NOTE ON RESTRICTED PARTS IN CYCLIC COMPOSITIONS

Meghann Moriah Gibson
Department of Mathematics, University of Mississippi, University, Mississippi
mmgibso1@go.olemiss.edu

Matthew Just
Department of Mathematics, University of Georgia, Athens, Georgia
justmatt@uga.edu

Hua Wang[1]
Department of Mathematical Sciences, Georgia Southern University, Statesboro, Georgia
hwang@georgiasouthern.edu

Received: 5/19/16, Accepted: 1/9/18, Published: 3/9/18

Abstract

Integer compositions and related enumeration problems have been extensively studied. The cyclic analogues of such questions, however, have significantly fewer results. In this note we follow the cyclic construction of Flajolet and Soria to obtain generating functions of parts under modulo conditions in cyclic compositions. With these generating functions we present some statistics of the parts in cyclic compositions. A combinatorial observation on this enumerative question is also provided.

1. Introduction

In 1893 P. A. MacMahon was one of the first to take an interest in what are now referred to as compositions [2, pg.1]. A *composition* of a given positive integer n is defined as a sequence of positive integers $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_k$ such that $\sum_{i=1}^{k} \sigma_i = n$. Each σ_i is called a *part* in the composition.

Cyclic compositions are a partition of the set of compositions under the equivalence relation S, where S is any cyclic shift of the parts of a composition. For example, Figure 1 shows a cyclic composition of 10 corresponding to the equivalence class $2+2+1+2+2+1 = 2+1+2+2+1+2 = 1+2+2+1+2+2 = 10$.

[1]This work was partially supported by grants from the Simons Foundation (#245307).

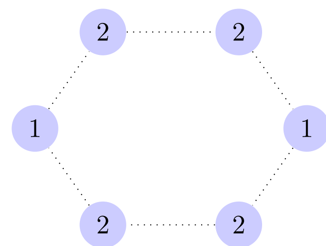

Figure 1: A cyclic composition of 10.

The cyclic compositions were first considered in [5], and enumerated via generating functions in [1]. As one can easily see from the above example, although similar in nature, the enumeration of various objects in cyclic compositions would dramatically differ from that in compositions. More recently some interesting properties of cyclic compositions were discussed in [4].

As a concept in combinatorial number theory, compositions have been extensively studied by both combinatorialists and number theorists. Much of such work focuses on the enumeration of parts or sub-word patterns under various restrictions. A nice summary of known results can be found in [2] and the references therein. In the recent work [3], the number of parts under congruence conditions in all compositions of n was studied and interesting combinatorial relations with the number of sub-word patters (also in all compositions of n) were found. Among other interesting observations, the following was presented as a consequence of previously known results. Here $P(k;n)$ denotes the number of parts equal to k in all compositions of n.

Theorem 1.1 ([3]).

$$P(k;n) = \begin{cases} (n-k+3)2^{n-k-2} & \textit{if } n > k, \\ 1 & \textit{if } n = k. \end{cases}$$

Following Theorem 1.1 above, one can obtain a formula for the number of parts congruent to i modulo m in all compositions of n, denoted by $P(i;m;n)$ for $i = 1, \ldots, m$ (note that we use m instead of 0 for the corresponding congruent class).

Theorem 1.2 ([3]).

$$P(i;m;n) = \left\lfloor \frac{2^{n+m-i-2}\Big((n-i+3)(2^m-1)-m\Big)}{(2^m-1)^2} \right\rceil,$$

where $\lfloor x \rceil$ *is the integer nearest to* x.

Recall that the total number of parts in all compositions of n is $2^{n-2}(n+1)$ [6, p. 120, Ex. 23], Theorem 1.2 provides a direct tool to study the statistics of the parts under congruence conditions.

On the other hand, there has been surprisingly little work on similar questions in cyclic compositions. We make some modest progress towards filling this gap in this note. The tool that is critical to our study is developed in [1], where the construction of cycles of combinatorial structures is examined analytically.

Let $\mathcal{CC}_n$ denote the set of cyclic compositions of n and $tpcc(n)$ denote the total number of parts in $\mathcal{CC}_n$. Direct application of the cyclic construction in [1] yields the bivariate generating function

$$CC(x,u) = \sum_{s\geq 1} \frac{\phi(s)}{s} \log\left(\frac{1}{1-u^s\frac{x^s}{1-x^s}}\right)$$

where the coefficient of $x^n u^k$ gives the number of cyclic compositions of n with k parts and ϕ is Euler's totient function. Hence the generating function for $tpcc(n)$ is

$$TPCC(x) = \frac{\partial}{\partial u} CC(x,u)\bigg|_{u=1} = \sum_{s\geq 1} \phi(s) \frac{x^s}{1-2x^s}.$$

Consequently we have the following.

