = \mathbb{N}_0 \setminus \Gamma(\{a, b\})$.

$\square$

Remark 1. We use the equivalent form of Proposition 5 (d):

$$\Gamma^c(\{a, b\}) = \left\{ by - ax : 1 \le y \le a - 1, 1 \le x < \tfrac{by}{a} \right\}.$$

Corollary 1. *Let a, b be positive integers, with* $\gcd(a, b) = 1$*. For $0 \le i \le a - 1$, let $\mathbf{m}_i$ denote the smallest integer in $\Gamma(\{a, b\})$ congruent to i mod a. Then $\big\{\mathbf{m}_i : 0 \le i \le a - 1\big\} = \big\{by : 0 \le y \le a - 1\big\}$.*

Proof. This immediately follows from Proposition 5 (c). $\square$

Proposition 6. *Let a, b be positive integers, with* $\gcd(a, b) = 1$*, and let c be any positive integer. The following are equivalent:*

(i) $c \in \Gamma(\{a,b\})$.

(ii) $\Gamma\big(\{a,b,c\}\big) = \Gamma\big(\{a,b\}\big)$.

(iii) $\mathtt{n}(a,b,c) = \mathtt{n}(a,b)$.

(iv) $\mathtt{g}(a,b,c) = \mathtt{g}(a,b)$.

Proof. We note that $\Gamma\big(\{a,b,c\}\big) \supseteq \Gamma\big(\{a,b\}\big)$, so that $\mathtt{g}(a,b,c) \le \mathtt{g}(a,b)$ and $\mathtt{n}(a,b,c) \le \mathtt{n}(a,b)$, for every positive integer c.

It is clear that (i) implies (ii), and that (ii) implies (iii). If $c \notin \Gamma(\{a,b\})$, then $c \in \Gamma^c(\{a,b\})$ but $c \notin \Gamma^c(\{a,b,c\})$. Hence $\mathtt{n}(a,b) - \mathtt{n}(a,b,c) \ge 1$, and so (iii) implies (i). This proves the equivalence of (i), (ii), and (iii).

It is clear that (ii) implies (iv). If $c \notin \Gamma(\{a,b\})$, then $c = by - ax$ for some $y \in [1, a-1]$ and $x \ge 1$ by Remark 1. But then $\mathtt{g}(a,b) = b(a-1) - a = a(x-1) + b(a-1-y) + c \in \Gamma(\{a,b,c\})$. Thus $\mathtt{g}(a,b,c) < \mathtt{g}(a,b)$, so that (iv) implies (i), which is equivalent to (ii). □

Let a, b be positive integers, with $\gcd(a,b) = 1$, and let $c \notin \Gamma(\{a,b\})$, $c > 0$. For each residue class $\mathbf{C}$ modulo a, let $\mathbf{m_C}$ denote the least integer in $\Gamma(\{a,b\}) \cap \mathbf{C}$ and let $\mathbf{m^\star_C}$ denote the least integer in $\Gamma(\{a,b,c\}) \cap \mathbf{C}$.

Theorem 1. *Let a, b be positive integers, with $\gcd(a,b) = 1$. Let $c = bs - ar$ with $1 \le r \le \left\lceil \frac{b(a-1)}{2a} \right\rceil$ and $\frac{a-1}{2} < s \le a-1$. Then*

$$\mathbf{m}^\star_{bi} = bi - ar \cdot \left\lfloor \frac{i}{s} \right\rfloor \ \textit{for}\ 1 \le i \le a-1.$$

Proof. Suppose $c = bs - ar$ with $1 \le r \le \left\lceil \frac{b(a-1)}{2a} \right\rceil$ and $\frac{a-1}{2} < s \le a-1$. Observe that $\mathbf{m}^\star_{bs} = c = bs - ar$, that $bi \in \Gamma(\{a,b\}) \subset \Gamma(\{a,b,c\})$ for $i < s$, and that $bi - ar = b(i-s) + c \in \Gamma(\{a,b,c\})$ for $i > s$. Therefore it suffices to prove that

$$bi - a \notin \Gamma(\{a,b,c\}) \ \text{ for } \ i < s, \quad bi - a(r+1) \notin \Gamma(\{a,b,c\}) \ \text{ for } \ i > s. \tag{1}$$

