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# $(k, l)$-UNIVERSALITY OF TERNARY QUADRATIC FORMS $ax^2 + by^2 + cz^2$

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**Abstract**

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The positive diagonal integral ternary quadratic form $ax^2+by^2+cz^2$ $(a, b, c \in \mathbb{N})$ is said to be $(k, l)$-universal if it represents every integer in the arithmetic progression $\{kn + l \mid n \in \mathbb{N}_0\}$, where $k, l \in \mathbb{N}$ are such that $l \le k$. We show that there are only finitely many $(k, l)$-universal positive integral ternary quadratic forms $ax^2 + by^2 + cz^2$ for a fixed pair $(k, l) \in \mathbb{N}^2$ with $l \le k$. We also prove the existence of a finite set $S = S(k, l)$ of positive integers congruent to $l$ modulo $k$ such that if $ax^2 + by^2 + cz^2$ represents every integer in $S$ then $ax^2 + by^2 + cz^2$ is $(k, l)$-universal. Assuming that certain ternaries are $(k, l)$-universal, we determine all the $(k, l)$-universal ternaries $ax^2 + by^2 + cz^2$ $(a, b, c \in \mathbb{N})$, as well as the sets $S(k, l)$, for all $k, l \in \mathbb{N}$ with $1 \le l \le k \le 11$.

## 1. Introduction

Let $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$. Let $k, l \in \mathbb{N}$ satisfy $l \le k$. A positive-definite quadratic form with integral coefficients is called $(k, l)$-universal if it represents every integer in the arithmetic progression $\{kn + l \mid n \in \mathbb{N}_0\}$. A $(1, 1)$-universal quadratic form represents all positive integers and is said to be universal. By Lagrange's theorem the quaternary quadratic form $x^2 + y^2 + z^2 + t^2$ is universal and by a theorem of Lebesgue [19] the ternary quadratic form $x^2 + y^2 + 2z^2$ is $(2, 1)$-universal. A $(k, l)$-universal quadratic form is said to be properly $(k, l)$-universal if it is not $(k', l - k'[\frac{l-1}{k'}])$-universal for any $k' \in \mathbb{N}$ satisfying $k' < k$ and $k' | k$. The

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form $x^2 + 2y^2 + 6z^2$ represents all positive integers not of the form $4^r(8s + 5)$ for some $r, s \in \mathbb{N}_0$ (see [10, pp. 111, 112]) so it represents all positive integers $8n + 7$ ($n \in \mathbb{N}_0$) and is thus $(8, 7)$-universal. However it is not properly $(8, 7)$-universal as it is $(4, 3)$-universal.

In this paper we consider positive diagonal integral ternary quadratic forms $ax^2 + by^2 + cz^2$, where $a, b$ and $c$ are positive integers. The ternary form $ax^2 + by^2 + cz^2$ is said to be primitive if $\gcd(a, b, c) = 1$. The primitive form associated with $ax^2 + by^2 + cz^2$ is the form $\frac{a}{d}x^2 + \frac{b}{d}y^2 + \frac{c}{d}z^2$, where $d = \gcd(a, b, c)$. If $ax^2 + by^2 + cz^2$ is $(k, l)$-universal and $m \in \mathbb{N}$ then $max^2 + mby^2 + mcz^2$ is $(mk, ml)$-universal. For example, as $x^2 + y^2 + 2z^2$ is $(2, 1)$-universal, $2x^2 + 2y^2 + 4z^2$ is $(4, 2)$-universal. Conversely, if $ax^2 + by^2 + cz^2$ is $(k, l)$-universal and $m \in \mathbb{N}$ divides each of $a, b, c, k$ and $l$ then $\frac{a}{m}x^2 + \frac{b}{m}y^2 + \frac{c}{m}z^2$ is $(\frac{k}{m}, \frac{l}{m})$-universal.

We now describe our approach to determining all ternary quadratic forms $ax^2 + by^2 + cz^2$ ($a, b, c \in \mathbb{N}, a \leq b \leq c$) which are $(k, l)$-universal for a given pair $(k, l) \in \mathbb{N}^2$ with $l \leq k$. Dickson [10, p. 104] has given an elementary proof that $ax^2 + by^2 + cz^2$ ($a, b, c \in \mathbb{N}$) is not $(1, 1)$-universal, so we do not need to consider the case $(k, l) = (1, 1)$. Panaitopol [23] has shown that there are no $(2, 2)$-universal forms $ax^2 + by^2 + cz^2$ ($a, b, c \in \mathbb{N}$). More generally we show that we do not need to consider the case $k = l$. Suppose that $ax^2 + by^2 + cz^2$ is $(k, k)$-universal, that is it represents all positive multiples of $k$. Then the rational form $\frac{1}{k}(ax^2 + by^2 + cz^2)$ represents all positive integers. But a positive ternary quadratic form with rational coefficients fails to represent rationally a full congruence class of positive integers. Thus $\frac{1}{k}(ax^2 + by^2 + cz^2)$ cannot represent all positive integers, a contradiction, see Conway [6, p. 143]. Hence we may suppose that $l < k$. Using the theorem of Alaca, Alaca, and Williams [1, p. 252] (see also [7]) that no binary quadratic form $ax^2 + by^2$ ($a, b \in \mathbb{N}$) can represent all members of an arithmetic progression of positive integers, we deduce in Theorem 2 upper bounds on the coefficients of a $(k, l)$-universal ternary quadratic form, so that for given pair $(k, l) \in \mathbb{N}^2$ with $l < k$ there are only finitely many $(k, l)$-universal ternary quadratic forms $ax^2 + by^2 + cz^2$ ($a, b, c \in \mathbb{N}$). In Section 3 we introduce the concept of the caliber $c(k, l)$ (see Definitions 4 and 5) and show that there are only finitely many $ax^2 + by^2 + cz^2$ representing the first $c(k, l)$ members of the arithmetic progression $\{nk + l \mid n \in \mathbb{N}_0\}$. These results allow us to prove the existence of an integer $r(k, l) > c(k, l)$ such that if $ax^2 + by^2 + cz^2$ ($a, b, c \in \mathbb{N}$) represents the first $r(k, l)$ members of $\{nk + l \mid n \in \mathbb{N}_0\}$ then $ax^2 + by^2 + cz^2$ is $(k, l)$-universal. This result is analogous to the 15-Theorem of Conway and Schneeberger, see [2], [3], [5], and can be proved in greater generality using the escalation technique of Bhargava, see [11], [12], [20], [24]. The proof of the existence of $r(k, l)$ is non-constructive since it is defined using the finite list of $(k, l)$-universal ternaries $ax^2 + by^2 + cz^2$ and thus cannot be used to prove $(k, l)$-universality of a given ternary $ax^2 + by^2 + cz^2$. However these results give an algorithm for producing a finite list of candidate forms that may be $(k, l)$-universal

(Theorem 10). For those candidate forms which are regular it is easy to decide whether they are $(k,l)$-universal or not. These forms are treated in Section 5. For candidate forms which are not regular it is often very difficult to determine whether they are $(k,l)$-universal or not. Those that we can apply genus theory to are treated in Section 6 with proofs in Appendix A. For each of these candidate forms that we cannot apply genus theory to, we list their $(k,l)$-universality as a conjecture. In Section 7 three forms are shown to be $(k,l)$-universal for certain values of $k$ and $l$ based upon the conjectured $(k,l)$-universality of three other candidate forms. The proof of our main result (Theorem 1) is given in Section 8 with full details in the cases $(k,l)=(8,1)$ and $(10,9)$ as the proofs in the other cases are quite similar.

We now state our main result.

**Theorem 1.** *Let $(k,l)\in\mathbb{N}^2$ satisfy $1\le l<k\le 11$. For each such pair $(k,l)$ the corresponding entry in the second column of Table 1 lists all the possible $(k,l)$-universal ternary quadratic forms $(a,b,c)=ax^2+by^2+cz^2 \quad (a,b,c\in\mathbb{N},\, a\le b\le c)$. For $(k,l)=(2,1)$, $(3,l)$ $(l=1,2)$, $(4,l)$ $(l=1,2,3)$, $(5,l)$ $(l=1,2,3,4)$, $(6,l)$ $(l=2,3,4)$, $(7,l)$ $(l=4,5,6)$, $(8,l)$ $(l=6,7)$, $(9,l)$ $(l=3,4,6,8)$, $(10,l)$ $(l=2,4,5,7,8)$, $(11,l)$ $(l=1,2,\ldots,10)$ the listed forms are precisely all the $(k,l)$-universal forms. In the remaining cases the listed forms contain one or more forms for which the $(k,l)$-universality has not yet been proved. These forms are indicated with an asterisk. The corresponding entry in the third column of Table 1 gives a set $H$ of positive integers such that if $ax^2+by^2+cz^2$ $(a,b,c\in\mathbb{N})$ represents each member of the set $H$ then $ax^2+by^2+cz^2$ is $(k,l)$-universal and moreover the set $H$ is minimal in the sense that for each $h\in H$ there is a form $ax^2+by^2+cz^2$ which does not represent $h$ but does represent all the integers of $H\backslash\{h\}$.*

Table 1: $(k,l)$-universal ternary quadratic forms $(a,b,c)=ax^2+by^2+cz^2 \quad (a,b,c\in\mathbb{N},\, a\le b\le c)$ for $1\le l<k\le 11$

| $(k,l)$ | $(a,b,c)$ | set $H$ |
|---|---|---|
| (2,1) | $(1,1,2)$, $(1,2,3)$, $(1,2,4)$ | 1,3,5,15 |
| (3,1) | $(1,1,3)$, $(1,1,6)$, $(1,3,3)$, $(1,3,9)$, $(1,6,9)$ | 1,7,10,13,19,22,34,46,70 |
| (3,2) | $(1,1,3)$, $(1,1,6)$, $(2,3,3)$ | 2,5,14,17,23,26,35,134 |
| (4,1) | $(1,1,1)$, $(1,1,2)$, $(1,1,4)$, $(1,1,5)$, $(1,1,8)$, $(1,1,16)$, $(1,2,2)$, $(1,2,3)$, $(1,2,4)$, $(1,4,4)$, $(1,4,5)$, $(1,4,8)$, $(1,4,16)$ | 1,5,21,33,53,65,77,201,721 |
| (4,2) | $(1,1,1)$, $(1,1,4)$, $(1,1,5)$, $(1,2,2)$, $(1,2,6)$, $(1,2,8)$, $(2,2,4)$, $(2,4,6)$, $(2,4,8)$ | 2,6,10,14,30 |

| | | |
|---|---|---|
| (4,3) | $(1,1,2)$, $(1,2,3)$, $(1,2,4)$, $(1,2,6)$ | 3,7,11,15,23,35 |
| (5,1) | $(1,2,5)$, $(1,5,10)$ | 1,6,11,21,26,46,91,111 |
| (5,2) | $(1,2,5)$ | 2,7,17,27,42,52,62 |
| (5,3) | $(1,2,5)$ | 3,13,23,38,58,78,133 |
| (5,4) | $(1,2,5)$, $(1,5,10)$ | 14,29,34,39,44,54,74 |
| (6,1) | $(1,1,2)$, $(1,1,3)$, $(1,1,6)$, $(1,2,3)$, $(1,2,4)$, $(1,3,3)$, $(1,3,4)$, $(1,3,6)$, $(1,3,7)^*$, $(1,3,9)$, $(1,3,12)$, $(1,3,18)$, $(1,3,24)$, $(1,3,36)$, $(1,3,42)^*$, $(1,3,54)^*$, $(1,4,6)$, $(1,6,9)$, $(1,6,12)$ | 1,7,13,19,31,55,85,91,115,145, 235,697 |
| (6,2) | $(1,1,3)$, $(1,1,6)$, $(1,1,12)$, $(2,2,3)$, $(2,2,6)$, $(2,2,12)$, $(2,3,3)$, $(2,3,18)$, $(2,6,6)$, $(2,6,18)$, $(2,12,18)$ | 2,14,20,26,44,68,92,134,140 |
| (6,3) | $(1,1,2)$, $(1,2,3)$, $(1,2,4)$, $(1,2,9)$, $(1,2,12)$, $(1,3,6)$, $(1,3,8)$, $(1,3,14)$, $(2,3,4)$, $(2,3,7)$, $(3,3,6)$, $(3,6,9)$, $(3,6,12)$ | 3,15,21,33,39,45,63 |
| (6,4) | $(1,1,3)$, $(1,1,6)$, $(1,1,12)$, $(1,3,3)$, $(1,3,9)$, $(1,6,6)$, $(1,6,9)$, $(1,9,12)$, $(2,2,3)$, $(2,2,6)$, $(2,2,12)$, $(3,3,4)$, $(4,6,6)$ | 4,10,22,28,34,46,52,70,268 |
| (6,5) | $(1,1,2)$, $(1,1,3)$, $(1,1,6)$, $(1,1,10)$, $(1,2,3)$, $(1,2,4)$, $(1,3,4)$, $(1,4,6)$, $(1,4,10)$, $(2,3,3)$, $(2,3,5)^*$, $(2,3,9)$, $(2,3,12)$ | 5,11,17,23,35,41 |
| (7,1) | $(1,2,7)^*$, $(1,7,14)^*$ | 15,22,29,43,57,78,106,267 |
| (7,2) | $(1,2,7)^*$ | 2,23,30,37,44,51,58,65,79 |
| (7,3) | $(1,2,7)^*$ | 3,10,24,31,38,52,94 |
| (7,4) | none | |
| (7,5) | none | |
| (7,6) | none | |

| | | |
|---|---|---|
| (8,1) | $(1,1,1)$, $(1,1,2)$, $(1,1,4)$, $(1,1,5)$, $(1,1,8)$, $(1,1,16)$, $(1,1,20)^*$, $(1,2,2)$, $(1,2,3)$, $(1,2,4)$, $(1,2,6)$, $(1,2,8)$, $(1,2,16)$, $(1,2,22)$, $(1,2,24)^*$, $(1,2,32)$, $(1,2,38)^*$, $(1,2,64)^*$, $(1,4,4)$, $(1,4,5)$, $(1,4,8)$, $(1,4,16)$, $(1,4,20)^*$, $(1,6,8)^*$, $(1,8,8)$, $(1,8,16)$, $(1,8,24)$, $(1,8,32)$, $(1,8,64)$, $(1,16,16)$ | 1,33,57,65,145,161,185,201,209, 217,377,481,721 |
| (8,2) | $(1,1,1)$, $(1,1,2)$, $(1,1,4)$, $(1,1,5)$, $(1,1,8)$, $(1,1,10)$, $(1,1,13)^*$, $(1,1,16)$, $(1,1,17)^*$, $(1,1,32)$, $(1,2,2)$, $(1,2,4)$, $(1,2,6)$, $(1,2,8)$, $(1,2,9)$, $(2,2,2)$, $(2,2,4)$, $(2,2,8)$, $(2,2,10)$, $(2,2,16)$, $(2,2,32)$, $(2,4,4)$, $(2,4,6)$, $(2,4,8)$, $(2,8,8)$, $(2,8,10)$, $(2,8,16)$, $(2,8,32)$ | 10,26,42,66,74,106,114,122,130, 154,258,282,402,1442 |
| (8,3) | $(1,1,1)$, $(1,1,2)$, $(1,2,2)$, $(1,2,3)$, $(1,2,4)$, $(1,2,6)$, $(1,2,8)$, $(1,2,16)$, $(1,2,18)$, $(1,2,24)$, $(1,2,32)$, $(1,2,34)^*$ | 3,19,35,43,91,115,187,195,395 |
| (8,4) | $(1,1,2)$, $(1,1,8)$, $(1,1,10)$, $(1,2,3)$, $(1,2,4)$, $(1,2,9)$, $(1,2,11)$, $(1,2,12)$, $(1,2,16)$, $(1,2,19)^*$, $(1,3,8)$, $(1,4,8)$, $(1,5,6)$, $(1,8,11)$, $(1,8,12)$, $(1,8,16)$, $(1,8,19)^*$, $(2,2,2)$, $(2,2,8)$, $(2,2,10)$, $(2,3,4)$, $(2,4,4)$, $(2,4,12)$, $(2,4,16)$, $(3,4,8)$, $(4,4,8)$, $(4,8,12)$, $(4,8,16)$ | 4,12,20,28,52,60,140,308 |
| (8,5) | $(1,1,1)$, $(1,1,2)$, $(1,1,4)$, $(1,1,5)$, $(1,1,8)$, $(1,1,13)^*$, $(1,1,16)$, $(1,2,2)$, $(1,2,3)$, $(1,2,4)$, $(1,4,4)$, $(1,4,5)$, $(1,4,8)$, $(1,4,13)^*$, $(1,4,16)$, $(2,3,10)^*$ | 5,13,21,29,37,69,77,85,133,581 |