Proposition 1.3.

$$tpcc(n) = \frac{1}{2}\sum_{s|n} \phi(s) 2^{n/s}.$$

Let $cp(i;m;n)$ $(1 \leq i \leq m)$ denote the number of parts congruent to i modulo m in $\mathcal{CC}_n$. Examining $cp(i;m;n)$ and the corresponding generating function is a bit more involved. We provide a detailed study of this generating function in Section 2. Based on our finding some statistical behaviors of the parts in $\mathcal{CC}_n$ are presented and justified in Section 3. In Section 4, we comment on an interesting combinatorial observation arising from our study.

2. Construction of the Generating Function of $cp(i;m;n)$

The essential idea still follows that in [1]. For completeness we provide the entire argument in this section.

First consider the series

$$x + x^2 + x^3 + \dots$$

that generates the parts used in a composition or cyclic composition. Multiplying by y each power that is congruent to $i \mod m$ (to mark the parts congruent to i

mod m) yields

$$x + x^2 + \cdots + yx^i + x^{i+1} + \cdots + yx^{i+m} + \ldots.$$

This forms the generating function for the number of particularly labeled parts of the compositions of n. Further multiplying each term by u (to mark all the parts), we have

$$\begin{aligned} ux + ux^2 + \cdots + yux^i + ux^{i+1} + \cdots + yux^{i+m} + \ldots \\ &= u(x + x^2 + \ldots) + u(y-1)\left(x^i + x^{i+m} + \ldots\right) \\ &= \frac{ux}{1-x} + \frac{(y-1)ux^i}{1-x^m}. \end{aligned}$$

Consequently we have the multivariable generating function of compositions

$$\begin{aligned} C(x,u,y) &= \sum_{k=1}^{\infty}\left(\frac{ux}{1-x} + \frac{(y-1)ux^i}{1-x^m}\right)^k \\ &= \frac{\frac{ux}{1-x} + \frac{(y-1)ux^i}{1-x^m}}{1 - \frac{ux}{1-x} + \frac{(y-1)ux^i}{1-x^m}} \\ &= \frac{ux(1-x^m) + ux^i(y-1)(1-x)}{(1-x)(1-x^m) - (ux(1-x^m) + ux^i(y-1)(1-x))} \end{aligned}$$

where the coefficient of $x^n u^r y^t$ is the number of compositions of n with r parts, t of which are congruent to $i \mod m$.

A primitive composition is a composition that is not composed of repeated copies of shorter compositions. For instance, 1326354 is a primitive composition of 24 while 132132132132 is a non-primitive or *periodic* composition of 24. It is clear that every composition is composed of d copies of a primitive composition for some $d \geq 1$. By letting

$$PC(x,u,y) = \sum_{n,r,t} pc(n,r,t)x^n u^r y^t$$

denote the generating function for primitive compositions (where the coefficient $pc(n,r,t)$ is the number of primitive compositions of n with r parts, t of which are congruent to $i \mod m$), we have

$$C(x,u,y) = \sum_{d\geq 1} PC(x^d, u^d, y^d).$$

Then $PC(x,u,y)$ can be implicitly derived using Möbius inversion as

$$PC(x,u,y) = \sum_{d\geq 1} \mu(d)C(x^d, u^d, y^d)$$

where $\mu(d)$ is the Möbius μ function.

We now let

$$PCC(x,u,y) = \sum_{n,r,t} pcc(n,r,t)x^n u^r y^t$$

denote the generating function for primitive cyclic compositions (where the coefficient $pcc(n,r,t)$ is the number of primitive cyclic compositions of n with r parts, t of which are congruent to $i \mod m$). First note that each primitive cyclic composition with r parts has r unique primitive composition representations. Thus there is a one-to-r mapping between primitive cyclic compositions and primitive compositions. Consequently

$$pcc(n,r,t)x^n u^r y^t = \frac{pc(n,r,t)}{r} x^n u^r y^t = \int_0^u pc(n,r,t)x^n v^{r-1} y^t dv$$