Suppose $i < s$. If $bi - a \in \Gamma(\{a,b,c\})$, then $bi - a - \lambda c = b(i - \lambda s) + a(\lambda r - 1) \in \Gamma(\{a,b\})$ for some $\lambda \ge 1$. Note that $\lambda = 0$ would imply $bi - a \in \Gamma(\{a,b\})$, and this is impossible since $\mathbf{m}_{bi} = bi$ for each i. Thus there exists $\mu \ge 1$ such that $i - \lambda s + \mu a \ge 0$ and $\lambda r - 1 - \mu b \ge 0$, or that

$$\lambda r \ge 1 + \mu b, \quad \lambda s \le i + \mu a \tag{2}$$

for some positive integers λ and μ. Thus $s(1 + \mu b) \le r(i + \mu a)$, so that

$$\mu c \le ri - s, \tag{3}$$

and

$$\frac{\mu a}{s} < \frac{\mu b}{r} < \frac{1 + \mu b}{r} \le \lambda \le \frac{i + \mu a}{s} < 1 + \frac{\mu a}{s}. \tag{4}$$

From eqn. (4), it follows that λ does not exist if $s \mid \mu a$. Henceforth suppose $s \nmid \mu a$. From eqn. (4), $\lambda = \lceil \frac{\mu a}{s} \rceil$, and since $\frac{\mu a}{s} < 2\mu$, we have $\lambda \leq 2\mu$.

Observe that $r \leq \lceil \frac{b(a-1)}{2a} \rceil = \lceil \frac{b}{2} - \frac{b}{2a} \rceil \leq \frac{b}{2}$. From eqn. (4), $\lambda > \frac{\mu b}{r} \geq 2\mu$. Therefore $\lambda = 2\mu$ when $s \nmid \mu a$. From eqn. (3) and eqn. (4),

$$r \geq \frac{(\mu b + 1)s}{\mu a + i} > \frac{\mu b + 1}{\lambda + 1} > \frac{\mu b}{\lambda} = \frac{b}{2},$$

which contradicts the assumption on r that leads to $r \leq \frac{b}{2}$.

This completes the proof of the claim in eqn. (1) for the case $i < s$.

Suppose $i > s$. If $bi - a(r+1) \in \Gamma(\{a,b,c\})$, then $bi - a(r+1) - \lambda c = b(i - \lambda s) + a\big((\lambda - 1)r - 1\big) \in \Gamma(\{a,b\})$ for some $\lambda \geq 1$. Again, note that $\lambda = 0$ would imply $bi - a(r+1) \in \Gamma(\{a,b\})$, and this is impossible since $\mathbf{m}_{bi} = bi$ for each i. Thus there exists $\mu \geq 1$ such that $i - \lambda s + \mu a \geq 0$ and $(\lambda - 1)r - 1 - \mu b \geq 0$, or that

$$(\lambda - 1)r \geq 1 + \mu b, \quad \lambda s \leq i + \mu a \tag{5}$$

for some positive integers λ and μ. Thus $s(r + 1 + \mu b) \leq r(i + \mu a)$, so that

$$\mu c \leq r(i - s) - s, \tag{6}$$

and

$$1 + \frac{\mu a}{s} < 1 + \frac{\mu b}{r} < 1 + \frac{1 + \mu b}{r} \leq \lambda \leq \frac{i + \mu a}{s} < 2 + \frac{\mu a}{s}. \tag{7}$$

From eqn. (7), it follows that λ does not exist if $s \mid \mu a$. Henceforth suppose $s \nmid \mu a$. From eqn. (7), $\lambda = 1 + \lceil \frac{\mu a}{s} \rceil$, and since $\frac{\mu a}{s} < 2\mu$, we have $\lambda \leq 1 + 2\mu$.

Observe that $r \leq \lceil \frac{b(a-1)}{2a} \rceil = \lceil \frac{b}{2} - \frac{b}{2a} \rceil \leq \frac{b}{2}$. From eqn. (7), $\lambda > 1 + \frac{\mu b}{r} \geq 1 + 2\mu$.
Therefore $\lambda = 1 + 2\mu$ when $s \nmid \mu a$.