| | | |
|---|---|---|
| (8,6) | $(1,1,1)$, $(1,1,4)$, $(1,1,5)$, $(1,2,2)$, $(1,2,3)$, $(1,2,6)$, $(1,2,8)$, $(1,2,11)$, $(1,2,12)$, $(1,5,6)$, $(2,2,4)$, $(2,3,4)$, $(2,4,6)$, $(2,4,8)$, $(2,4,12)$ | 6,14,22,30,38,46,70,78,110 |
| (8,7) | $(1,1,2)$, $(1,1,5)$, $(1,2,3)$, $(1,2,4)$, $(1,2,6)$, $(1,6,8)$, $(2,2,5)$, $(2,5,8)$ | 7,15,23,31,39,55,71,119,167 |
| (9,1) | $(1,1,3)$, $(1,1,6)$, $(1,3,3)$, $(1,3,7)^*$, $(1,3,9)$, $(1,6,9)$, $(1,6,15)^*$ | 1,10,19,37,46,55,82,91,118 |
| (9,2) | $(1,1,3)$, $(1,1,6)$, $(2,3,3)$, $(2,3,5)^*$ | 2,11,29,38,56,65,74,119,668 |
| (9,3) | $(1,1,3)$, $(1,3,3)$, $(1,3,9)$, $(1,3,27)$, $(2,3,3)$, $(2,3,27)$, $(3,3,9)$, $(3,3,18)$, $(3,9,9)$, $(3,9,27)$, $(3,18,27)$ | 3,21,30,39,57,66,102,138,174,210 |
| (9,4) | $(1,1,3)$, $(1,1,6)$, $(1,3,3)$, $(1,3,9)$, $(1,6,9)$ | 4,13,22,40,49,58,76,130,139,166, 175,238,445 |
| (9,5) | $(1,1,3)$, $(1,1,6)$, $(2,3,3)$, $(2,3,5)^*$ | 5,14,23,41,50,68,86,122,140 |
| (9,6) | $(1,1,6)$, $(1,3,3)$, $(1,6,9)$, $(2,3,3)$, $(3,3,9)$, $(3,3,18)$, $(6,9,9)$ | 6,15,42,51,69,78,105,114,402 |
| (9,7) | $(1,1,3)$, $(1,1,6)$, $(1,3,3)$, $(1,3,7)^*$, $(1,3,9)$, $(1,6,7)^*$, $(1,6,9)$ | 7,34,43,52,61,70,106,142,187,385 |
| (9,8) | $(1,1,3)$, $(1,1,6)$, $(2,3,3)$ | 8,17,26,35,44,53,62,71,98,134 |
| (10,1) | $(1,1,2)$, $(1,2,3)$, $(1,2,4)$, $(1,2,5)$, $(1,2,12)^*$, $(1,3,8)^*$, $(1,5,10)$ | 1,11,21,31,41,51,71,91,111,431 |
| (10,2) | $(1,2,5)$, $(1,2,10)$, $(2,4,10)$, $(2,5,10)$, $(2,10,20)$ | 2,12,22,42,52,62,92,132 |
| (10,3) | $(1,1,2)$, $(1,2,3)$, $(1,2,4)$, $(1,2,5)$, $(1,2,11)^*$, $(1,3,5)^*$, $(1,3,14)^*$ | 3,13,23,33,53,73,103,133 |
| (10,4) | $(1,2,5)$, $(1,2,10)$, $(1,5,10)$, $(2,4,10)$ | 14,34,44,54,84,104,124,314 |
| (10,5) | $(1,1,2)$, $(1,1,10)$, $(1,2,3)$, $(1,2,4)$, $(1,4,10)$, $(1,5,6)$, $(1,5,14)$, $(2,5,7)$, $(5,5,10)$, $(5,10,15)$, $(5,10,20)$ | 5,15,25,35,55,65,75,85 |

| | | |
|---|---|---|
| (10,6) | $(1,2,5)$, $(1,2,10)$, $(1,3,5)^*$, $(1,5,10)$, $(2,4,10)$ | 6,26,46,76,116,136,156,266 |
| (10,7) | $(1,1,2)$, $(1,2,3)$, $(1,2,4)$, $(1,2,5)$ | 7,17,27,47,77,87,97,287 |
| (10,8) | $(1,2,5)$, $(1,2,10)$, $(2,4,10)$, $(2,5,10)$, $(2,10,20)$ | 18,28,38,68,78,88,108,148,168 |
| (10,9) | $(1,1,2)$, $(1,2,3)$, $(1,2,4)$, $(1,2,5)$, $(1,2,12)^*$, $(1,2,17)^*$, $(1,2,20)^*$, $(1,3,8)^*$, $(1,5,10)$, $(2,3,4)^*$ | 19,29,39,69,79,89,119,129,149, 179,209 |
| (11,$l$) | none for $l$=1,...,10 | |

For all the forms $ax^2 + by^2 + cz^2$ in Table 1 which are conjectured to be $(k,l)$-universal, it has been checked that they represent all the positive integers congruent to $l$ modulo $k$ up to $100,000$.

The case $(k,l) = (2,1)$ of Theorem 1 is not new. Dickson [9, Theorems V, X and VII] showed that the three ternary quadratic forms $x^2+y^2+2z^2$, $x^2+2y^2+3z^2$ and $x^2 + 2y^2 + 4z^2$ represent all positive integers $n \equiv 1 \pmod 2$, and Kaplansky [18, pp. 212-213] proved that there are no other such ternary forms with this property; see also Panaitopol [23, Theorem 1]. Williams [27, Theorem A] showed that if $ax^2 + by^2 + cz^2$ $(a,b,c \in \mathbb{N})$ represents $1,3,5,15$ then it is $(2,1)$-universal and that the set $\{1,3,5,15\}$ is minimal as $2x^2 + 3y^2 + 4z^2$ represents $3,5$ and $15$ but not $1$, $x^2 + y^2 + 5z^2$ represents $1,5$ and $15$ but not $3$, $x^2 + 2y^2 + 6z^2$ represents $1,3$ and $15$ but not $5$, and $x^2+y^2+z^2$ represents $1,3$ and $5$ but not $15$. The cases $(k,l) = (4,2)$ and $(8,4)$ are due to Williams [26], [27].

It is shown in Section 7 that $x^2+4y^2+20z^2$ is $(8,1)$-universal under the assumption that $x^2 + y^2 + 20z^2$ is $(8,1)$-universal, that $x^2 + 8y^2 + 19z^2$ is $(8,4)$-universal assuming that $x^2+2y^2+19z^2$ is, and that $x^2+4y^2+13z^2$ is $(8,5)$-universal assuming that $x^2 + y^2 + 13z^2$ is.

The computer programs used in this paper required little more than a program to determine whether or not there are integers $x, y$ and $z$ satisfying $ax^2+by^2+cz^2 = n$ for certain ranges of positive integers $a, b, c$ and $n$. This was done by a search through integers $x, y$ and $z$ satisfying $0 \le x \le [\sqrt{n/a}]$, $0 \le y \le [\sqrt{n/b}]$, and $0 \le z \le [\sqrt{n/c}]$.

## 2. $(k,l)$-universal Forms $ax^2 + by^2 + cz^2$

Let $k$ and $l$ be positive integers with $l < k$. In this section we prove that there are only finitely many ternary quadratic forms $ax^2 + by^2 + cz^2$ $(a,b,c \in \mathbb{N})$ which are $(k,l)$-universal, see Theorem 3. Indeed we bound the size of the coefficients $a, b$ and

$c$ of a $(k,l)$-universal ternary quadratic form $ax^2+by^2+cz^2$ in terms of $k$ and $l$ (see Theorem 2).

The binary quadratic form $ax^2+by^2$ $(a,b \in \mathbb{N},\ a \leq b)$ has discriminant $d=-4ab$. As $a,b \in \mathbb{N}$ we have $d<0$ so that $d$ is not a perfect square. Thus, by a theorem of Alaca, Alaca and Williams [1, p. 252] $ax^2+by^2$ cannot represent all the integers in an infinite arithmetic progression of positive integers. Thus we can make the following definition.

**Definition 1.** Let $a,b,k$ and $l$ be positive integers with $l<k$. We define $N(a,b,k,l)$ to be the smallest member of the infinite arithmetic progression $\{nk+l \mid n \in \mathbb{N}_0\}$ which is not represented by $ax^2+by^2$.

Clearly $N(a,b,k,l)$ is a positive integer greater than or equal to $l$ and congruent to $l \pmod{k}$.

**Definition 2.** Let $k$ and $l$ be positive integers with $l<k$. We define

$$N(k,l) := \max_{\substack{(a,b)\,\in\,\mathbb{N}^2 \\ a \leq b \\ a \leq l \\ b \leq 3k+l}} N(a,b,k,l).$$

As $N(a,b,k,l) \equiv l \pmod{k}$ and $N(a,b,k,l) \geq l$ for all $a,b \in \mathbb{N}$, we have $N(k,l) \equiv l \pmod{k}$ and $N(k,l) \geq l$.

The values of $N(k,l)$ for $1 \leq l < k \leq 11$ are given in Table 2.

Table 2: Values of $N(k,l)$ for $1 \leq l < k \leq 11$.

| | $l$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | | | | | | | | | | | |
| 2 | | 5 | | | | | | | | | |
| 3 | | 13 | 17 | | | | | | | | |
| 4 | | 21 | 10 | 11 | | | | | | | |
| 5 | | 21 | 12 | 13 | 24 | | | | | | |
| 6 | | 55 | 26 | 15 | 34 | 17 | | | | | |
| 7 | | 15 | 23 | 17 | 39 | 19 | 20 | | | | |
| 8 | | 65 | 42 | 35 | 28 | 21 | 22 | 23 | | | |
| 9 | | 28 | 38 | 39 | 31 | 23 | 51 | 34 | 35 | | |
| 10 | | 21 | 42 | 23 | 34 | 25 | 26 | 27 | 48 | 29 | |
| 11 | | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 |

We illustrate the determination of the values of $N(k,l)$ in Table 2 by showing that $N(3,1) = 13$.

The pairs $(a,b) \in \mathbb{N}^2$ satisfying $a \leq b$, $a \leq 1$ and $b \leq 10$ are $(1,b)$ ($b = 1,2,3,4,5,6,7,8,9,10$). We have

$$\begin{aligned} N(1,b,3,1) &= 7 \text{ for } b = 1,2,4,5,8,9,10, \\ &= 10 \text{ for } b = 3,7, \\ &= 13 \text{ for } b = 6. \end{aligned}$$

Hence

$$N(3,1) := \max_{\substack{(a,b) \in \mathbb{N}^2 \\ a \leq b \\ a \leq 1 \\ b \leq 10}} N(a,b,3,1) = 13.$$

**Theorem 2.** *Let $k$ and $l$ be positive integers satisfying $l < k$. Then any $(k,l)$-universal ternary quadratic form $ax^2 + by^2 + cz^2$ $(a,b,c \in \mathbb{N},\ a \leq b \leq c)$ satisfies*

$$a \leq l,\ b \leq 3k + l,\ c \leq N(k,l),$$

*where $N(k,l)$ is defined in Definition 2.*

*Proof.* Suppose that the ternary quadratic form $ax^2 + by^2 + cz^2$ $(a,b,c \in \mathbb{N},\ a \leq b \leq c)$ is $(k,l)$-universal. Then $ax^2 + by^2 + cz^2$ represents $nk + l$ for all $n \in \mathbb{N}_0$. Hence there exist integers $x_n, y_n$, and $z_n$ such that

$$ax_n^2 + by_n^2 + cz_n^2 = nk + l,\ n \in \mathbb{N}_0.$$

In particular we have

$$ax_0^2 + by_0^2 + cz_0^2 = l.$$

As $l \neq 0$ we have $(x_0, y_0, z_0) \neq (0,0,0)$ so that $x_0^2 + y_0^2 + z_0^2 \geq 1$. Hence

$$a \ \leq a(x_0^2 + y_0^2 + z_0^2) = ax_0^2 + ay_0^2 + az_0^2 \ \leq ax_0^2 + by_0^2 + cz_0^2 \ = l,$$

which is the first inequality.

Now suppose that $(y_n, z_n) = (0,0)$ for $n = 0,1,2,3$. Then

$$ax_0^2 = l,\ ax_1^2 = k + l,\ ax_2^2 = 2k + l,\ ax_3^2 = 3k + l.$$

As $l \neq 0$ we have $x_0 \neq 0$ and we can define a positive integer $l_1$ by $l_1 = x_0^2$ so that $l = al_1$. As $a, k \geq 1$ we have $x_1^2 > x_0^2$, and we can define a positive integer $k_1$ by $k_1 = x_1^2 - x_0^2$, so that $k = (k + l) - l = ax_1^2 - ax_0^2 = a(x_1^2 - x_0^2) = ak_1$. Thus

$$x_0^2 = l_1,\ x_1^2 = k_1 + l_1,\ x_2^2 = 2k_1 + l_1,\ x_3^2 = 3k_1 + l_1,$$

showing that $x_0^2, x_1^2, x_2^2$ and $x_3^2$ are four squares in arithmetic progression. But four squares cannot be in arithmetic progression [21, Theorem 3, p. 21]. Thus our supposition is incorrect and we must have $(y_n, z_n) \neq (0,0)$ for some $n \in \{0,1,2,3\}$. For this $n$ we have $y_n^2 + z_n^2 \geq 1$, and so

$$b \ \leq b(y_n^2 + z_n^2) = by_n^2 + bz_n^2 \ \leq by_n^2 + cz_n^2 \ \leq ax_n^2 + by_n^2 + cz_n^2 \ = nk + l.$$

But $n \leq 3$ so we have

$$b \leq 3k + l,$$

which is the second inequality of the theorem.