and we have

$$\begin{aligned}
PCC(x,u,y) &= \int_0^u \frac{PC(x,v,y)}{v}\, dv \\
&= \int_0^u \frac{1}{v} \sum_{d\geq 1} \mu(d)\ C(x^d, v^d, y^d)\ dv \\
&= \int_0^u \frac{1}{v} \sum_{d\geq 1} \mu(d)\ \frac{v^d x^d(1-x^{md}) + v^d x^{di}(y^d-1)(1-x^d)}{(1-x^d)(1-x^{md}) - (v^d x^d(1-x^{md}) + v^d x^{di}(y^d-1)(1-x^d))}\ dv \\
&= \sum_{d\geq 1} \mu(d) \int_0^u \frac{v^{d-1}(x^d(1-x^{md}) + x^{di}(y^d-1)(1-x^d))}{(1-x^d)(1-x^{md}) - v^d(x^d(1-x^{md}) + x^{di}(y^d-1)(1-x^d))}\ dv.
\end{aligned}$$

Integrating through substitution and plugging in $u=1$, we have

$$\begin{aligned}
PCC(x,y) = PCC(x,u,y)\Big|_{u=1} &= \left(\int_0^u \frac{PC(x,v,y)}{v}\ dv\right)\Big|_{u=1} \\
&= \left(\sum_{d\geq 1} \frac{\mu(d)}{d} \log\left(\frac{(1-x^d)(1-x^{md})}{(1-x^d)(1-x^{md}) - u^d(x^d(1-x^{md}) + x^{di}(y^d-1)(1-x^d))}\right)\right)\Big|_{u=1} \\
&= \sum_{d\geq 1} \frac{\mu(d)}{d} \log\left(\frac{(1-x^d)(1-x^{md})}{(1-x^d)(1-x^{md}) - (x^d(1-x^{md}) + x^{di}(y^d-1)(1-x^d))}\right).
\end{aligned}$$

Since every Cyclic Composition is composed of q adjacent copies of primitive compositions, the bivariate generating functions for cyclic compositions is

$$CC(x,y) = \sum_{q\geq 1} PCC(x^q, y^q)$$
$$= \sum_{q\geq 1}\sum_{d\geq 1} \frac{\mu(d)}{d} \log\left(\frac{(1-x^{qd})(1-x^{mqd})}{(1-x^{qd})(1-x^{mqd}) - (x^{qd}(1-x^{mqd}) + x^{qdi}(y^{qd}-1)(1-x^{qd}))}\right).$$

Using the variable substitution $s = qd$ and given the identity $\sum_{d|s} \frac{\mu(d)}{d} = \frac{\phi(s)}{s}$, where $\phi(s)$ is the Euler totient function, we have the generating function for cyclic compositions as

$$CC(x,y) = \sum_{q\geq 1} PCC(x^q, y^q)$$
$$= \sum_{s\geq 1}\sum_{d|s} \frac{\mu(d)}{d} \log\left(\frac{(1-x^{qd})(1-x^{mqd})}{(1-x^{qd})(1-x^{mqd}) - (x^{qd}(1-x^{mqd}) + x^{qdi}(y^{qd}-1)(1-x^{qd}))}\right)$$
$$= \sum_{s\geq 1} \frac{\phi(s)}{s} \log\left(\frac{(1-x^{s})(1-x^{sm})}{(1-x^{s})(1-x^{sm}) - (x^{s}(1-x^{sm}) + x^{si}(y^{s}-1)(1-x^{s}))}\right).$$

Here the coefficient of $x^n y^t$ is the number of cyclic compositions of n with t parts that are congruent to $i \mod m$.

We now have the generating function $GPCC(x)$ for the number of parts congruent to $i \mod m$ in $\mathcal{CC}_n$ as

$$GPCC(x) = \frac{\partial}{\partial y}\Big(CC(x,y)\Big)\Big|_{y=1}$$
$$= \sum_{s\geq 1} \frac{\phi(s)}{s} \left(\frac{\partial}{\partial y}\left(\log\left(\frac{(1-x^{s})(1-x^{sm})}{(1-x^{s})(1-x^{sm}) - (x^{s}(1-x^{sm}) + x^{si}(y^{s}-1)(1-x^{s}))}\right)\right)\right)\Bigg|_{y=1}$$
$$= \sum_{s\geq 1} \frac{\phi(s)}{s} \left(\frac{x^{si}(1-x^{s}) s y^{s-1}}{(1-x^{s})(1-x^{sm}) - (x^{s}(1-x^{sm}) + x^{si}(y^{s}-1)(1-x^{s}))}\right)\Bigg|_{y=1}$$
$$= \sum_{s\geq 1} \left(\phi(s) \frac{x^{si}(1-x^{s})}{(1-2x^{s})(1-x^{sm})}\right). \tag{2.1}$$

3. Some Statistics of the Parts in $\mathcal{CC}_n$

The generating function provided in (2.1) yields the Table 1 of values for $cp(i;m;n)$ with $m = 10$.