From eqn. (6) and eqn. (7),

$$r \geq \frac{(\mu b + 1)s}{\mu a + i - s} > \frac{\mu b + 1}{\lambda} = \frac{\mu b + 1}{2\mu + 1} > \frac{\mu b}{2\mu} = \frac{b}{2},$$

which contradicts the assumption on r that leads to $r \leq \frac{b}{2}$.

This completes the proof of the claim in eqn. (1) for the case $i > s$, and the proof of the Theorem. $\square$

Theorem 2. *Let a, b be positive integers, with $\gcd(a,b) = 1$. Let $c = bs - ar$ with $1 \leq r \leq \lceil \frac{b(a-1)}{2a} \rceil$ and $\frac{a-1}{2} < s \leq a - 1$. Then*

(i) $\mathtt{g}(a,b,c) = \mathtt{g}(a,b) - \min\{ar, b(a-s)\} = \max\{b(a-1) - ar, b(s-1)\} - a$.

(ii) $\mathtt{n}(a,b,c) = \mathtt{n}(a,b) - r(a-s) = \dfrac{1}{2}(a-1)(b-1) - r(a-s)$.

(iii) $\mathbf{s}(a,b,c) = \mathbf{s}(a,b) - \frac{1}{2}r(a-s)\big(c + \mathtt{g}(a,b)\big)$
$$= \frac{1}{12}(a-1)(b-1)(2ab-a-b-1) - \frac{1}{2}r(a-s)\big(c+ab-a-b\big).$$

(iv) $\mathcal{S}^\star(\{a,b,c\}) = \{b(a-1)-a(r+1), b(s-1)-a\}$.

Proof. Suppose $c = bs - ar$ with $1 \le r \le \left\lceil \frac{b(a-1)}{2a} \right\rceil$ and $\frac{a-1}{2} < s \le a-1$. By Theorem 1,

$$\Gamma(\{a,b,c\}) \setminus \Gamma(\{a,b\}) = \bigcup_{s \le i \le a-1} \{bi - at : 1 \le t \le r\}. \tag{8}$$

From Proposition 1 and Theorem 1, we have

$$\begin{aligned} \mathtt{g}(a,b,c) &= \max_{1\le i\le a-1} \mathbf{m}^\star_{bi} - a \\ &= \max_{1\le i\le a-1} \left(bi - ar \cdot \left\lfloor \frac{i}{s} \right\rfloor \right) - a \\ &= \max\big\{ b(a-1) - ar, b(s-1) \big\} - a. \end{aligned}$$

This gives the result in part (i).

We determine each of the functions $\mathtt{n}(a,b,c)$, $\mathbf{s}(a,b,c)$ from eqn. (8).

The result in part (ii) is a direct consequence of $\big|\Gamma(\{a,b,c\}) \setminus \Gamma(\{a,b\})\big| = r(a-s)$, and the result in part (iii) is a direct consequence of

$$\begin{aligned} \sum_{i=s}^{a-1}\sum_{t=1}^{r}(bi-at) &= \sum_{i=s}^{a-1}\left(bri - \frac{1}{2}ar(r+1) \right) \\ &= \frac{1}{2}br(a-s)(a+s-1) - \frac{1}{2}ar(r+1)(a-s) \\ &= \frac{1}{2}r(a-s)\Big(b(a+s-1) - a(r+1) \Big) \\ &= \frac{1}{2}r(a-s)(ab-a-b+c). \end{aligned}$$

To prove the result in part (iv), we show that

$$\mathbf{m}^\star_{bj} > \mathbf{m}^\star_{b(j+i)} - \mathbf{m}^\star_{bi} \ \text{ for all } \ i \in \{1,\ldots,a-1\}, \tag{9}$$

when $j \in \{s-1, a-1\}$, and

$$\mathbf{m}^\star_{bj} = \mathbf{m}^\star_{b(j+1)} - \mathbf{m}^\star_{b} \tag{10}$$

when $j \notin \{s-1, a-1\}$.