Recall from Definition 1 that $N(a,b,k,l)$ is the smallest member of the infinite arithmetic progression $\{nk + l \mid n \in \mathbb{N}_0\}$, which is not represented by $ax^2 + by^2$. As $N(a,b,k,l) \in \{nk + l \mid n \in \mathbb{N}_0\}$ we can define $n(a,b,k,l) \in \mathbb{N}_0$ such that $N(a,b,k,l) = kn(a,b,k,l) + l$. Thus writing $n$ for $n(a,b,k,l)$ we have

$$ax_n^2 + by_n^2 + cz_n^2 = N(a,b,k,l).$$

But $ax^2 + by^2$ does not represent $N(a,b,k,l)$ so we must have $z_n \neq 0$. Thus

$$c \leq cz_n^2 \leq ax_n^2 + by_n^2 + cz_n^2 = N(a,b,k,l).$$

But $a \leq b$, $a \leq l$ and $b \leq 3k + l$ so by Definition 2 we have $c \leq N(k,l)$, which is the third inequality of the theorem. □

The following theorem is an immediate consequence of Theorem 2.

**Theorem 3.** *Let $k$ and $l$ be positive integers satisfying $l < k$. Then there are only finitely many ternary quadratic forms*

$$ax^2 + by^2 + cz^2 \ (a,b,c \in \mathbb{N}),$$

*which are $(k,l)$-universal.*

More generally Oh [22] has shown by more advanced techniques that there are only finitely many positive-definite integral $(k,l)$-universal ternary quadratic forms $ax^2 + by^2 + cz^2 + dxy + exz + fyz$.

Theorem 3 allows us to make the following definition.

**Definition 3.** Let $k$ and $l$ be positive integers with $l < k$. We define

$$A(k,l) := \text{number of ternary quadratic forms } ax^2 + by^2 + cz^2$$
$$(a,b,c \in \mathbb{N},\ a \leq b \leq c) \text{ that are } (k,l)\text{-universal.}$$

From Theorem 2 we obtain the following crude upper bound for $A(k,l)$, namely,

$$A(k,l) \leq l(3k + l)N(k,l).$$

Theorem 2 also gives us a method of determining all possible $(k,l)$-universal ternary quadratic forms $ax^2 + by^2 + cz^2$ $(a,b,c \in \mathbb{N},\ a \leq b \leq c)$.

### 2.1. Method for Determining All Possible $(k,l)$-universal Ternaries $ax^2+by^2+cz^2$ for a Given Pair of Positive Integers $k$ and $l$ with $l<k$

First, for each pair $(a,b)\in\mathbb{N}^2$ satisfying $a\le b$, $a\le l$, $b\le 3k+l$, we determine the positive integer $N(a,b,k,l)$ as in Definition 1, and then determine $N(k,l)$ as in Definition 2. Secondly, we determine for each ternary quadratic form $ax^2+by^2+cz^2$ with $a,b$ and $c$ positive integers satisfying

$$a\le b\le c,\ a\le l,\ b\le 3k+l,\ c\le N(k,l),$$

whether or not there is an integer among $l,k+l,2k+l,\ldots,Mk+l$, where $M$ is a large positive integer (we used $M=10^4$), which is not represented by $ax^2+by^2+cz^2$. Thirdly, we discard those forms $ax^2+by^2+cz^2$ for which such an integer exists. The forms that remain (if any) are by Theorem 2 the only possibilities to be $(k,l)$-universal, and they must be examined for $(k,l)$-universality on an individual basis. If no forms remain then there are no ternaries $ax^2+by^2+cz^2$ which are $(k,l)$-universal.

We illustrate the method by proving the following result.

**Theorem 4.** *No ternary quadratic form $ax^2+by^2+cz^2$ $(a,b,c\in\mathbb{N})$ is $(k,l)$-universal for*

$$(k,l)=(7,l),\ (l=4,5,6),\ (11,l),\ (l=1,2,\ldots,10).$$

*Proof.* We just treat the case $(k,l)=(7,4)$ as the other cases can be handled in a similar manner. From Table 2 we have $N(7,4)=39$. By Theorem 2 the only possible ternary quadratic forms $ax^2+by^2+cz^2$ $(a,b,c\in\mathbb{N},\ a\le b\le c)$, which are $(7,4)$-universal, are those satisfying

$$a\le 4,\ b\le 25,\ c\le 39.$$

Each of these 2470 forms fails to represent at least one of the integers in the set

$$\{4,\ 11,\ 18,\ 25,\ 32,\ 39,\ 46,\ 53,\ 60,\ 67,\ 74,\ 158,\ 214\}$$

so none of them is $(7,4)$-universal. $\square$

## 3. Determination of $(k,l)$-universal Ternaries From a Finite Set

Our objective in this section is to prove the following result.

**Theorem 5.** *Let $k$ and $l$ be positive integers satisfying $l<k$. Then there exists a finite subset $S=S(k,l)$ of $\{nk+l\mid n\in\mathbb{N}_0\}$ such that if $ax^2+by^2+cz^2$ $(a,b,c\in\mathbb{N})$ represents all the members of $S$ then $ax^2+by^2+cz^2$ is $(k,l)$-universal.*

We begin with a lemma, which can be proved in a manner similar to Theorem 2.

**Lemma 1.** *Let $k$ and $l$ be positive integers with $l < k$. Suppose that the binary quadratic form $ax^2 + by^2$ $(a, b \in \mathbb{N},\ a \le b)$ represents $l$, $k + l$, $2k + l$ and $3k + l$. Then*

$$a \le l,\ \ b \le 3k + l.$$

**Definition 4.** Let $k$ and $l$ be positive integers such that $l < k$. The finite arithmetic progression

$$\{l,\ k + l,\ 2k + l, \ldots, (h - 1)k + l\},$$

where $h \in \mathbb{N}$, is called a $(k, l)$-core of length $h$ if there do not exist positive integers $a$ and $b$ such that the form $ax^2 + by^2$ represents all of $l,\ k+l,\ 2k+l, \ldots, (h-1)k+l$.

The form $lx^2 + ky^2$ represents $l$ so that $\{l\}$ is not a $(k, l)$-core. The form $lx^2 + ky^2$ also represents $k + l$ so that $\{l, k + l\}$ is not a $(k, l)$-core. Hence if $\{l,\ k + l,\ 2k + l, \ldots, (h - 1)k + l\}$ is a $(k, l)$-core then $h \ge 3$.

**Theorem 6.** *Let $k$ and $l$ be positive integers such that $l < k$. Then a $(k, l)$-core always exists.*

*Proof.* Suppose that $k$ and $l$ are positive integers with $l < k$ for which a $(k, l)$-core does not exist. Thus for every $h \in \mathbb{N}$ there exist positive integers $a$ and $b$ such that $ax^2 + by^2$ represents all of $l, k + l, 2k + l, \ldots, (h - 1)k + l$. Take

$$h_0 = \frac{N(k, l) - l}{k} + 4 \in \mathbb{N},$$

where $N(k, l)$ is defined in Definition 2. We note that

$$h_0 \ge 4,\ \ h_0k + l > N(k, l).$$

Let $a$ and $b$ be positive integers with $a \le b$ such that $ax^2 + by^2$ represents all of

$$l,\ k + l,\ 2k + l, \ldots, (h_0 - 1)k + l.$$

By Definition 1 we have

$$h_0k + l \le N(a, b, k, l).$$

As $h_0 \ge 4$ the form $ax^2+by^2$ represents all of $l,\ k+l,\ 2k+l, 3k+l$. Thus, by Lemma 1, we have $a \le l,\ \ b \le 3k + l$. Hence, by Definitions 1 and 2 we have

$$h_0k + l \le N(a, b, k, l) \le N(k, l),$$

contradicting $h_0k + l > N(k, l)$. Hence a $(k, l)$-core always exists. □

In view of Theorem 6 we can make the following definition.

**Definition 5.** Let $k$ and $l$ be positive integers such that $l < k$. Let $c(k,l)$ denote the smallest positive integer $h$ such that $\{l,\, k+l,\, 2k+l, \ldots, (h-1)k+l\}$ is a $(k,l)$-core. The positive integer $c(k,l)$ is called the caliber of the $(k,l)$-core.

By the comment following Definition 4, we have $c(k,l) \geq 3$. In Section 4 we determine the caliber $c(k,l)$ for all positive integers $k$ and $l$ satisfying $1 \leq l < k \leq 11$.

**Theorem 7.** *Let $k$ and $l$ be positive integers with $l < k$. Then the only ternary quadratic forms $ax^2 + by^2 + cz^2$ $(a,b,c \in \mathbb{N},\, a \leq b \leq c)$ representing the $c(k,l)$ integers*

$$l,\, k+l,\, 2k+l, \ldots, (c(k,l)-1)k+l,$$

*satisfy*

$$a \ \leq b \ \leq c \ \leq (c(k,l)-1)k+l,$$

*where the caliber $c(k,l)$ is defined in Definition 5.*

*Proof.* Suppose that the ternary quadratic form $ax^2 + by^2 + cz^2$ $(a,b,c \in \mathbb{N},\, a \leq b \leq c)$ represents the $c(k,l)$ integers $l,\, k+l,\, 2k+l, \ldots, (c(k,l)-1)k+l$. We prove that $c \leq (c(k,l)-1)k+l$. Suppose on the contrary that $c > (c(k,l)-1)k+l$. Then, for any integers $x, y$ and $z \neq 0$, we have

$$ax^2 + by^2 + cz^2 \geq c > (c(k,l)-1)k+l.$$

But $ax^2 + by^2 + cz^2$ represents all of $l, k+l, 2k+l, \ldots, (c(k,l)-1)k+l$. Hence in each case it must do so with $z = 0$. Thus $ax^2 + by^2$ represents $l, k+l, 2k+l, \ldots, (c(k,l)-1)k+l$. This contradicts that $l, k+l, 2k+l, \ldots, (c(k,l)-1)k+l$ is a $(k,l)$-core. Hence the only possible ternary quadratic forms $ax^2+by^2+cz^2$ $(a,b,c \in \mathbb{N},\, a \leq b \leq c)$ representing $l, k+l, 2k+l, \ldots, (c(k,l)-1)k+l$ satisfy $a \leq b \leq c \leq (c(k,l)-1)k+l$. □

Theorem 7 shows that for each pair of positive integers $k$ and $l$ with $l < k$ there are only finitely many ternary quadratic forms $ax^2+by^2+cz^2$ $(a,b,c \in \mathbb{N},\, a \leq b \leq c)$ representing $l, k+l, 2k+l, \ldots, (c(k,l)-1)k+l$, where $c(k,l)$ is the caliber. This allows us to make the following definition.

**Definition 6.** Let $k$ and $l$ be positive integers with $l < k$. We define

$$\begin{aligned} B(k,l) :=& \text{number of quadratic forms } ax^2+by^2+cz^2 \ (a,b,c \in \mathbb{N},\, a \leq b \leq c) \\ & \text{representing all of the integers } l, k+l, 2k+l, \ldots, (c(k,l)-1)k+l. \end{aligned}$$

By Definitions 3 and 6, and Theorem 7, we have

$$A(k,l) \le B(k,l) \le \frac{1}{6}((c(k,l)-1)k+l)((c(k,l)-1)k+l+1) \times ((c(k,l)-1)k+l+2) \tag{3.1}$$

and

$$B(k,l) - A(k,l) = \text{number of ternary quadratic forms } ax^2+by^2+cz^2 \ (a,b,c \in \mathbb{N},\ a \le b \le c) \text{ that represent all the integers } l, k+l, 2k+l, \ldots, (c(k,l)-1)k+l \text{ but are not } (k,l)\text{-universal.} \tag{3.2}$$

**Definition 7.** Let $k$ and $l$ be positive integers with $l < k$. Suppose that $B(k,l) > A(k,l)$. Let

$$a_i x^2 + b_i y^2 + c_i z^2 \, (a_i, b_i, c_i \in \mathbb{N},\, a_i \le b_i \le c_i,\ i = 1, 2, \ldots, B(k,l) - A(k,l))$$

be the $B(k,l) - A(k,l)$ ternary quadratic forms that represent all the integers

$$l,\ k+l,\ 2k+l, \ldots (c(k,l)-1)k+l$$

but which are not $(k,l)$-universal. For $i = 1, 2, \ldots, B(k,l) - A(k,l)$ we let $(r_i - 1)k + l$ be the least integer of the arithmetic progression $l,\ k+l,\ 2k+l, \ldots$ not represented by $a_i x^2 + b_i y^2 + c_i z^2$. Clearly we have

$$r_i > c(k,l),\ i = 1, 2, \ldots, B(k,l) - A(k,l).$$

Let

$$r(k,l) := \max_{1 \le i \le B(k,l) - A(k,l)} r_i$$

so that

$$r(k,l) > c(k,l).$$

If $B(k,l) - A(k,l) = 0$ we set $r(k,l) = c(k,l)$.

**Theorem 8.** *Let $k$ and $l$ be positive integers with $l < k$. If the ternary quadratic form $ax^2 + by^2 + cz^2$ $(a,b,c \in \mathbb{N},\ a \le b \le c)$ represents all the positive integers*

$$l, k+l, 2k+l, \ldots, (r(k,l)-1)k+l$$

*then it is $(k,l)$-universal.*

*Proof.* If $B(k,l) - A(k,l) = 0$, then $r(k,l) = c(k,l)$, and $ax^2 + by^2 + cz^2$ represents all the positive integers $l,\ k+l, \ldots (c(k,l)-1)k+l$, and so by (3.2) $ax^2+by^2+cz^2$ is $(k,l)$-universal. Hence we may assume that $B(k,l) > A(k,l)$. Suppose that the ternary quadratic form $ax^2 + by^2 + cz^2$ ($a, b, c \in \mathbb{N}$, $a \le b \le c$) represents all the integers

$$l,\ k+l,\ 2k+l, \ldots, (r(k,l)-1)k+l$$

but is not $(k,l)$-universal. As $r(k,l) > c(k,l)$, $ax^2 + by^2 + cz^2$ represents all the positive integers

$$l,\ k+l,\ 2k+l, \ldots (c(k,l)-1)k+l.$$

By Definition 7 we have $ax^2 + by^2 + cz^2 = a_ix^2 + b_iy^2 + c_iz^2$ for some $i \in \{1, 2, \ldots, B(k,l) - A(k,l)\}$. Also from Definition 7 $a_ix^2 + b_iy^2 + c_iz^2$ does not represent $(r_i(k,l) - 1)k + l$. As $r_i \le r(k,l)$ the integer $(r_i - 1)k + l$ is in the list $l,\ k+l,\ 2k+l,\ \ldots, (r(k,l)-1)k+l$, and so is represented by $ax^2 + by^2 + cz^2 = a_ix^2 + b_iy^2 + c_iz^2$. This is a contradiction. Hence $ax^2+by^2+cz^2$ is $(k,l)$-universal. □

*Proof of Theorem* 5. By Theorem 8 we may take $S(k,l)$ to be the set $\{l, k+l, 2k+l, \ldots (r(k,l)-1)k+l\}$. □

## 4. Explicit $(k,l)$-cores for $1 \le l < k \le 11$

In this section we give a table of $(k,l)$-cores and calibers $c(k,l)$ for $1 \le l < k \le 11$, see Table 3. For each such pair $(k,l)$ a $(k,l)$-core of minimal length is given in the third column of Table 3, the caliber $c(k,l)$ in the fourth column, and in the fifth column a form $ax^2+by^2$ ($a, b \in \mathbb{N}, a \le b$) representing the first $c(k,l)-1$ members of the $(k,l)$-core (thereby establishing the minimality of the length of the $(k,l)$-core).