A number of interesting observations immediately follow:

	i	1	2	3	4	5	6	7
n								
1		1	0	0	0	0	0	0
2		2	1	0	0	0	0	0
3		4	1	1	0	0	0	0
4		7	3	1	1	0	0	0
5		12	4	2	1	1	0	0
6		22	11	5	2	1	1	0
7		38	16	8	4	2	1	1
8		74	36	17	9	4	2	1
9		138	66	34	16	8	4	2
10		272	136	66	33	17	8	4
11		521	256	128	64	32	16	8
12		1057	527	264	132	65	33	16
13		2058	1023	511	256	128	64	32
14		4136	2068	1031	515	258	129	65
15		8216	4104	2054	1025	513	256	128
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
50		$2.81 \cdot 10^{14}$	$1.41 \cdot 10^{14}$	$7.03 \cdot 10^{13}$	$3.52 \cdot 10^{13}$	$1.76 \cdot 10^{13}$	$8.79 \cdot 10^{12}$	$4.39 \cdot 10^{12}$
51		$5.62 \cdot 10^{14}$	$2.81 \cdot 10^{14}$	$1.41 \cdot 10^{14}$	$7.03 \cdot 10^{13}$	$3.52 \cdot 10^{13}$	$1.76 \cdot 10^{13}$	$8.79 \cdot 10^{12}$
52		$1.12 \cdot 10^{15}$	$5.62 \cdot 10^{14}$	$2.81 \cdot 10^{14}$	$1.41 \cdot 10^{14}$	$7.03 \cdot 10^{13}$	$3.52 \cdot 10^{13}$	$1.76 \cdot 10^{13}$
53		$2.25 \cdot 10^{15}$	$1.12 \cdot 10^{15}$	$5.62 \cdot 10^{14}$	$2.81 \cdot 10^{14}$	$1.41 \cdot 10^{14}$	$7.03 \cdot 10^{13}$	$3.52 \cdot 10^{13}$
54		$4.50 \cdot 10^{15}$	$2.25 \cdot 10^{15}$	$1.12 \cdot 10^{15}$	$5.62 \cdot 10^{14}$	$2.81 \cdot 10^{14}$	$1.41 \cdot 10^{14}$	$7.03 \cdot 10^{13}$
55		$9.00 \cdot 10^{15}$	$4.50 \cdot 10^{15}$	$2.25 \cdot 10^{15}$	$1.12 \cdot 10^{15}$	$5.62 \cdot 10^{14}$	$2.81 \cdot 10^{14}$	$1.41 \cdot 10^{14}$

	i	8	9	10
n				
1		0	0	0
2		0	0	0
3		0	0	0
4		0	0	0
5		0	0	0
6		0	0	0
7		0	0	0
8		1	0	0
9		1	1	0
10		2	1	1
11		4	2	1
12		8	4	2
13		16	8	4
14		32	16	8
15		64	32	16
$\vdots$		$\vdots$	$\vdots$	$\vdots$
50		$2.20 \cdot 10^{12}$	$1.10 \cdot 10^{12}$	$5.49 \cdot 10^{11}$
51		$4.39 \cdot 10^{12}$	$2.20 \cdot 10^{12}$	$1.10 \cdot 10^{12}$
52		$8.79 \cdot 10^{12}$	$4.39 \cdot 10^{12}$	$2.20 \cdot 10^{12}$
53		$1.76 \cdot 10^{13}$	$8.79 \cdot 10^{12}$	$4.39 \cdot 10^{12}$
54		$3.52 \cdot 10^{13}$	$1.76 \cdot 10^{13}$	$8.79 \cdot 10^{12}$
55		$7.03 \cdot 10^{13}$	$3.52 \cdot 10^{13}$	$1.76 \cdot 10^{13}$

Table 1: Values of $cp(i;m;n)$ for $m = 10$.