Suppose $j \notin \{s-1, a-1\}$. By Theorem 1,

$$\mathbf{m}^\star_{bj} + \mathbf{m}^\star_{b} = \left(bj - ar \cdot \left\lfloor \frac{j}{s} \right\rfloor \right) + b = b(j+1) - ar \cdot \left\lfloor \frac{j}{s} \right\rfloor = b(j+1) - ar \cdot \left\lfloor \frac{j+1}{s} \right\rfloor = \mathbf{m}^\star_{b(j+1)}.$$

Therefore eqn. (10) holds.

Suppose $j = s-1$. To show that eqn. (9) holds, we consider three cases for i: (I) $i \in [1, a-s]$; (II) $i \in (a-s, s-1]$; (III) $i = s$; (IV) $i \in (s, a-1]$.
For $i \in [1, a-s]$,

$$\mathbf{m}^{\star}_{b(s-1)} + \mathbf{m}^{\star}_{bi} = b(s-1) + bi > b(i+s-1) - ar = \mathbf{m}^{\star}_{b(i+s-1)}.$$

For $i \in (a-s, s-1]$,

$$\mathbf{m}^{\star}_{b(s-1)} + \mathbf{m}^{\star}_{bi} = b(s-1) + bi > b(i+s-1-a) = \mathbf{m}^{\star}_{b(i+s-1)}.$$

For $i = s$,

$$\mathbf{m}^{\star}_{b(s-1)} + \mathbf{m}^{\star}_{bs} = b(s-1) + (bs - ar) > b(2s-1-a) - ar = \mathbf{m}^{\star}_{b(2s-1)}.$$

For $i \in (s, a-1]$,

$$\mathbf{m}^{\star}_{b(s-1)} + \mathbf{m}^{\star}_{bi} = b(s-1) + (bi - ar) > b(i+s-1-a) = \mathbf{m}^{\star}_{b(i+s-1)}.$$

Suppose $j = a-1$. To show that eqn. (9) holds, we consider three cases for i: (V) $i \in [1, s)$; (VI) $i = s$; (VII) $i \in (s, a-1]$.
For $i \in [1, s)$,

$$\mathbf{m}^{\star}_{b(a-1)} + \mathbf{m}^{\star}_{bi} = \big(b(a-1) - ar\big) + bi > b(i-1) = \mathbf{m}^{\star}_{b(i-1)}.$$

For $i = s$,

$$\mathbf{m}^{\star}_{b(a-1)} + \mathbf{m}^{\star}_{bs} = \big(b(a-1) - ar\big) + (bs - ar) > b(s-1) = \mathbf{m}^{\star}_{b(s-1)}.$$

For $i \in (s, a-1]$,

$$\mathbf{m}^{\star}_{b(a-1)} + \mathbf{m}^{\star}_{bi} = \big(b(a-1) - ar\big) + (bi - ar) > b(i-1) - ar = \mathbf{m}^{\star}_{b(i-1)}.$$

This completes the proof of part (iv). □

Theorem 3. *Let a, b be positive integers, with $a < b$ and $\gcd(a, b) = 1$. If $c = bs - ar \notin \Gamma(\{a, b\})$, then*

$$\mathtt{g}(a, b) - \mathtt{g}(a, b, c) \geq \min\big\{ar, b(a-s)\big\}, \quad \mathtt{n}(a, b) - \mathtt{n}(a, b, c) \geq r(a-s), \tag{11}$$

with equality in each case if and only if $1 \leq r \leq \left\lceil \frac{b(a-1)}{2a} \right\rceil$ and $s > \frac{a-1}{2}$.

Proof. Suppose $c = bs - ar$ with $1 \leq s \leq a-1$, $1 \leq r < \frac{bs}{a}$. Then $c \notin \Gamma(\{a, b\})$, and so $\mathtt{n}(a, b) - \mathtt{n}(a, b, c) > 0$ by Proposition 6. We note that equality in eqn. (11) holds if $1 \leq r \leq \left\lceil \frac{b(a-1)}{2a} \right\rceil$ and $s > \frac{a-1}{2}$ by Theorem 1, parts (ii) and (iii).

Therefore, we must show (I) that the inequalities in eqn. (11) always hold, and (II) that there is no equality if either $\left\lceil \frac{b(a-1)}{2a} \right\rceil < r < \frac{bs}{a}$ or $1 \leq s \leq \frac{a-1}{2}$.