Table 3: Explicit $(k,l)$-cores for $1 \le l < k \le 11$

| $k$ | $l$ | $(k,l)$-core | $c(k,l)$ | $ax^2+by^2$ |
|---|---|---|---|---|
| 2 | 1 | $\{1,3,5\}$ | 3 | $x^2+2y^2$ |
| 3 | 1 | $\{1,4,7,10,13\}$ | 5 | $x^2+6y^2$ |
| 3 | 2 | $\{2,5,8,11,14,17\}$ | 6 | $2x^2+3y^2$ |
| 4 | 1 | $\{1,5,9,13,17,21\}$ | 6 | $x^2+y^2$ |
| 4 | 2 | $\{2,6,10\}$ | 3 | $x^2+2y^2$ |
| 4 | 3 | $\{3,7,11\}$ | 3 | $x^2+3y^2$ |
| 5 | 1 | $\{1,6,11,16,21\}$ | 5 | $x^2+2y^2$ |
| 5 | 2 | $\{2,7,12\}$ | 3 | $2x^2+5y^2$ |
| 5 | 3 | $\{3,8,13\}$ | 3 | $x^2+2y^2$ |
| 5 | 4 | $\{4,9,14,19,24\}$ | 5 | $x^2+10y^2$ |
| 6 | 1 | $\{1,7,13,19,25,31,37,43,49,55\}$ | 10 | $x^2+3y^2$ |
| 6 | 2 | $\{2,8,14,20,26\}$ | 5 | $2x^2+3y^2$ |
| 6 | 3 | $\{3,9,15\}$ | 3 | $x^2+2y^2$ |
| 6 | 4 | $\{4,10,16,22,28,34\}$ | 6 | $x^2+6y^2$ |

| | | | | |
|---|---|---|---|---|
| 6 | 5 | $\{5, 11, 17\}$ | 3 | $2x^2 + 3y^2$ |
| 7 | 1 | $\{1, 8, 15\}$ | 3 | $x^2 + y^2$ |
| 7 | 2 | $\{2, 9, 16, 23\}$ | 4 | $x^2 + y^2$ |
| 7 | 3 | $\{3, 10, 17\}$ | 3 | $3x^2 + 7y^2$ |
| 7 | 4 | $\{4, 11, 18, 25, 32, 39\}$ | 6 | $x^2 + 2y^2$ |
| 7 | 5 | $\{5, 12, 19\}$ | 3 | $2x^2 + 3y^2$ |
| 7 | 6 | $\{6, 13, 20\}$ | 3 | $6x^2 + 7y^2$ |
| 8 | 1 | $\{1, 9, 17, 25, 33, 41, 49, 57, 65\}$ | 9 | $x^2 + 2y^2$ |
| 8 | 2 | $\{2, 10, 18, 26, 34, 42\}$ | 6 | $x^2 + y^2$ |
| 8 | 3 | $\{3, 11, 19, 27, 35\}$ | 5 | $x^2 + 2y^2$ |
| 8 | 4 | $\{4, 12, 20, 28\}$ | 4 | $x^2 + 11y^2$ |
| 8 | 5 | $\{5, 13, 21\}$ | 3 | $x^2 + y^2$ |
| 8 | 6 | $\{6, 14, 22\}$ | 3 | $x^2 + 5y^2$ |
| 8 | 7 | $\{7, 15, 23\}$ | 3 | $x^2 + 6y^2$ |
| 9 | 1 | $\{1, 10, 19, 28\}$ | 4 | $x^2 + 10y^2$ |
| 9 | 2 | $\{2, 11, 20, 29, 38\}$ | 5 | $2x^2 + 3y^2$ |
| 9 | 3 | $\{3, 12, 21, 30, 39\}$ | 5 | $2x^2 + 3y^2$ |
| 9 | 4 | $\{4, 13, 22, 31\}$ | 4 | $x^2 + 13y^2$ |
| 9 | 5 | $\{5, 14, 23\}$ | 3 | $x^2 + 5y^2$ |
| 9 | 6 | $\{6, 15, 24, 33, 42, 51\}$ | 6 | $x^2 + 6y^2$ |
| 9 | 7 | $\{7, 16, 25, 34\}$ | 4 | $x^2 + 3y^2$ |
| 9 | 8 | $\{8, 17, 26, 35\}$ | 4 | $x^2 + y^2$ |
| 10 | 1 | $\{1, 11, 21\}$ | 3 | $x^2 + 2y^2$ |
| 10 | 2 | $\{2, 12, 22, 32, 42\}$ | 5 | $x^2 + 2y^2$ |
| 10 | 3 | $\{3, 13, 23\}$ | 3 | $x^2 + 3y^2$ |
| 10 | 4 | $\{4, 14, 24, 34\}$ | 4 | $x^2 + 5y^2$ |
| 10 | 5 | $\{5, 15, 25\}$ | 3 | $5x^2 + 10y^2$ |
| 10 | 6 | $\{6, 16, 26\}$ | 3 | $x^2 + 2y^2$ |
| 10 | 7 | $\{7, 17, 27\}$ | 3 | $7x^2 + 10y^2$ |
| 10 | 8 | $\{8, 18, 28, 38, 48\}$ | 5 | $2x^2 + 5y^2$ |
| 10 | 9 | $\{9, 19, 29\}$ | 3 | $x^2 + 2y^2$ |
| 11 | 1 | $\{1, 12, 23\}$ | 3 | $x^2 + 2y^2$ |
| 11 | 2 | $\{2, 13, 24\}$ | 3 | $x^2 + y^2$ |
| 11 | 3 | $\{3, 14, 25\}$ | 3 | $2x^2 + 3y^2$ |
| 11 | 4 | $\{4, 15, 26\}$ | 3 | $x^2 + 6y^2$ |
| 11 | 5 | $\{5, 16, 27\}$ | 3 | $x^2 + y^2$ |
| 11 | 6 | $\{6, 17, 28\}$ | 3 | $x^2 + 2y^2$ |
| 11 | 7 | $\{7, 18, 29\}$ | 3 | $2x^2 + 5y^2$ |
| 11 | 8 | $\{8, 19, 30\}$ | 3 | $x^2 + 2y^2$ |
| 11 | 9 | $\{9, 20, 31\}$ | 3 | $x^2 + y^2$ |
| 11 | 10 | $\{10, 21, 32\}$ | 3 | $10x^2 + 11y^2$ |

We just prove that $\{1, 4, 7, 10, 13\}$ is a $(3, 1)$-core with caliber $c(3, 1) = 5$. The rest of the table can be justified in a similar manner. Suppose that $ax^2 + by^2$ $(a, b \in \mathbb{N},\ a \leq b)$ represents all the integers $1, 4, 7, 10$, and $13$. As $ax^2+by^2$ represents 1, we must have $a = 1$. Then $x^2 + by^2$ represents 7 so $b \in \{1, 2, 3, 4, 5, 6, 7\}$. But $x^2 + by^2$ does not represent 7 for $b = 1, 2, 4, 5$; $x^2 + by^2$ does not represent 10 for $b = 3, 7$; and $x^2 + by^2$ does not represent 13 for $b = 6$. Thus there is no form $ax^2 + by^2$ $(a, b \in \mathbb{N},\ a \leq b)$ representing all the integers $1, 4, 7, 10$, and $13$. Hence $\{1, 4, 7, 10, 13\}$ is a $(3, 1)$-core. The form $x^2 + 6y^2$ represents $1, 4, 7, 10$ but not 13 so $\{1, 4, 7, 10, 13\}$ is a $(3, 1)$-core of minimal length. Hence the caliber $c(3, 1) = 5$.

A comparison of the values of $N(k, l)$ in Table 2 and the values of $c(k, l)$ in Table 3 suggests that $N(k, l) = (c(k, l)-1)k+l$ for all positive integers $k$ and $l$ with $l < k$. We cannot quite prove this, but we can prove the following result.

**Theorem 9.** *Let $k$ and $l$ be positive integers with $l < k$. Then*

$$N(k, l) = (c(k, l) - 1)k + l \text{ if } c(k, l) = 3 \text{ or } c(k, l) \geq 5,$$
$$N(k, l) \leq (c(k, l) - 1)k + l \text{ if } c(k, l) = 4.$$

*Proof.* Let $a, b \in \mathbb{N}$ satisfy $a \leq b$. The smallest integer in the sequence $\{nk + l \mid n \in \mathbb{N}_0\}$ which is not represented by $ax^2 + by^2$ is $N(a, b, k, l)$ (Definition 1). If $N(a, b, k, l) = l, k + l$ or $2k + l$ then $N(a, b, k, l) \leq 2k + l \leq (c(k, l) - 1)k + l$ as $c(k, l) \geq 3$. If $N(a, b, k, l)$ is not one of $l, k + l$ and $2k + l$ then $N(a, b, k, l) \geq 3k + l$ so that $N(a, b, k, l) - k \geq 2k + l$ and thus

$$l, k + l, 2k + l, \ldots, N(a, b, k, l) - k \tag{4.1}$$

are all represented by $ax^2 + by^2$. Hence (4.1) is not a $(k, l)$-core, and so

$$N(a, b, k, l) - k < (c(k, l) - 1)k + l,$$

and thus as $N(a, b, k, l) \equiv l \pmod{k}$, we have

$$N(a, b, k, l) \leq (c(k, l) - 1)k + l$$

for all positive integers $a$ and $b$ with $a \leq b$. Hence

$$N(k, l) = \max_{\substack{(a,b) \in \mathbb{N}^2 \\ a \leq b \\ a \leq l \\ b \leq 3k+l}} N(a, b, k, l) \leq (c(k, l) - 1)k + l.$$

This completes the proof in the case $c(k, l) = 4$. To complete the proof of the theorem we must prove that

$$N(k, l) \geq (c(k, l) - 1)k + l$$

if $c(k,l) = 3$ or $c(k,l) \geq 5$.

First we suppose that $c(k,l) \geq 5$. By Definition 5

$$l, k+l, \ldots, (c(k,l)-2)k+l$$

is not a $(k,l)$-core. Hence, by Definition 4, there exist positive integers $a$ and $b$ with $a \leq b$ such that the form $ax^2 + by^2$ represents all of $l, k+l, \ldots, (c(k,l)-2)k+l$. Thus, by Definition 1, we have

$$N(a,b,k,l) > (c(k,l)-2)k+l$$

so that

$$N(a,b,k,l) \geq (c(k,l)-1)k+l.$$

As $c(k,l) \geq 5$ we have $(c(k,l)-2)k+l \geq 3k+l$, and so $ax^2+by^2$ represents $l, k+l, 2k+l, 3k+l$. Thus, by Lemma 1, we have $a \leq l$ and $b \leq 3k+l$. Hence

$$N(k,l) \geq N(a,b,k,l) \geq (c(k,l)-1)k+l.$$

This completes the proof in the case $c(k,l) \geq 5$.

Secondly, we suppose that $c(k,l) = 3$. In this case

$$\{l, k+l, 2k+l\}$$

is a $(k,l)$-core. Hence there do not exist positive integers $a$ and $b$ such that $ax^2+by^2$ represents $l, k+l$, and $2k+l$. But $lx^2+ky^2$ represents $l$ and $k+l$ so it must be the case that $lx^2+ky^2$ does not represent $2k+l$. Therefore the least integer in the arithmetic progression $\{nk+l \,|\, n \in \mathbb{N}_0\}$ which is not represented by $lx^2+ky^2$ is $2k+l$, and thus $N(l,k,k,l) = 2k+l$. Hence

$$N(k,l) = \max_{\substack{(a,b) \in \mathbb{N}^2 \\ a \leq b \\ a \leq l \\ b \leq 3k+l}} N(a,b,k,l) \geq N(l,k,k,l) = 2k+l$$

so

$$N(k,l) \geq (c(k,l)-1)k+l.$$

This completes the proof of the theorem in the case $c(k,l) = 3$. The theorem is now established. $\square$

The difficulty in determining $N(k,l)$ when $c(k,l) = 4$ appears to result from the fact that although four squares cannot be in arithmetic progression three squares can be, for example $1, 25, 49$.

From Theorems 2 and 9 we have the following result.

**Theorem 10.** *Let $k$ and $l$ be positive integers satisfying $l < k$. Then any $(k,l)$-universal ternary quadratic form $ax^2+by^2+cz^2$ $(a,b,c \in \mathbb{N}, a \leq b \leq c)$ satisfies*

$$a \leq l,\ b \leq 3k+l,\ c \leq (c(k,l)-1)k+l,\ \textit{if } c(k,l) \geq 4,$$

*and*

$$a \leq l,\ b \leq c \leq 2k+l,\ \textit{if } c(k,l)=3.$$

The $(k,l)$-cores given in Table 3 will be used in Section 8. In preparation for the proof of Theorem 1 given in Section 8, we establish the $(k,l)$-universality of certain ternary quadratic forms $ax^2+by^2+cz^2$ $(a,b,c \in \mathbb{N}, a \leq b \leq c)$ in Sections 5, 6, and 7. In Section 5 we treat regular forms, in Section 6 non-regular forms, and in Section 7 we give some conditional results. By the remarks in Section 1 we need only consider primitive forms and proper $(k,l)$-universality.

## 5. Regular Primitive Properly $(k,l)$-universal Ternary Quadratic Forms $ax^2+by^2+cz^2$

Let $G$ denote a genus of classes of primitive positive-definite integral ternary quadratic forms. It is a theorem of Jones (see [15, Theorem, p. 99], [16, Theorem 5, p. 123]) that the positive integers not represented by the form classes of $G$ lie in a certain set of arithmetic progressions. Moreover every positive integer not in one of these arithmetic progressions is represented by at least one form (class) of $G$. If it is the case that one form of $G$ represents all the positive integers not in these progressions, that form is called regular. Thus, if $G$ contains exactly one class, then that class (and hence every form in the class) is regular.

For example, the classes of the ternary quadratic forms $x^2+8y^2+32z^2$ and $4x^2+8y^2+9z^2-4xz$ comprise a genus of discriminant 256, and the integers not represented by either $x^2+8y^2+32z^2$ or $4x^2+8y^2+9z^2-4xz$ are precisely the integers in the arithmetic progressions

$$4n+2,\ 8n+3,\ 8n+5,\ 32n+20,\ 4^m(8n+7),\ m,n \in \mathbb{N}_0; \tag{5.1}$$

see Jones and Pall [17, p. 191] and Dickson [10, Table 5, p. 112]. The form $x^2+8y^2+32z^2$ represents all the integers not in the progressions (5.1), and so is regular. On the other hand, the integer 1 does not belong to any of the arithmetic progressions (5.1) but it is not represented by the form $4x^2+8y^2+9z^2-4xz$ so $4x^2+8y^2+9z^2-4xz$ is not regular.

There are precisely 102 regular primitive forms $ax^2+by^2+cz^2$ $(a,b,c \in \mathbb{N},\ a \leq b \leq c)$. These are listed by Jones and Pall [17, Table 1, p. 190] and the positive integers not represented by them in Dickson [10, Table 5, pp. 112-113]. Knowing the integers not represented by a regular primitive ternary quadratic form, it is then easy to determine values of $k$ and $l$ for which the form is properly $(k,l)$-universal. For example, the integers $8t+1$ $(t \in \mathbb{N}_0)$ do not belong to any of the arithmetic progressions in (5.1). Thus $x^2+8y^2+32z^2$ (being regular) is $(8,1)$-universal. As $x^2+8y^2+32z^2$ does not represent 5 it is not $(4,1)$-universal, and so is properly $(8,1)$-universal. Examining the 102 regular primitive forms $ax^2+by^2+cz^2$ in [10, Table 5, pp. 112-113] for proper $(k,l)$-universality, where $1 \le l < k \le 11$, we obtain Table 4.