- For n sufficiently large,
$$\frac{cp(i;m;n+1)}{cp(i;m;n)} \sim 2.$$
That is, going down a column in the table by one step doubles the value of the next entry;

- For n sufficiently large,
$$\frac{cp(i+1;m;n)}{cp(i;m;n)} \sim \frac{1}{2}.$$
That is, moving right in a row in the table by one step halves the value of the next entry;

- For n sufficiently large,
$$\frac{cp(i+1;m;n+1)}{cp(i;m;n)} \sim 1.$$
That is, the values on each of the diagonals or sub-diagonals are asymptotically the same.

To see the reasoning behind these observations, we start with the following asymptotic formula for $cp(i;m;n)$.

Theorem 3.1. *We have*
$$cp(i;m;n) \sim \frac{2^{n+m-i-1}}{2^m-1}$$
as $n \to \infty$.

Proof. First rewrite (2.1) as
$$GPCC(x) = \sum_{s\geq 1} \phi(s)A(x^s)$$
where
$$A(x) = x^i \cdot \frac{1-x}{(1-2x)(1-x^m)}.$$
Next, we will consider the partial fraction decomposition of $\frac{1-x}{(1-2x)(1-x^m)}$. Observe that
$$\frac{1-x}{(1-2x)(1-x^m)} = \frac{1}{(1-2x)(1+x+\ldots+x^{m-1})} = \frac{a_0}{1-2x} + \sum_{j=1}^{m-1} \frac{a_j}{x-w_j}$$
with constants $a_0, a_1, \ldots, a_{m-1}$, where w_j for $1 \leq j \leq m-1$ are the complex roots of $x^m = 1$. Through the traditional "Cover-up" method (i.e., multiplying both

sides by $(1-2x)(1-x^m)$ and plugging in $x=\frac{1}{2}$), we have $a_0 = \frac{2^{m-1}}{2^m-1}$. Since $x=\frac{1}{2}$ is a distinct root of the denominator of smallest magnitude, it follows that the coefficient of x^n in $A(x)$ is asymptotically $\frac{2^{n+m-i-1}}{2^m-1}$ (contributed from $x^i \cdot \frac{a_0}{1-2x}$) as $n \to \infty$. Thus considering $cp(i;m;n)$, the coefficient of x^n in $\sum_{s\geq 1}\phi(s)A(x^s)$, we have

$$cp(i;m;n) \sim \frac{1}{2^m-1}\left(\sum_{s|n}\phi(s)2^{\frac{n}{s}+m-i-1}\right). \tag{3.1}$$

By taking the $s=1$ term, it is obvious that

$$\frac{1}{2^m-1}\left(\sum_{s|n}\phi(s)2^{\frac{n}{s}+m-i-1}\right) \geq \frac{2^{n+m-i-1}}{2^m-1}.$$

On the other hand

$$\begin{aligned}
\frac{1}{2^m-1}\left(\sum_{s|n}\phi(s)2^{\frac{n}{s}+m-i-1}\right) &= \sum_{s|n}\phi(s)\frac{2^{m-i-1}}{2^m-1}2^{\frac{n}{s}} \\
&= \frac{2^{m-i-1}}{2^m-1}\sum_{s|n}\phi(s)2^{\frac{n}{s}} \\
&= \frac{2^{m-i-1}}{2^m-1}\left(2^n + \sum_{s|n, s\neq 1}\phi(s)2^{\frac{n}{s}}\right) \\
&\leq \frac{2^{m-i-1}}{2^m-1}\left(2^n + (n-1)2^{\frac{n}{2}}\right) \\
&\sim \frac{2^{n+m-i-1}}{2^m-1}
\end{aligned}$$

when n is large. □

Similarly, it is easy to see from Proposition 1.3, that

$$tpcc(n) \sim 2^{n-1}$$

as $n \to \infty$. Consequently we have

Corollary 3.2. *We have*

$$\frac{cp(i;m;n)}{tpcc(n)} \sim \frac{2^{m-i}}{2^m-1}$$

as $n \to \infty$.

Among other things, Theorem 3.1 and Corollary 3.2 imply that for large n:

- $cp(i;m;n+1) \sim 2cp(i;m;n)$;

- $cp(i+1;m;n) \sim \frac{1}{2}cp(i;m;n)$;
- $cp(i+1;m;n+1) \sim cp(i;m;n)$.