Since $\mathbf{m}^{\star}_{bs} = c$, $\big(\Gamma(\{a,b,c\}) \setminus \Gamma(\{a,b\})\big) \bigcap (bs) = \{bs-a, bs-2a, \dots, bs-ar\}$.

For each $i > s$, $bi - ar = b(i-s) + c \in \Gamma(\{a,b,c\})$. Therefore $bi - aj \in \Gamma(\{a,b,c\})$ for $j \in \{1,\dots,r-1\}$, so that $\big(\Gamma(\{a,b,c\}) \setminus \Gamma(\{a,b\})\big) \bigcap (bi) \supseteq \{bi-a, bi-2a, \dots, bi-ar\}$.

Hence each of the $a-s$ congruence classes $(bs), (b(s+1)), \dots, (b(a-1))$ modulo a contributes at least r to the difference $\mathtt{n}(a,b) - \mathtt{n}(a,b,c)$, and we have the required lower bound given by eqn. (11) for $\mathtt{n}(a,b) - \mathtt{n}(a,b,c)$.

As shown before, $\mathbf{m}_{bi} - ar = bi - ar = b(i-s) + c \in \Gamma(\{a,b,c\})$ for $i > s$. Hence $\mathbf{m}^{\star}_{bi} \le bi - ar$ when $i > s$. Therefore

$$
\begin{aligned}
\mathtt{g}(a,b,c) &= \max_{1\le i\le a-1} \mathbf{m}^{\star}_{bi} - a \\
&\le \max\left\{\max_{1\le i\le s-1} bi, \max_{s\le i\le a-1}(bi-ar)\right\} - a \\
&\le \max\big\{b(s-1), b(a-1)-ar\big\} - a.
\end{aligned}
$$

This proves the inequality in eqn. (11) for $\mathtt{g}(a,b) - \mathtt{g}(a,b,c)$, and completes the proof of our assertion in (I).

To prove the assertion in (II), we shall first show that

$$
bi - a(r+1) \in \Gamma(\{a,b,c\}) \ \text{ for at least one } \ i > s \tag{12}
$$

whenever either $\left\lceil \frac{b(a-1)}{2a} \right\rceil < r < \frac{bs}{a}$ or $1 \le s \le \frac{a-1}{2}$.

Suppose $r > \left\lceil \frac{b(a-1)}{2a} \right\rceil$. Define κ by $\kappa c \equiv -a(r+1) \pmod b$, $0 \le \kappa \le b-1$, and set $i = \big(\kappa c + a(r+1)\big)/b$. Then $bi - a(r+1) = \kappa c \in \Gamma(\{a,b,c\})$. Note that $\kappa = 0$ implies $a \mid i$, which is impossible. Hence $\kappa \ge 1$. Now $bi = a(r+1) + \kappa c \ge a(r+1) + c = bs + a > bs$, so that $i > s$. Hence i satisfies both conditions $i > s$ and $bi - a(r+1) = \kappa c \in \Gamma(\{a,b,c\})$, as desired.

Suppose $s \le \frac{a-1}{2}$. For $i = 2s$, we have $bi - a(r+1) = a(r-1) + 2c \in \Gamma(\{a,b,c\})$. Note that any choice of $i > 2s$ may be invalid since $i \le a-1$.

Hence eqn. (12) holds.

Since $\mathbf{m}_{bi} - \mathbf{m}^{\star}_{bi} \ge a(r+1)$ for some $i_0 > s$ and $\mathbf{m}_{bi} - \mathbf{m}^{\star}_{bi} \ge ar$ for all $i \ge s$, $i \ne i_0$, we have

$$
\begin{aligned}
\mathtt{n}(a,b) - \mathtt{n}(a,b,c) &= \frac{1}{a}\sum_{i=1}^{a-1}\big(\mathbf{m}_{bi} - \mathbf{m}^{\star}_{bi}\big) \\
&\ge \frac{1}{a}\sum_{i=s}^{a-1}\big(\mathbf{m}_{bi} - \mathbf{m}^{\star}_{bi}\big) \\
&> r(a-s).
\end{aligned}
$$

Let $i \ge i_0$. Then $bi - a(r+1) = b(i-i_0) + \big(bi_0 - a(r+1)\big) \in \Gamma(\{a,b,c\})$. Hence $\mathbf{m}^{\star}_{bi} \le bi - a(r+1)$, so that $\mathbf{m}_{bi} - \mathbf{m}^{\star}_{bi} = b(a-1) - \mathbf{m}^{\star}_{bi} \ge b(a-1-i) + a(r+1) \ge a(r+1) > \min\big\{ar, b(a-s)\big\}$.