Table 4: Regular primitive ternary quadratic forms $ax^2+by^2+cz^2$ which are properly $(k,l)$-universal where $1 \le l < k \le 11$

| $(k,l)$ | $ax^2+by^2+cz^2$ |
|---|---|
| (2,1) | $x^2+y^2+2z^2$, $x^2+2y^2+3z^2$, $x^2+2y^2+4z^2$ |
| (3,1) | $x^2+y^2+3z^2$, $x^2+y^2+6z^2$, $x^2+3y^2+3z^2$, $x^2+3y^2+9z^2$, $x^2+6y^2+9z^2$ |
| (3,2) | $x^2+y^2+3z^2$, $x^2+y^2+6z^2$, $2x^2+3y^2+3z^2$ |
| (4,1) | $x^2+y^2+z^2$, $x^2+y^2+4z^2$, $x^2+y^2+5z^2$, $x^2+y^2+8z^2$, $x^2+y^2+16z^2$, $x^2+2y^2+2z^2$, $x^2+4y^2+4z^2$, $x^2+4y^2+8z^2$, $x^2+4y^2+16z^2$ |
| (4,2) | $x^2+y^2+z^2$, $x^2+y^2+4z^2$, $x^2+y^2+5z^2$, $x^2+2y^2+2z^2$, $x^2+2y^2+6z^2$, $x^2+2y^2+8z^2$ |
| (4,3) | $x^2+2y^2+6z^2$ |
| (5,1) | $x^2+2y^2+5z^2$, $x^2+5y^2+10z^2$ |
| (5,2) | $x^2+2y^2+5z^2$ |
| (5,3) | $x^2+2y^2+5z^2$ |
| (5,4) | $x^2+2y^2+5z^2$, $x^2+5y^2+10z^2$ |
| (6,1) | $x^2+3y^2+4z^2$, $x^2+3y^2+6z^2$, $x^2+3y^2+12z^2$, $x^2+3y^2+18z^2$, $x^2+3y^2+36z^2$, $x^2+4y^2+6z^2$ |
| (6,2) | $x^2+y^2+12z^2$, $2x^2+2y^2+3z^2$, $2x^2+3y^2+18z^2$ |
| (6,3) | $x^2+3y^2+6z^2$ |
| (6,4) | $x^2+y^2+12z^2$, $x^2+6y^2+6z^2$, $x^2+9y^2+12z^2$, $2x^2+2y^2+3z^2$, $3x^2+3y^2+4z^2$ |
| (6,5) | $x^2+3y^2+4z^2$, $x^2+4y^2+6z^2$, $2x^2+3y^2+9z^2$, $2x^2+3y^2+12z^2$ |
| (8,1) | $x^2+2y^2+6z^2$, $x^2+2y^2+8z^2$, $x^2+2y^2+16z^2$, $x^2+2y^2+32z^2$, $x^2+8y^2+8z^2$, $x^2+8y^2+16z^2$, $x^2+8y^2+24z^2$, $x^2+8y^2+32z^2$, $x^2+8y^2+64z^2$, $x^2+16y^2+16z^2$ |
| (8,2) | $x^2+y^2+2z^2$, $x^2+y^2+8z^2$, $x^2+y^2+16z^2$, $x^2+2y^2+4z^2$ |
| (8,3) | $x^2+y^2+z^2$, $x^2+2y^2+2z^2$, $x^2+2y^2+8z^2$, $x^2+2y^2+16z^2$, $x^2+2y^2+32z^2$ |

| | |
|---|---|
| (8,4) | $x^2+y^2+2z^2$, $x^2+y^2+8z^2$, $x^2+2y^2+3z^2$, $x^2+2y^2+4z^2$, $x^2+2y^2+16z^2$, $x^2+4y^2+8z^2$, $x^2+8y^2+16z^2$ |
| (8,6) | $x^2+2y^2+3z^2$ |
| (8,7) | $x^2+y^2+5z^2$ |
| (9,3) | $x^2+y^2+3z^2$, $x^2+3y^2+3z^2$, $x^2+3y^2+9z^2$, $2x^2+3y^2+3z^2$ |
| (9,6) | $x^2+y^2+6z^2$, $x^2+3y^2+3z^2$, $x^2+6y^2+9z^2$, $2x^2+3y^2+3z^2$ |
| (10,2) | $x^2+2y^2+10z^2$, $2x^2+5y^2+10z^2$ |
| (10,4) | $x^2+2y^2+10z^2$ |
| (10,6) | $x^2+2y^2+10z^2$ |
| (10,8) | $x^2+2y^2+10z^2$, $2x^2+5y^2+10z^2$ |

There are 55 pairs $(k,l)$ of positive integers satisfying $1 \leq l < k \leq 11$. In view of Theorem 4 there are 13 pairs $(k,l)$ with $1 \leq l < k \leq 11$ for which there are no ternaries $ax^2+by^2+cz^2$ $(a,b,c \in \mathbb{N},\ a \leq b \leq c)$ which are $(k,l)$-universal, and thus not properly $(k,l)$-universal. Table 4 gives regular ternaries which are properly $(k,l)$-universal for 27 pairs $(k,l)$. This leaves $55-13-27=15$ pairs of $(k,l)$. For the 3 pairs $(k,l)=(7,1)$, $(7,2)$, and $(7,3)$ there are no forms among the 102 regular primitive ternary quadratic forms $ax^2+by^2+cz^2$ $(a,b,c \in \mathbb{N},\ a \leq b \leq c)$ which are $(k,l)$-universal. Finally for the 12 pairs $(k,l)=(8,5)$, $(9,1)$, $(9,2)$, $(9,4)$, $(9,5)$, $(9,7)$, $(9,8)$, $(10,1)$, $(10,3)$, $(10,5)$, $(10,7)$, and $(10,9)$, all the forms among the 102 regular primitive ternary quadratic forms $ax^2+by^2+cz^2$ which are $(k,l)$-universal are not properly $(k,l)$-universal. Thus, for example, the only ternaries among the 102 regular ternaries that are $(10,7)$-universal are $x^2+y^2+2z^2$, $x^2+2y^2+3z^2$, $x^2+2y^2+4z^2$ and $x^2+2y^2+5z^2$, but the first three of these are $(2,1)$-universal and the fourth is $(5,2)$-universal.

## 6. Non-regular Primitive Properly $(k,l)$-universal Ternary Quadratic Forms $ax^2+by^2+cz^2$

A computer search based on Theorem 10 was carried out to determine possible non-regular, primitive, properly $(k,l)$-universal, ternary quadratic forms $ax^2+by^2+cz^2$ $(a,b,c \in \mathbb{N},\ a \leq b \leq c)$ for $k$ and $l$ in the range $1 \leq l < k \leq 11$. Eighty such forms were found. These are listed in Table 5. All of these forms were found to represent all the positive integers that are congruent to $l$ modulo $k$ up to $100,000$. In general it is very difficult to determine the integers represented by a non-regular ternary quadratic form and we are able to supply proofs for only 41 of the 80 forms. These proofs are given in Appendix A.

Table 5: Non-regular primitive ternary quadratic forms $ax^2+by^2+cz^2$ which are properly $(k,l)$-universal where $1 \le l < k \le 11$. In the column headed "status" P indicates that the proper $(k,l)$-universality of the form has been proved while C indicates that it is conjectured.

| $(k,l)$ | $ax^2+by^2+cz^2$ | status | $(k,l)$ | $ax^2+by^2+cz^2$ | status |
|---|---|---|---|---|---|
| $(4,1)$ | $x^2+4y^2+5z^2$ | P | $(8,4)$ | $x^2+5y^2+6z^2$ | P |
| $(6,1)$ | $x^2+3y^2+7z^2$ | C | | $x^2+8y^2+11z^2$ | P |
| | $x^2+3y^2+24z^2$ | P | | $x^2+8y^2+12z^2$ | P |
| | $x^2+3y^2+42z^2$ | C | | $x^2+8y^2+19z^2$ | C |
| | $x^2+3y^2+54z^2$ | C | | $2x^2+3y^2+4z^2$ | P |
| | $x^2+6y^2+12z^2$ | P | | $3x^2+4y^2+8z^2$ | P |
| $(6,3)$ | $x^2+2y^2+9z^2$ | P | $(8,5)$ | $x^2+y^2+13z^2$ | C |
| | $x^2+2y^2+12z^2$ | P | | $x^2+4y^2+13z^2$ | C |
| | $x^2+3y^2+8z^2$ | P | | $2x^2+3y^2+10z^2$ | C |
| | $x^2+3y^2+14z^2$ | P | $(8,6)$ | $x^2+2y^2+11z^2$ | P |
| | $2x^2+3y^2+4z^2$ | P | | $x^2+2y^2+12z^2$ | P |
| | $2x^2+3y^2+7z^2$ | P | | $x^2+5y^2+6z^2$ | P |
| $(6,5)$ | $x^2+y^2+10z^2$ | P | | $2x^2+3y^2+4z^2$ | P |
| | $x^2+4y^2+10z^2$ | P | $(8,7)$ | $x^2+6y^2+8z^2$ | P |
| | $2x^2+3y^2+5z^2$ | C | | $2x^2+2y^2+5z^2$ | P |
| $(7,1)$ | $x^2+2y^2+7z^2$ | C | | $2x^2+5y^2+8z^2$ | P |
| | $x^2+7y^2+14z^2$ | C | $(9,1)$ | $x^2+3y^2+7z^2$ | C |
| $(7,2)$ | $x^2+2y^2+7z^2$ | C | | $x^2+6y^2+15z^2$ | C |
| $(7,3)$ | $x^2+2y^2+7z^2$ | C | $(9,2)$ | $2x^2+3y^2+5z^2$ | C |
| $(8,1)$ | $x^2+y^2+20z^2$ | C | $(9,3)$ | $x^2+3y^2+27z^2$ | P |
| | $x^2+2y^2+22z^2$ | P | | $2x^2+3y^2+27z^2$ | P |
| | $x^2+2y^2+24z^2$ | C | $(9,5)$ | $2x^2+3y^2+5z^2$ | C |
| | $x^2+2y^2+38z^2$ | C | $(9,7)$ | $x^2+3y^2+7z^2$ | C |
| | $x^2+2y^2+64z^2$ | C | | $x^2+6y^2+7z^2$ | C |
| | $x^2+4y^2+20z^2$ | C | $(10,1)$ | $x^2+2y^2+12z^2$ | C |
| | $x^2+6y^2+8z^2$ | C | | $x^2+3y^2+8z^2$ | C |
| $(8,2)$ | $x^2+y^2+10z^2$ | P | $(10,3)$ | $x^2+2y^2+11z^2$ | C |
| | $x^2+y^2+13z^2$ | C | | $x^2+3y^2+5z^2$ | C |
| | $x^2+y^2+17z^2$ | C | | $x^2+3y^2+14z^2$ | C |
| | $x^2+y^2+32z^2$ | P | $(10,5)$ | $x^2+y^2+10z^2$ | P |
| | $x^2+2y^2+9z^2$ | P | | $x^2+4y^2+10z^2$ | P |
| $(8,3)$ | $x^2+2y^2+18z^2$ | P | | $x^2+5y^2+6z^2$ | P |
| | $x^2+2y^2+24z^2$ | P | | $x^2+5y^2+14z^2$ | P |
| | $x^2+2y^2+34z^2$ | C | | $2x^2+5y^2+7z^2$ | P |

| | | | | | |
|---|---|---|---|---|---|
| $(8,4)$ | $x^2+y^2+10z^2$ | P | $(10,6)$ | $x^2+3y^2+5z^2$ | C |
| | $x^2+2y^2+9z^2$ | P | $(10,9)$ | $x^2+2y^2+12z^2$ | C |
| | $x^2+2y^2+11z^2$ | P | | $x^2+2y^2+17z^2$ | C |
| | $x^2+2y^2+12z^2$ | P | | $x^2+2y^2+20z^2$ | C |
| | $x^2+2y^2+19z^2$ | C | | $x^2+3y^2+8z^2$ | C |
| | $x^2+3y^2+8z^2$ | P | | $2x^2+3y^2+4z^2$ | C |

## 7. Conditional Results

In this section we derive the $(k,l)$-universality for $(k,l) = (8,1)$, $(8,4)$, $(8,5)$ of three forms from that of the conjectured $(k,l)$-universality of three other forms.

**Proposition 1.** *Assuming that the form $x^2+y^2+20z^2$ is $(8,1)$-universal then so is the form $x^2+4y^2+20z^2$.*

*Proof.* Let $n \in \mathbb{N}$ be such that $n \equiv 1 \pmod 8$. As $x^2+y^2+20z^2$ is assumed to be $(8,1)$-universal, there exist integers $u$, $v$, and $w$ such that $n = u^2+v^2+20w^2$. Hence $1 \equiv u+v \pmod 2$ so that one of $u$ and $v$ is even. Without loss of generality we may suppose that $v$ is even. Then $v = 2h$ $(h \in \mathbb{Z})$ and $n = u^2+4h^2+20w^2$, proving that $x^2+4y^2+20z^2$ is $(8,1)$-universal under the assumption that $x^2+y^2+20z^2$ is $(8,1)$-universal. $\square$

**Proposition 2.** *Assuming that the form $x^2+2y^2+19z^2$ is $(8,4)$-universal then so is the form $x^2+8y^2+19z^2$.*

*Proof.* Let $n \in \mathbb{N}$ be such that $n \equiv 4 \pmod 8$. As $x^2+2y^2+19z^2$ is assumed to be $(8,4)$-universal, there exist integers $u$, $v$, and $w$ such that $n = u^2+2v^2+19w^2$. As $n$ is even, we have $u \equiv w \pmod 2$, so that $u^2+19w^2 \equiv 0 \pmod 4$, and thus

$$0 \equiv n = u^2+2v^2+19w^2 \equiv 2v^2 \pmod 4;$$

so $v \equiv 0 \pmod 2$. Hence $v = 2h\,(h \in \mathbb{Z})$ and $n = u^2+8h^2+19w^2$ proving that $x^2+8y^2+19z^2$ is $(8,4)$-universal under the assumption that $x^2+2y^2+19z^2$ is $(8,4)$-universal. $\square$

Proposition 2 should be compared with Lemma 2.6 of [26].