These observations are somewhat surprising, but they follow directly from the generating functions.

4. A Combinatorial Observation

We conclude this note by exploring an interesting observation. Recall that

$$GPCC(x) = \sum_{s\geq 1} \phi(s)A(x^s). \tag{4.1}$$

We first make the following claim.

Proposition 4.1. *The function*

$$A(x) = x^i \cdot \frac{1-x}{(1-2x)(1-x^m)}$$

is the generating function for the number of compositions with last part congruent to $i \mod m$.

Proof. Let A_n denote the set of compositions of n with last part congruent to $i \mod m$. We show that

$$|A_n| = \sum_{j=1}^{m} |A_{n-j}| + |A_{n-m}|.$$

The generating function as claimed immediately follows from standard arguments. Consider the compositions of n with last part congruent to $i \mod m$ in the following different cases and apply the corresponding operations:

- The last part x is greater than m. In this case we simply replace x with $x-m$, yielding a composition in A_{n-m};
- The last part x is exactly i. Consider the second to last part $y \equiv j \mod m$ for $j = 1, 2, \ldots, m$. In this case replace x and y with a single part $y - j + i$ yields a composition in A_{n-j}.

It is not hard to see that this map is a bijection between A_n and $\left(\cup_{j=1}^{m} A_{n-j}\right) \cup A'_{n-m}$, where A'_{n-m} and A_{n-m} are two copies of the same set. □

This fact inspires a combinatorial argument that establishes $GPCC(x)$ directly from $A(x)$. The detailed justification seems to be rather tedious, but we will briefly discuss the idea in the rest of this section.

For a cyclic composition with some part congruent to $i \mod m$, "cutting" this composition right after this part yields a composition with the last part congruent to $i \mod m$. Such regular compositions have generating function $A(x)$, by the previous proposition. Now we consider all the regular compositions of n of length k formed from cutting cyclic compositions at every part congruent to $i \mod m$. They can be broken into different cases as follows:

- Every regular composition (and its corresponding last part congruent to $i \mod m$) corresponds to at least one (exactly one if the regular composition is primitive) such "cutting" of some cyclic composition, counted by the $s = 1$ term of (4.1).

- If the resulting (after the "cutting") regular composition (not primitive) is made of two identical copies of compositions of length $\frac{k}{2}$, then further cutting this composition in the middle (right after a part congruent to $i \mod m$) yields two compositions of length $\frac{k}{2}$. This case is counted by a pair of compositions ending with parts congruent to $i \mod m$, with generating function $A(x^2)$ as the $s = 2$ term of (4.1).

- If the resulting regular composition is made of three identical copies of compositions of length $\frac{k}{3}$, then further cutting this composition at the parts congruent to $i \mod m$ yields three compositions of length $\frac{k}{3}$. This case, and the corresponding two parts congruent to $i \mod m$, are counted by such a triple of compositions ending with parts congruent to $i \mod m$, with generating function $2A(x^3)$ as the $s = 3$ term of (4.1);

- If the resulting regular composition is made of four identical copies of compositions of length $\frac{k}{4}$, then further cutting this composition at the parts congruent to $i \mod m$ yields four compositions of length $\frac{k}{4}$. This case, and the corresponding two parts congruent to $i \mod m$ (other than the part at the original cut, counted in the first case, and the middle part, counted in the second case), are counted by such a 4-tuple of compositions ending with parts congruent to $i \mod m$, with generating function $2A(x^4)$ as the $s = 4$ term of (4.1);

- In general, if the resulting regular composition is made of s identical copies of compositions of length $\frac{k}{s}$, then further cutting this composition at the parts congruent to $i \mod m$ yields s compositions of length $\frac{k}{s}$. Among these s parts congruent to $i \mod m$, $s - \phi(s)$ are already counted in previous cases. The other $\phi(s)$ such parts are counted by the generating function $\phi(s)A(x^s)$ as the sth term of (4.1).