Thus there is no equality in eqn. (11) if either $\lceil \frac{b(a-1)}{2a} \rceil < r < \frac{bs}{a}$ or $1 \le s \le \frac{a-1}{2}$, completing the proof of our assertion in (II). □

Corollary 2. *Let a, b be positive integers, with $a < b$ and $\gcd(a,b) = 1$. If $c \notin \Gamma(\{a,b\})$, then*

(i)
$$\mathtt{g}(a,b) - \mathtt{g}(a,b,c) \ge a,$$
with equality if and only if $c = bs - a$ with $\frac{a-1}{2} < s \le a-1$.

(ii)
$$\mathtt{n}(a,b) - \mathtt{n}(a,b,c) \ge 1,$$
with equality if and only if $c = b(a-1) - a = \mathtt{g}(a,b)$.

4. Concluding Remarks

For the cases covered by Theorem 2, Proposition 4 may also be used to determine the functions $\mathtt{g}(A)$ and $\mathtt{n}(A)$, and also the set $\mathcal{S}^\star(A)$. This requires determining the three constants c_1, c_2, c_3, and then the six constants $r_{12}, r_{13}, r_{21}, r_{23}, r_{31}, r_{32}$. We list this in the following result, without proof.

Proposition 7. *Let a, b be positive integers, with $\gcd(a,b) = 1$. Let $c = bs - ar$ with $1 \le r \le \lceil \frac{b(a-1)}{2a} \rceil$ and $\frac{a}{2} < s \le a-1$. Then*

$$\begin{array}{lll} c_1 = b - r, & r_{12} = a - s, & r_{13} = 1, \\ c_2 = s, & r_{21} = r, & r_{23} = 1, \\ c_3 = 2, & r_{31} = b - 2r, & r_{32} = 2s - a. \end{array}$$

The omission of the case $s = \frac{a}{2}$ from the assumption in Theorem 2 provides no serious setback, since this case is only possible when a is even and then $c = bs - ar$ implies that $\frac{a}{2}$ is a common divisor of a and c. Proposition 2 assures that the condition $\gcd(a,c) = 1$ is without loss of generality when $|A| = 3$ when computing the functions $\mathtt{g}(A)$ and $\mathtt{n}(A)$. The results of Theorem 2, except for the formula for $\mathcal{S}(A)$, now follow from Proposition 4 and Proposition 7.

Theorem 1 provides a simple result for the change $\mathbf{m}_{bi} - \mathbf{m}^\star_{bi}$ for each $i \in \{1, \ldots, a-1\}$ in the case $c = bs - ar$ with $1 \le r \le \lceil \frac{b(a-1)}{2a} \rceil$ and $\frac{a-1}{2} < s \le a-1$. This enables us to determine explicitly $\mathtt{g}(a,b,c)$, $\mathtt{n}(a,b,c)$, $\mathtt{s}(a,b,c)$, and $\mathcal{S}^\star(\{a,b,c\})$ for c in these cases.

Observe that if $s \le \frac{a-1}{2}$, then $r < \frac{bs}{a} \le \frac{b(a-1)}{2a}$. Therefore the cases that remain are:

(i) $1 \le r \le \lceil \frac{b(a-1)}{2a} \rceil$, $1 \le s \le \frac{a-1}{2}$;

(ii) $\left\lceil \frac{b(a-1)}{2a} \right\rceil < r < \frac{bs}{a}$, $\frac{a-1}{2} < s \leq a-1$.

Obtaining results for the change $\mathbf{m}_{bi} - \mathbf{m}^{\star}_{bi}$ for each $i \in \{1, \ldots, a-1\}$ for these two cases appear to be considerably more difficult.

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