**Proposition 3.** *Assuming that the form $x^2+y^2+13z^2$ is $(8,5)$-universal then so is the form $x^2+4y^2+13z^2$.*

*Proof.* Let $n \in \mathbb{N}$ be such that $n \equiv 5 \pmod 8$. As $x^2 + y^2 + 13z^2$ is assumed to be $(8, 5)$-universal, there are integers $u$, $v$, and $w$ such that $n = u^2 + v^2 + 13w^2$. Suppose that both of $u$ and $v$ are odd. Then $u^2 \equiv v^2 \equiv 1 \pmod 8$ and

$$5 \equiv n = u^2 + v^2 + 13w^2 \equiv 2 + 5w^2 \pmod 8,$$

so that $w^2 \equiv 7 \pmod 8$, which is impossible. Hence at least one of $u$ and $v$ is even. Without loss of generality we may suppose that $v$ is even, say $v = 2h$ $(h \in \mathbb{Z})$. Then $n = u^2 + 4h^2 + 13w^2$, proving that the form $x^2 + 4y^2 + 13z^2$ is $(8, 5)$-universal under the assumption that $x^2 + y^2 + 13z^2$ is $(8, 5)$-universal. □

## 8. Proof of Theorem 1

As already mentioned the cases $(k, l) = (2, 1), (4, 2)$ and $(8, 4)$ are not new so we do not need to prove them here. The cases $(k, l) = (7, l)\,(l = 4, 5, 6)$ and $(11, l)$ $(l = 1, 2, \ldots, 10)$ were proved in Theorem 4. The proofs of the remaining parts of Theorem 1 are very similar so we just treat the cases $(k, l) = (8, 1)$ and $(10, 9)$ in complete detail to illustrate the method. The information needed to prove all the remaining cases of Theorem 1 is given in this paper.

*Proof of Theorem 1 for* $(k, l) = (8, 1)$. Suppose that $ax^2+by^2+cz^2$ $(a, b, c \in \mathbb{N},\ a \leq b \leq c)$ is $(8, 1)$-universal. From Table 3 we have $c(8, 1) = 9$. By Theorem 10 we have

$$a \leq l = 1, \quad b \leq 3k + l = 3.8 + 1 = 25, \quad c \leq (c(8, 1) - 1)k + l = 65.$$

Thus all the $(8, 1)$-universal ternaries lie among the 1000 forms

$$x^2 + by^2 + cz^2,\ 1 \leq b \leq 25 < c \leq 65,$$

and the 325 forms

$$x^2 + by^2 + cz^2,\ 1 \leq b \leq c \leq 25.$$

A simple computer program found that of these 1325 forms only the following 30 forms represented all the positive integers $n$ with $n$ congruent to 1 modulo 8 and $1 \leq n \leq 9993$, namely

| | | |
|---|---|---|
| (1) $x^2 + y^2 + z^2$ | (11) $x^2 + 2y^2 + 6z^2$ | (21) $x^2 + 4y^2 + 8z^2$ |
| (2) $x^2 + y^2 + 2z^2$ | (12) $x^2 + 2y^2 + 8z^2$ | (22) $x^2 + 4y^2 + 16z^2$ |
| (3) $x^2 + y^2 + 4z^2$ | (13) $x^2 + 2y^2 + 16z^2$ | (23) $x^2 + 4y^2 + 20z^2$ |
| (4) $x^2 + y^2 + 5z^2$ | (14) $x^2 + 2y^2 + 22z^2$ | (24) $x^2 + 6y^2 + 8z^2$ |

| | | |
|---|---|---|
| (5) $x^2+y^2+8z^2$ | (15) $x^2+2y^2+24z^2$ | (25) $x^2+8y^2+8z^2$ |
| (6) $x^2+y^2+16z^2$ | (16) $x^2+2y^2+32z^2$ | (26) $x^2+8y^2+16z^2$ |
| (7) $x^2+y^2+20z^2$ | (17) $x^2+2y^2+38z^2$ | (27) $x^2+8y^2+24z^2$ |
| (8) $x^2+2y^2+2z^2$ | (18) $x^2+2y^2+64z^2$ | (28) $x^2+8y^2+32z^2$ |
| (9) $x^2+2y^2+3z^2$ | (19) $x^2+4y^2+4z^2$ | (29) $x^2+8y^2+64z^2$ |
| (10) $x^2+2y^2+4z^2$ | (20) $x^2+4y^2+5z^2$ | (30) $x^2+16y^2+16z^2$ |

These thirty ternaries are our candidates for $(8,1)$-universality. From Table 4 we see that forms (2), (9), and (10) are regular and properly $(2,1)$-universal, and thus are $(8,1)$-universal. Again from Table 4, forms (1), (3)$-$(6), (8), (19), (21), and (22) are regular and properly $(4,1)$-universal, and thus are $(8,1)$-universal. Also from Table 4 forms (11) $-$ (13), (16), and (25) $-$ (30) are regular and properly $(8,1)$-universal. Form (14) is non-regular but proved to be $(8,1)$-universal in Proposition 15. Form (20) is non-regular but proved to be properly $(4,1)$-universal in Proposition 4 so is $(8,1)$-universal. We have been unable to prove that the five non-regular forms (7), (15), (17), (18), and (24) are $(8,1)$-universal, but as they all represent every positive integer $n$ congruent to 1 modulo 8 in the range $1 \le n \le 99{,}993$, we conjecture that they are $(8,1)$-universal. Under the assumption that (7) is $(8,1)$-universal, we proved in Proposition 1 that (23) is $(8,1)$-universal. This completes the proof that the $(8,1)$-entry in the second column of Table 1 is complete.

By Table 3 we have $c(8,1)=9$. By Theorem 7 the only ternary quadratic forms $ax^2+by^2+cz^2$ ($a,b,c\in\mathbb{N}$, $a\le b\le c$) representing the nine integers

$$1,9,17,25,33,41,49,57,65$$

satisfy

$$a\le b\le c\le 65.$$

Clearly for $ax^2+by^2+cz^2$ ($a,b,c\in\mathbb{N}$, $a\le b\le c$) to represent 1 we must have $a=1$. A computer program searched the 2145 ternaries $x^2+by^2+cz^2$ ($1\le b\le c\le 65$) and found just ninety-four of them represent all of the nine integers $1,9,17,25,33,41,49,57,65$. Under the assumptions of part (i) we have $A(8,1)=30$ and $B(8,1)=94$ so there are $B(8,1)-A(8,1)=94-30=64$ that represent $1,9,17,25,33,41,49,57,65$, but which are not $(8,1)$-universal. These sixty-four forms fail to represent at least one of the fourteen integers $73,97,105,145,161,177,185,201,209,217,265,377,481,721$. Hence if $ax^2+by^2+cz^2$ ($a,b,c,\in\mathbb{N}$, $a\le b\le c$) represents all the twenty-three integers

$$1,9,17,25,33,41,49,57,65,73,97,105,145,161,177,185,201,209,217,265,$$
$$377,481,721$$

then it must be one of the thirty forms in the $(8,1)$-entry of Theorem 1, and so is $(8,1)$-universal. However, we do not need all of these integers since if $ax^2+by^2+cz^2$

represents $l \in \mathbb{N}$ then it also represents $lm^2$ for any $m \in \mathbb{N}$. Hence we can delete $9, 25$, and 49 from the set to obtain the smaller set of twenty integers

$$1, 17, 33, 41, 57, 65, 73, 97, 105, 145, 161, 177, 185, 201, 209, 217, 265,$$
$$377, 481, 721$$

with the same property. Can we eliminate any more of the integers from this set and still have the same property holding for the reduced set? Removing successively one integer at a time, and testing with a computer program if representation of the integers of the smaller set still ensures that $ax^2 + by^2 + cz^2$ must be one of the $(8, 1)$-universal ternaries, we find on the first pass through the integers that we can eliminate 265, and then successively $177, 105, 97, 73, 41$, and 17. We cannot remove any further integers from the resulting set of thirteen integers

$$1, 33, 57, 65, 145, 161, 185, 201, 209, 217, 377, 481, 721$$

as from Table 6 we see that

$2x^2 + 3y^2 + 4z^2$ represents all of these except 1
$x^2 + y^2 + 3z^2$ represents all of these except 33,
$x^2 + y^2 + 29z^2$ represents all of these except 57,
$x^2 + 2y^2 + 9z^2$ represents all of these except 65,
$x^2 + 2y^2 + 10z^2$ represents all of these except 145,
$x^2 + y^2 + 23z^2$ represents all of these except 161,
$x^2 + 6y^2 + 32z^2$ represents all of these except 185,
$x^2 + 10y^2 + 14z^2$ represents all of these except 201,
$x^2 + 3y^2 + 14z^2$ represents all of these except 209,
$x^2 + y^2 + 31z^2$ represents all of these except 217,
$x^2 + 2y^2 + 17z^2$ represents all of these except 377,
$x^2 + 2y^2 + 11z^2$ represents all of these except 481,
$x^2 + y^2 + 13z^2$ represents all of these except 721.

This proves the minimality of the set $\{1, 33, 57, 65, 145, 161, 185, 201, 209,$ $217, 377, 481, 721\}$. □

*Proof of Theorem 1 for $(k, l) = (10, 9)$* . Suppose that $ax^2 + by^2 + cz^2$ $(a, b, c, \in \mathbb{N},$ $a \leq b \leq c)$ is $(10, 9)$-universal. From Table 3 we have $c(10, 9) = 3$. By Theorem 10 the $(10, 9)$-universal ternaries $ax^2 + by^2 + cz^2$ $(a, b, c, \in \mathbb{N},\ a \leq b \leq c)$ lie among the 2955 forms

$$ax^2 + by^2 + cz^2,\ a = 1, 2, \ldots, 9,\ a \leq b \leq c \leq 29.$$

A simple computer program found that of these 2955 forms only the following ten forms

$$\begin{array}{lll}
(1)\, x^2+y^2+2z^2 & (2)\, x^2+2y^2+3z^2 & (3)\, x^2+2y^2+4z^2 \\
(4)\, x^2+2y^2+5z^2 & (5)\, x^2+2y^2+12z^2 & (6)\, x^2+2y^2+17z^2 \\
(7)\, x^2+2y^2+20z^2 & (8)\, x^2+3y^2+8z^2 & (9)\, x^2+5y^2+10z^2 \\
(10)\, 2x^2+3y^2+4z^2 & &
\end{array}$$

represented all the positive integers $n$ congruent to 9 modulo 10 with $9 \leq n \leq 9999$. By our assumption, forms $(5), (6), (7), (8)$, and $(10)$ are $(10, 9)$-universal. By Theorem 1 for $(k, l) = (2, 1)$ the forms $(1), (2)$ and $(3)$ are $(2, 1)$-universal, and so are $(10, 9)$-universal. By Theorem 1 for $(k, l) = (5, 4)$ the forms $(4)$ and $(9)$ are $(5, 4)$-universal, and so are $(10, 9)$-universal. This completes the proof that the $(10, 9)$-entry in the second column of Table 1 is complete.

As $c(10, 9) = 3$, by Theorem 7 the only ternary quadratic forms $ax^2 + by^2 + cz^2$ $(a, b, c, \in \mathbb{N},\ a \leq b \leq c)$ representing the three integers $9, 19, 29$ satisfy $a \leq b \leq c \leq 29$. A computer program found in this range 111 ternaries $ax^2 + by^2 + cz^2$ which represent these three integers. Under the assumption that forms $(5), (6), (7), (8)$, and $(10)$ are $(10, 9)$-universal, we have $A(10, 9) = 10$ and $B(10, 9) = 111$. Thus there are $B(10, 9) - A(10, 9) = 111 - 10 = 101$ forms that represent $9, 19$, and $29$ but which are not $(10, 9)$-universal. The least integers of the infinite arithmetic progression $\{10j + 9 \,|\, j \in \mathbb{N}_0\}$ not represented by these 101 forms are the eleven integers $39, 49, 59, 69, 79, 89, 119, 129, 149, 179, 209$. Hence if $ax^2+by^2+cz^2$ $(a, b, c, \in \mathbb{N},\ a \leq b \leq c)$ represents all the fourteen integers

$$9, 19, 29, 39, 49, 59, 69, 79, 89, 119, 129, 149, 179, 209,$$

then it must be one of the forms $(1) - (10)$, and so is $(10, 9)$-universal. Removing successively one integer at a time, and testing with a computer program if representation of the integers of the smaller set still ensures that $ax^2 + by^2 + cz^2$ must be one of the $(10, 9)$-universal ternaries, we find that we can first eliminate 59, then 49, and finally 9. We cannot remove any further integers from the resulting set of eleven integers

$$19, 29, 39, 69, 79, 89, 119, 129, 149, 179, 209$$

as by Table 6

$2x^2 + 4y^2 + 5z^2$ represents all of these except 19,
$x^2 + 2y^2 + 9z^2$ represents all of these except 29,
$x^2 + 2y^2 + 11z^2$ represents all of these except 39,
$x^2 + y^2 + 3z^2$ represents all of these except 69,
$x^2 + y^2 + 10z^2$ represents all of these except 79,

$2x^2 + 3y^2 + 7z^2$ represents all of these except 89,
$x^2 + y^2 + 14z^2$ represents all of these except 119,
$x^2 + 7y^2 + 10z^2$ represents all of these except 129,
$x^2 + 10y^2 + 20z^2$ represents all of these except 149,
$x^2 + 3y^2 + 5z^2$ represents all of these except 179,
$x^2 + 5y^2 + 18z^2$ represents all of these except 209.

This proves the minimality of the set $\{19, 29, 39, 69, 79, 89, 119, 129, 149, 179, 209\}$. □

The proofs of the remaining cases can be carried out in a similar manner. It only remains to give the ternary quadratic forms necessary to establish the minimality of the sets $H$ given in Theorem 1. These are given in Table 6.

Table 6: Forms $ax^2 + by^2 + cz^2$ $(a, b, c \in \mathbb{N})$ representing all but one member of the set $H$ given in Table 1

| $(k, l)$ | forms $(a, b, c) = ax^2 + by^2 + cz^2$ representing all members of $H$ except for $h$ is indicated as $(a, b, c)$ $(h)$ |
|---|---|
| $(2, 1)$ | $(2, 3, 4)$ (1) $(1, 1, 5)$ (3) $(1, 2, 6)$ (5) $(1, 1, 1)$ (15) |
| $(3, 1)$ | $(2, 2, 3)$ (1) $(1, 1, 1)$ (7) $(1, 2, 3)$ (10) $(1, 2, 6)$ (13) $(1, 1, 5)$ (19) $(1, 3, 7)$ (22) $(1, 4, 6)$ (34) $(1, 1, 2)$ (46) $(1, 2, 7)$ (70) |
| $(3, 2)$ | $(1, 4, 6)$ (2) $(1, 2, 6)$ (5) $(1, 1, 2)$ (14) $(2, 2, 3)$ (17) $(1, 1, 1)$ (23) $(1, 2, 3)$ (26) $(1, 1, 5)$ (35) $(1, 1, 10)$ (134) |
| $(4, 1)$ | $(2, 3, 4)$ (1) $(1, 2, 10)$ (5) $(1, 1, 32)$ (21) $(1, 1, 8)$ (33) $(1, 5, 9)$ (53) $(1, 2, 5)$ (65) $(1, 1, 20)$ (77) $(1, 1, 17)$ (20) $(1, 1, 13)$ (72) |
| $(4, 2)$ | $(1, 5, 5)$ (2) $(1, 1, 3)$ (6) $(1, 2, 3)$ (10) $(1, 2, 7)$ (14) $(1, 1, 6)$ (30) |
| $(4, 3)$ | $(1, 1, 6)$ (3) $(1, 2, 11)$ (7) $(1, 3, 14)$ (11) $(1, 1, 3)$ (15) $(1, 3, 8)$ (23) $(1, 2, 7)$ (35) |
| $(5, 1)$ | $(5, 6, 10)$ (1) $(1, 1, 3)$ (6) $(1, 3, 5)$ (11) $(1, 2, 6)$ (21) $(1, 2, 3)$ (26) $(1, 1, 2)$ (46) $(1, 5, 6)$ (91) $(1, 1, 1)$ (111) |
| $(5, 2)$ | $(1, 3, 5)$ (2) $(1, 1, 1)$ (7) $(2, 3, 7)$ (17) $(1, 1, 5)$ (27) $(1, 1, 3)$ (42) $(1, 2, 6)$ (52) $(1, 1, 2)$ (62) |
| $(5, 3)$ | $(1, 1, 5)$ (3) $(1, 2, 15)$ (13) $(1, 1, 1)$ (23) $(1, 3, 5)$ (38) $(1, 2, 3)$ (58) $(1, 1, 2)$ (78) $(1, 2, 7)$ (133) |
| $(5, 4)$ | $(1, 1, 2)$ (14) $(1, 2, 6)$ (29) $(1, 5, 15)$ (34) $(1, 1, 1)$ (39) $(1, 1, 5)$ (44) $(1, 1, 3)$ (54) $(1, 2, 3)$ (74) |