The above explanation, although tedious, essentially presents the following idea. Consider any cyclic composition of n of length k, made up of d copies of a primitive composition containing p parts congruent to $i \mod m$. Then there are dp total parts congruent to $i \mod m$. Of the $\frac{k}{d}$ compositions corresponding to this one cyclic composition, p of them have last part congruent to $i \mod m$. For any divisor s of d, consider the composition formed by s copies of the primitive composition constituting the cyclic composition in question ending in one of the p parts congruent to $i \mod m$. Each of these compositions will be counted by $A(x^s)$. Now summing over all these compositions and multiplying by $\phi(s)$ for those counted in $A(x^s)$, we obtain

$$\sum_{s|d} \phi(s)p = dp$$

which is exactly the number of parts congruent to $i \mod m$ in this composition. Thus summing over all cyclic compositions of n we see the relationship between $GPCC(x)$ and $A(x)$.

As an example, consider $n = 6$, $i = 1$ and $m = 3$. For the purpose of illustration we will use subscripts to denote the location of an entry in a (cyclic) composition; i.e., a cyclic composition of 6 with all parts of size 1 can be "cut" into a composition $1_1 1_2 1_3 1_4 1_5 1_6$ or $1_6 1_1 1_2 1_3 1_4 1_5$ depending on the location of the cut.

For each part in a cyclic composition of 6 that is congruent to 1 modulo 3, performing the operation described above yields:

(a) $1_1 1_2 1_3 1_4 1_5 1_6$, $1_6 1_1 1_2 1_3 1_4 1_5$, $1_5 1_6 1_1 1_2 1_3 1_4$, $1_4 1_5 1_6 1_1 1_2 1_3$, $1_3 1_4 1_5 1_6 1_1 1_2$, $1_2 1_3 1_4 1_5 1_6 1_1$

(b) 11121, 11211, 12111, 21111, 2211, 1221, 1131, 1311, 3111, 231, 321, 411, 141, 114, 24

(c) $21_2 21_4$, $21_4 21_2$

- All the compositions of case (b), together with $1_1 1_2 1_3 1_4 1_5 1_6$ and $21_2 21_4$, provide us exactly the set of compositions of 6 that ends with a part congruent to 1 modulo 3. They are counted by the coefficient of x^6 in $\phi(1)A(x)$.

- Since 2121 and 111111 can both be considered as repeating two copies of compositions of 3 (i.e., 21 and 111), they can be further cut in the middle, yielding $21; 21$ and $111; 111$ and counting $21_4 21_2$ and $1_4 1_5 1_6 1_1 1_2 1_3$. This is the coefficient of x^6 in $\phi(2)A(x^2)$.

- Furthermore, 111111 can also be considered as repeating three copies of compositions of 2 (i.e, 11), it can be further cut to obtain $11; 11; 11$, counting $1_5 1_6 1_1 1_2 1_3 1_4$ and $1_3 1_4 1_5 1_6 1_1 1_2$ through the coefficient of x^6 in $\phi(3)A(x^3)$.

- Lastly, 111111 can be considered as repeating six copies of compositions of 1, it can be further cut to obtain $1; 1; 1; 1; 1; 1$, counting $1_6 1_1 1_2 1_3 1_4 1_5$ and $1_2 1_3 1_4 1_5 1_6 1_1$ through the coefficient of x^6 in $\phi(6)A(x^6)$.

5. Concluding Remarks

In this note we studied the total number of parts congruent to $i \mod m$ in all cyclic compositions of n. This was done by following the cyclic construction of the generating function illustrated in general in [1]. From the generating function we provide justification for some rather surprising behaviors of the asymptotic values of this counting sequence. In the end we note an interesting relation between our generating function and the generating function of the number of compositions that end with a part congruent to $i \mod m$. We present combinatorial reasonings for this observation, which is interesting on its own as it provides a direct and combinatorial way of constructing the cyclic version of the generating function.

References

[1] P. Flajolet and M. Soria, The cycle construction, *SIAM J. Discrete Math.* **4** (1991), 58-60.

[2] S. Heubach and T. Mansour, *Combinatorics of Compositions and Words*, CRC Press, 2009.

[3] B. Hopkins, M. Shattuck, A. V. Sills, T. Thanatipanonda, and H. Wang, Parts and subword patterns in compositions, to appear in *J. Comb. Number Theory.*

[4] A. Knopfmacher and N. Robbins, Some properties of cyclic compositions, *Fibonacci Quart.* **48** (2010), 249-255.

[5] D. M. Y. Sommerville, On certain periodic properties of cyclic compositions of numbers, *Proc. Lond. Math. Soc.* **s2-7(1)** (1909), 263-313.

[6] R. P. Stanley, *Enumerative Combinatorics*, vol. 1, 2nd edition, Cambridge University Press, 2011.