| $(k,l)$ | forms $(a,b,c) = ax^2 + by^2 + cz^2$ representing all members of $H$ except for $h$ is indicated as $(a,b,c)$ $(h)$ |
|---|---|
| $(6,1)$ | $(2,3,5)\,(1)$ $(1,1,18)\,(7)$ $(1,6,15)\,(13)$ $(1,6,7)\,(19)$ $(1,7,12)\,(31)$ $(1,3,8)\,(55)$ $(1,3,13)\,(85)$ $(1,2,7)\,(91)$ $(1,3,16)\,(115)$ $(1,3,19)\,(145)$ $(1,3,28)\,(235)$ $(1,3,48)\,(697)$ |
| $(6,2)$ | $(4,4,6)\,(2)$ $(1,1,2)\,(14)$ $(1,2,6)\,(20)$ $(1,2,3)\,(26)$ $(2,6,14)\,(44)$ $(2,3,8)\,(68)$ $(1,1,1)\,(92)$ $(1,1,10)\,(134)$ $(1,2,5)\,(140)$ |
| $(6,3)$ | $(1,1,5)\,(3)$ $(1,2,5)\,(15)$ $(1,2,7)\,(21)$ $(1,3,11)\,(33)$ $(1,2,13)\,(39)$ $(1,3,3)\,(45)$ $(2,3,3)\,(63)$ |
| $(6,4)$ | $(2,3,8)\,(4)$ $(1,2,3)\,(10)$ $(1,1,11)\,(22)$ $(1,1,1)\,(28)$ $(1,4,6)\,(34)$ $(1,1,2)\,(46)$ $(1,2,6)\,(52)$ $(1,2,7)\,(70)$ $(1,1,5)\,(268)$ |
| $(6,5)$ | $(1,2,6)\,(5)$ $(1,1,15)\,(11)$ $(2,3,11)\,(17)$ $(1,1,1)\,(23)$ $(1,1,7)\,(35)$ $(2,3,15)\,(41)$ |
| $(7,1)$ | $(1,1,1)\,(15)$ $(1,1,7)\,(22)$ $(1,2,6)\,(29)$ $(1,6,29)\,(43)$ $(1,1,6)\,(57)$ $(1,1,2)\,(78)$ $(1,2,3)\,(106)$ $(1,1,14)\,(267)$ |
| $(7,2)$ | $(1,7,14)\,(2)$ $(1,1,26)\,(23)$ $(1,1,2)\,(30)$ $(1,2,6)\,(37)$ $(1,1,14)\,(44)$ $(1,1,3)\,(51)$ $(1,2,3)\,(58)$ $(1,2,5)\,(65)$ $(1,1,10)\,(79)$ |
| $(7,3)$ | $(1,1,5)\,(3)$ $(1,2,3)\,(10)$ $(1,1,3)\,(24)$ $(1,1,1)\,(31)$ $(1,3,3)\,(38)$ $(1,2,6)\,(52)$ $(1,1,2)\,(94)$ |
| $(8,1)$ | $(2,3,4)\,(1)$ $(1,1,3)\,(33)$ $(1,1,29)\,(57)$ $(1,2,9)\,(65)$ $(1,2,10)\,(145)$ $(1,1,23)\,(161)$ $(1,6,32)\,(185)$ $(1,10,14)\,(201)$ $(1,3,14)\,(209)$ $(1,1,31)\,(217)$ $(1,2,17)\,(377)$ $(1,2,11)\,(481)$ $(1,1,13)\,(721)$ |
| $(8,2)$ | $(1,2,5)\,(10)$ $(1,4,5)\,(26)$ $(1,1,53)\,(42)$ $(1,4,10)\,(66)$ $(1,8,9)\,(74)$ $(2,10,18)\,(106)$ $(1,1,37)\,(114)$ $(1,6,11)\,(122)$ $(1,2,10)\,(130)$ $(1,1,25)\,(154)$ $(1,1,41)\,(258)$ $(1,1,29)\,(282)$ $(1,1,34)\,(402)$ $(1,1,26)\,(1442)$ |
| $(8,3)$ | $(1,1,6)\,(3)$ $(2,3,22)\,(19)$ $(1,2,9)\,(35)$ $(2,3,7)\,(43)$ $(1,2,11)\,(91)$ $(1,2,23)\,(115)$ $(1,3,8)\,(187)$ $(1,1,3)\,(195)$ $(1,2,19)\,(395)$ |
| $(8,4)$ | $(2,3,7)\,(4)$ $(1,1,6)\,(12)$ $(1,2,10)\,(20)$ $(1,10,11)\,(28)$ $(1,2,17)\,(52)$ $(1,1,3)\,(60)$ $(1,1,7)\,(140)$ $(1,1,11)\,(308)$ |
| $(8,5)$ | $(1,2,10)\,(5)$ $(1,5,16)\,(13)$ $(1,1,6)\,(21)$ $(2,2,5)\,(29)$ $(2,3,13)\,(37)$ $(1,1,3)\,(69)$ $(1,1,11)\,(77)$ $(1,2,5)\,(85)$ $(1,5,9)\,(133)$ $(1,1,17)\,(581)$ |
| $(8,6)$ | $(2,5,7)\,(6)$ $(1,2,19)\,(14)$ $(1,5,9)\,(22)$ $(1,1,6)\,(30)$ $(1,4,5)\,(38)$ $(1,6,14)\,(46)$ $(1,2,14)\,(70)$ $(1,2,13)\,(78)$ $(1,2,5)\,(110)$ |

| $(k,l)$ | forms $(a,b,c) = ax^2 + by^2 + cz^2$ representing all members of $H$ except for $h$ is indicated as $(a,b,c)$ $(h)$ |
|---|---|
| $(8,7)$ | $(1,2,12)\,(7)$ $(1,1,3)\,(15)$ $(1,6,38)\,(23)$ $(2,5,7)\,(31)$ $(1,1,6)\,(39)$ $(1,3,11)\,(55)$ $(1,6,23)\,(71)$ $(1,2,7)\,(119)$ $(1,6,14)\,(167)$ |
| $(9,1)$ | $(2,2,3)\,(1)$ $(1,2,3)\,(10)$ $(1,1,38)\,(19)$ $(1,2,6)\,(37)$ $(1,1,2)\,(46)$ $(1,1,1)\,(55)$ $(1,4,6)\,(82)$ $(1,1,14)\,(91)$ $(1,1,10)\,(118)$ |
| $(9,2)$ | $(1,3,7)\,(2)$ $(1,1,5)\,(11)$ $(1,2,6)\,(29)$ $(1,1,10)\,(38)$ $(1,1,2)\,(56)$ $(1,2,5)\,(65)$ $(1,2,3)\,(74)$ $(1,1,7)\,(119)$ $(2,3,11)\,(668)$ |
| $(9,3)$ | $(1,1,5)\,(3)$ $(1,2,6)\,(21)$ $(1,2,15)\,(30)$ $(1,1,1)\,(39)$ $(2,2,3)\,(57)$ $(1,3,11)\,(66)$ $(1,3,14)\,(102)$ $(1,2,3)\,(138)$ $(1,3,29)\,(174)$ $(1,2,5)\,(210)$ |
| $(9,4)$ | $(2,3,5)\,(4)$ $(1,2,31)\,(13)$ $(1,1,11)\,(22)$ $(1,2,5)\,(40)$ $(2,2,3)\,(49)$ $(1,3,13)\,(58)$ $(1,1,22)\,(76)$ $(1,2,13)\,(130)$ $(1,6,6)\,(139)$ $(1,3,21)\,(166)$ $(1,1,1)\,(175)$ $(1,1,2)\,(238)$ $(1,3,19)\,(445)$ |
| $(9,5)$ | $(1,2,6)\,(5)$ $(1,1,2)\,(14)$ $(1,1,1)\,(23)$ $(2,2,3)\,(41)$ $(1,4,6)\,(50)$ $(2,3,21)\,(68)$ $(1,1,10)\,(86)$ $(1,2,3)\,(122)$ $(1,1,5)\,(140)$ |
| $(9,6)$ | $(1,1,10)\,(6)$ $(1,1,1)\,(15)$ $(1,2,3)\,(42)$ $(1,1,5)\,(51)$ $(1,2,6)\,(69)$ $(1,1,2)\,(78)$ $(1,2,15)\,(105)$ $(1,6,47)\,(114)$ $(1,6,15)\,(402)$ |
| $(9,7)$ | $(1,1,1)\,(7)$ $(1,3,39)\,(34)$ $(2,3,5)\,(43)$ $(1,6,33)\,(52)$ $(1,6,10)\,(61)$ $(1,3,15)\,(70)$ $(1,2,3)\,(106)$ $(1,1,2)\,(142)$ $(1,3,13)\,(187)$ $(1,2,5)\,(385)$ |
| $(9,8)$ | $(1,10,13)\,(8)$ $(2,2,3)\,(17)$ $(1,2,3)\,(26)$ $(1,1,7)\,(35)$ $(1,1,30)\,(44)$ $(1,2,6)\,(53)$ $(1,1,2)\,(62)$ $(1,1,1)\,(71)$ $(1,4,6)\,(98)$ $(1,1,10)\,(134)$ |
| $(10,1)$ | $(2,3,4)\,(1)$ $(1,3,14)\,(11)$ $(1,2,6)\,(21)$ $(1,7,10)\,(31)$ $(1,3,11)\,(41)$ $(1,1,3)\,(51)$ $(1,2,9)\,(71)$ $(1,2,13)\,(91)$ $(1,2,20)\,(111)$ $(1,2,15)\,(431)$ |
| $(10,2)$ | $(1,5,6)\,(2)$ $(1,1,5)\,(12)$ $(1,1,10)\,(22)$, $(1,2,3)\,(42)$ $(1,2,6)\,(52)$ $(1,1,2)\,(62)$ $(1,1,1)\,(92)$ $(2,3,10)\,(132)$ |
| $(10,3)$ | $(1,1,5)\,(3)$ $(1,2,15)\,(13)$ $(1,2,9)\,(23)$ $(1,1,3)\,(33)$ $(1,3,20)\,(53)$ $(3,5,10)\,(73)$ $(1,2,12)\,(103)$ $(1,2,7)\,(133)$ |
| $(10,4)$ | $(1,1,2)\,(14)$ $(1,3,5)\,(34)$ $(1,1,5)\,(44)$ $(1,1,3)\,(54)$ $(1,1,6)\,(84)$ $(1,2,13)\,(104)$ $(1,1,1)\,(124)$ $(1,5,7)\,(314)$ |
| $(10,5)$ | $(1,2,11)\,(5)$ $(1,1,3)\,(15)$ $(2,5,13)\,(25)$ $(1,1,14)\,(35)$ $(1,5,11)\,(55)$ $(2,3,13)\,(65)$ $(1,1,6)\,(75)$ $(2,3,7)\,(85)$ |
| $(10,6)$ | $(1,1,3)\,(6)$ $(1,2,30)\,(26)$ $(1,1,2)\,(46)$ $(1,1,5)\,(76)$ $(1,2,6)\,(116)$ $(1,5,6)\,(136)$ $(1,1,1)\,(156)$ $(1,2,7)\,(266)$ |

| $(k,l)$ | forms $(a,b,c) = ax^2 + by^2 + cz^2$ representing all members of $H$ except for $h$ is indicated as $(a,b,c)$ $(h)$ |
|---|---|
| $(10,7)$ | $(1,2,9)\,(7)$ $(2,3,4)\,(17)$ $(1,1,5)\,(27)$ $(1,3,14)\,(47)$ $(1,2,6)\,(77)$ $(1,1,3)\,(87)$ $(2,4,5)\,(97)$ $(1,1,7)\,(287)$ |
| $(10,8)$ | $(1,5,6)\,(18)$ $(1,1,1)\,(28)$ $(1,3,5)\,(38)$ $(2,5,14)\,(68)$ $(1,1,2)\,(78)$ $(1,3,11)\,(88)$ $(1,1,5)\,(108)$ $(1,2,6)\,(148)$ $(1,1,7)\,(168)$ |
| $(10,9)$ | $(2,4,5)\,(19)$ $(1,2,9)\,(29)$ $(1,2,11)\,(39)$ $(1,1,3)\,(69)$ $(1,1,10)\,(79)$ $(2,3,7)\,(89)$ $(1,1,14)\,(119)$ $(1,7,10)\,(129)$ $(1,10,20)\,(149)$ $(1,3,5)\,(179)$ $(1,5,18)\,(209)$ |

## 9. Concluding Remarks

The elementary method described in this paper for determining the candidate $(k,l)$-universal ternary forms $ax^2 + by^2 + cz^2$ $(a,b,c \in \mathbb{N}, a \leq b \leq c)$ is perfectly general and, for a given $(k,l) \in \mathbb{N}^2$, Theorem 10 gives an upper bound for the number of such candidate forms.

A preliminary calculation in the case $(k,l) = (12,1)$ suggests that there are one hundred candidate forms $ax^2 + by^2 + cz^2$ $(a,b,c \in \mathbb{N},\ a \leq b \leq c)$, which are $(12,1)$-universal. Of these forms thirty-five are regular and sixty-five non-regular.

There is another direction of this research that remains to be pursued, namely that of establishing the least number of integers in a minimal set. For example in Theorem 1, we saw that

$$\{1, 33, 57, 65, 145, 161, 185, 201, 209, 217, 377, 481, 721\}$$

is a minimal set containing 13 integers for the $(8,1)$-universal ternaries. However

$$\{17, 33, 41, 57, 65, 105, 161, 209, 217, 377, 481, 721\}$$

is also a minimal set for the $(8,1)$-universal ternaries having 12 members. Does there exist such a minimal set with less than 12 members?

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## References

[1] A. Alaca, Ş. Alaca and K. S. Williams, Arithmetic progressions and binary quadratic forms, *Amer. Math. Monthly* **115.3** (2008), 252–254.

[2] M. Bhargava, On the Conway-Schneeberger fifteen theorem, *Contemp. Math.* **272**, *American Mathematical Society, Providence, RI,* (2000), 27–37.

[3] M. Bhargava and J. Hanke, Universal quadratic forms and the 290-Theorem, *http://math.stanford.edu/∼ vakil/files/290-Theorem-preprint.pdf*

[4] H. Brandt, O. Intrau and A. Schiemann, Brandt-Intrau-Schiemann table of ternary quadratic forms, *http://www.math.rwth-aachen.de/∼Gabriele.Nebe/LATTICES/Brandt_1.html*, *http://www.math.rwth-aachen.de/∼Gabriele.Nebe/LATTICES/Brandt_2.html.*

[5] J. H. Conway, Universal quadratic forms and the fifteen theorem, *Contemp. Math.* **272**, *American Mathematical Society, Providence, RI,* (2000), 23–26.

[6] J. H. Conway, *The Sensual (Quadratic) Form,* The Carus Mathematical Monographs. No **26**, Mathematical Association of America, Washington, DC, 2005.

[7] P. K. Dey and R. Thangadurai, The length of an arithmetic progression represented by a binary quadratic form, *Amer. Math. Monthly* **121.10** (2014), 932–936.

[8] L. E. Dickson, Ternary quadratic forms and congruences, *Ann. of Math.* **28** No. 1/4 (1926-1927), 333–341. [Collected Mathematical Papers, Vol. I, Chelsea, New York, 1975, 641–649.]

[9] L. E. Dickson, Integers represented by positive ternary quadratic forms, *Bull. Amer. Math. Soc.* **33.1** (1927), 63–70. [Collected Mathematical Papers, Vol. V, Chelsea, New York, 1975, 255-262.]

[10] L. E. Dickson, *Modern Elementary Theory of Numbers,* University of Chicago Press, Chicago, IL, 1939.

[11] A. J. Hahn, Quadratic forms over $\mathbb{Z}$ from Diophantus to the 290 theorem, *Adv. Appl. Clifford Algebr.* **18** (2008), 665-676.

[12] J. S. Hsia, Y. Kitaoka and M. Kneser, Representations of positive definite quadratic forms, *J. Reine Angew. Math.* **301** (1978), 132-141.

[13] W. C. Jagy, Integral positive ternary quadratic forms, *see Jagy Encyclopedia at https://wayback.archive-it.org/1688/20101130163702/ http://zakuski.math.utsa.edu/∼ kap/forms.html#*

[14] W. C. Jagy, *Personal communication*, April 8, 2015.

[15] B. W. Jones, A new definition of genus for ternary quadratic forms, *Trans. Amer. Math. Soc.* **33.1** (1931), 92-110.

[16] B. W. Jones, The regularity of a genus of positive ternary quadratic forms, *Trans. Amer. Math. Soc.* **33.1** (1931), 111-124.

[17] B. W. Jones and G. Pall, Regular and semi-regular positive ternary quadratic forms, *Acta Math.* **70** (1939), 165-191.

[18] I. Kaplansky, Ternary positive quadratic forms that represent all odd positive integers, *Acta. Arith.* **70.3** (1995), 209–214.

[19] V.-A. Lebesque, Démonstration de ce théorème: tout nombre impair est la somme de quatre carrés dont deux sont égaux, *J. Pures Appl. Math.* **2** (1857), 149–152.

[20] Y. S. Moon, Universal quadratic forms and the 15-theorem and 290-theorem, *preprint,* 2008 :http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1. 1.588.7595&rep=rep1&type=pdf

[21] L. J. Mordell, *Diophantine Equations*, Academic Press, London and New York, 1969.

[22] B.-K. Oh, Representations of arithmetic progressions by positive definite quadratic forms, *Int. J. Number Theory* **7.6** (2011), 1603-1614.

[23] L. Panaitopol, On the representation of natural numbers as sums of squares, *Amer. Math. Monthly* **112.2** (2005), 168–171.

[24] J. Rouse, Quadratic forms representing all odd positive integers, *Amer. J. Math.* **136.6** (2014), 1693–1745.

[25] Z.-W. Sun, On universal sums of polygonal numbers, *Sci. China Math.* **58** (2015), doi:10.1007/s11425-015-4994-4.

[26] K. S. Williams, Ternary quadratic forms $ax^2 + by^2 + cz^2$ representing all positive integers $8k + 4$, *Acta Arith.* **166.4** (2014), 391–396.

[27] K. S. Williams, A "four integers" theorem and a " five integers" theorem, *Amer. Math. Monthly* **122.6** (2015), 528-536.

## Appendix A

We give the proofs of the proper $(k, l)$-universality for all those forms marked P in Table 5. The ideas contained in [13], [14] were very helpful in proving these results. For the forms marked C their proper $(k, l)$-universality has been verified for all $n \equiv l \pmod{k}$ with $n \leq 10^5$.

**Proposition 4.** *$x^2 + 4y^2 + 5z^2$ is properly $(4, 1)$-universal.*

*Proof.* From Table 4 we see that $x^2 + y^2 + 5z^2$ is $(4, 1)$-universal. Let $n \in \mathbb{N}$ satisfy $n \equiv 1 \pmod 4$ so there exist integers $u, v$, and $w$ such that $n = u^2 + v^2 + 5w^2$. If $u \equiv v \equiv 1 \pmod 2$ then $w^2 \equiv 5w^2 \equiv n - u^2 - v^2 \equiv 1 - 1 - 1 \equiv 3 \pmod 4$, which is impossible. Hence $u$ or $v \equiv 0 \pmod 2$. Interchanging $u$ and $v$ if necessary, we may suppose that $v \equiv 0 \pmod 2$, so that $v = 2v_1$, $v_1 \in \mathbb{Z}$. Thus $n = u^2 + 4v_1^2 + 5w^2$ proving that $x^2 + 4y^2 + 5z^2$ is $(4, 1)$-universal. It is not $(2, 1)$-universal as it does not represent 3. Thus $x^2 + 4y^2 + 5z^2$ is properly $(4, 1)$-universal. □

**Proposition 5.** *$x^2 + 3y^2 + 24z^2$ is properly $(6, 1)$-universal.*

*Proof.* By [25, Theorem 1.7(i)] $x^2 + 3y^2 + 24z^2$ is $(6, 1)$-universal. It is not $(2, 1)$-universal as it does not represent 5, and it is not $(3, 1)$-universal as it does not represent 10. Hence $x^2 + 3y^2 + 24z^2$ is properly $(6, 1)$-universal. □

**Proposition 6.** *$x^2 + 6y^2 + 12z^2$ is properly $(6, 1)$-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 1 \pmod 6$. If $n$ is a perfect square, say $n = m^2$ $(m \in \mathbb{Z})$ then $x^2 + 6y^2 + 12z^2$ represents $n$ as $m^2 + 6 \cdot 0^2 + 12 \cdot 0^2$. Now suppose that $n$ is not a square. By [25, Lemma 3.3] there are integers $a, b$, and $c$ with $a \equiv 1 \pmod 2$ such that $n = a^2 + 3b^2 + 6c^2$. Clearly, $b \equiv 0 \pmod 2$. Define integers $u$, $v$, and $w$ by $u = a$, $v = c$, and $w = b/2$. Then $n = u^2 + 6v^2 + 12w^2$ so that $x^2 + 6y^2 + 12z^2$ represents $n$. Hence $x^2 + 6y^2 + 12z^2$ is $(6, 1)$-universal. Clearly $x^2 + 6y^2 + 12z^2$ does not represent 3, so that $x^2 + 6y^2 + 12z^2$ is not $(2, 1)$-universal. Also $x^2 + 6y^2 + 12z^2$ does not represent 46, so that $x^2 + 6y^2 + 12z^2$ is not $(3, 1)$-universal. Hence $x^2 + 6y^2 + 12z^2$ is properly $(6, 1)$-universal. □

**Proposition 7.** *$x^2 + 2y^2 + 9z^2$ is properly $(6, 3)$-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 3 \pmod 6$. As $n$ is odd, we see from Table 4 that $x^2 + y^2 + 2z^2$ represents $n$. Hence there exist integers $u$, $v$, and $w$ such that $n = u^2 + v^2 + 2w^2$. As $n \equiv 0 \pmod 3$ we have $u^2 + v^2 + 2w^2 \equiv 0 \pmod 3$. Suppose $u \not\equiv 0 \pmod 3$ and $v \not\equiv 0 \pmod 3$. Then $u^2 \equiv v^2 \equiv 1 \pmod 3$ so $w^2 \equiv u^2 + v^2 \equiv 2 \pmod 3$, which is impossible. Hence either $u$ or $v$ is divisible by 3. Interchanging $u$ and $v$ if necessary, we may suppose that $v \equiv 0 \pmod 3$, say $v = 3v_1$ ($v_1 \in \mathbb{Z}$). Then $n = u^2 + 2w^2 + 9v_1^2$. Thus $x^2 + 2y^2 + 9z^2$ is $(6, 3)$-universal. As $x^2 + 2y^2 + 9z^2$ does not represent 30 it is not $(3, 3)$-universal. Therefore $x^2 + 2y^2 + 9z^2$ is properly $(6, 3)$-universal. □

**Proposition 8.** *$x^2 + 2y^2 + 12z^2$ is properly $(6, 3)$-universal.*

*Proof.* By Table 4 the form $x^2 + 2y^2 + 3z^2$ represents every odd positive integer $n$, and so represents every $n \equiv 3 \pmod 6$. Hence $n = u^2 + 2v^2 + 3w^2$ for integers $u$, $v$, and $w$. If $w$ is even, say $w = 2w_1$, then $n = u^2 + 2v^2 + 12w_1^2$, and $n$ is represented by the form $n = x^2 + 2y^2 + 12z^2$. If $w$ is odd then $u$ is even, say $u = 2u_1$, so $n = 4u_1^2 + 2v^2 + 3w^2$. Hence $0 \equiv u_1^2 - v^2 \pmod 3$ so $v \equiv u_1 \pmod 3$ or $v \equiv -u_1 \pmod 3$. Replacing $u_1$ by $-u_1$ if necessary, we may suppose that $v \equiv u_1 \pmod 3$. Then

$$X = \frac{2v + 3w + 4u_1}{3},\ Y = \frac{v - 3w + 2u_1}{3},\ Z = \frac{v - u_1}{3}$$

are integers and $n = X^2 + 2Y^2 + 12Z^2$ so that $n$ is represented by the form $x^2 + 2y^2 + 12z^2$. Thus $x^2 + 2y^2 + 12z^2$ is $(6, 3)$-universal. As $x^2 + 2y^2 + 12z^2$ does not represent 5 it is not $(2, 1)$-universal, and as it does not represent 42 it is not $(3, 3)$-universal. Thus $x^2 + 2y^2 + 12z^2$ is properly $(6, 3)$-universal. □

**Proposition 9.** *$x^2 + 3y^2 + 8z^2$ is properly $(6, 3)$-universal.*

*Proof.* By Table 4 the form $x^2 + 2y^2 + 3z^2$ represents every odd positive integer $n$, and so represents every $n \equiv 3 \pmod 6$. Thus there are integers $u$, $v$, and $w$ such that $n = u^2 + 2v^2 + 3w^2$. If $v \equiv 0 \pmod 2$ then $v = 2v_1$ for some $v_1 \in \mathbb{Z}$, and so $n = u^2 + 3w^2 + 8v_1^2$. If $v \equiv 1 \pmod 2$, as $(u - v)(u + v) \equiv u^2 - v^2 \equiv n \equiv 0 \pmod 3$, we may replace $v$ by $-v$ if necessary so that $u + v \equiv 0 \pmod 3$. Define $X$, $Y$, and $Z \in \mathbb{Z}$ by

$$X = \frac{2u + 2v - 3w}{3},\ Y = \frac{u + v + 3w}{3},\ Z = \frac{u - 2v}{3}.$$

Then

$$\begin{aligned} X^2 + 2Y^2 + 3Z^2 &= \frac{1}{9}((2u + 2v - 3w)^2 + 2(u + v + 3w)^2 + 3(u - 2v)^2) \\ &= u^2 + 2v^2 + 3w^2 = n. \end{aligned}$$

Now $u + w \equiv u^2 + 2v^2 + 3w^2 = n \equiv 1 \pmod 2$ so $3Y = u + v + 3w \equiv (u + w) + v \equiv 1 + 1 \equiv 0 \pmod 2$, and thus $Y \equiv 0 \pmod 2$. Hence

$$n = X^2 + 3Z^2 + 8(Y/2)^2.$$

This proves that $x^2+3y^2+8z^2$ is $(6,3)$-universal. Now $x^2+3y^2+8z^2$ does not represent 5, so it is not $(2,1)$-universal, and it does not represent 6 so it is not $(3,3)$-universal. Hence $x^2+3y^2+8z^2$ is properly $(6,3)$-universal. $\square$

**Proposition 10.** *$x^2+3y^2+14z^2$ is properly $(6,3)$-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 3 \pmod 6$ so that $n/3 \equiv 1 \pmod 2$. From [18, p. 213] the positive ternary quadratic form $x^2+3y^2+2yz+5z^2$ is $(2,1)$-universal. Hence there exist integers $r, s$, and $t$ such that

$$\frac{n}{3} = r^2+3s^2+2st+5t^2.$$

Thus

$$n = u^2+3v^2+14w^2,$$

where the integers $u, v$, and $w$ are given by

$$u = 3s+t, \quad v = r, \quad w = t.$$

Hence $x^2+3y^2+14z^2$ is $(6,3)$-universal. It is not $(2,1)$-universal as it does not represent 5, and is not $(3,3)$-universal as it does not represent 6. Thus $x^2+3y^2+14z^2$ is properly $(6,3)$-universal. $\square$

**Proposition 11.** *$2x^2+3y^2+4z^2$ is properly $(6,3)$-universal.*

*Proof.* By Table 4 the form $x^2+2y^2+3z^2$ represents every odd positive integer, and so represents every positive integer $n \equiv 3 \pmod 6$. Thus there are integers $u$, $v$, and $w$ such that $n = u^2+2v^2+3w^2$. Now $(u+v)(u-v) = u^2-v^2 \equiv u^2+2v^2+3w^2 = n \equiv 0 \pmod 3$ so replacing $v$ by $-v$ if necessary, we may suppose that $u+v \equiv 0 \pmod 3$. If $u \equiv 0 \pmod 2$ then $u = 2U$ for some $U \in \mathbb{Z}$ and $n = 2v^2+3w^2+4U^2$. If $u \equiv 1 \pmod 2$ then $w \equiv 0 \pmod 2$, say $w = 2W$ ($W \in \mathbb{Z}$), and we can define integers $u_1$, $v_1$, and $w_1$ by

$$u_1 = \frac{u+v+6W}{3},\ v_1 = \frac{u-2v}{3},\ w_1 = \frac{u+v-3W}{3}.$$

We have

$$\begin{aligned} 2u_1^2+3v_1^2+4w_1^2 &= \frac{1}{9}(2(u+v+6W)^2+3(u-2v)^2+4(u+v-3W)^2) \\ &= u^2+2v^2+12W^2 \\ &= u^2+2v^2+3w^2 = n. \end{aligned}$$

Thus in both cases we see that $n$ is represented by the form $2x^2+3y^2+4z^2$, proving that $2x^2+3y^2+4z^2$ is $(6,3)$-universal. Now $2x^2+3y^2+4z^2$ does not represent 1 so it is not $(2,1)$-universal, and it does not represent 42 so it is not $(3,3)$-universal. Hence $2x^2+3y^2+4z^2$ is properly $(6,3)$-universal. $\square$

**Proposition 12.** $2x^2 + 3y^2 + 7z^2$ *is properly* $(6, 3)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 3 \pmod 6$ so that $n/3 \equiv 1 \pmod 2$. From [18, p. 213] the positive ternary quadratic form $x^2 + 3y^2 + 2yz + 5z^2$ is $(2, 1)$-universal. Hence there exist integers $r, s$, and $t$ such that

$$\frac{n}{3} = r^2 + 3s^2 + 2st + 5t^2.$$

Thus

$$n = 2u^2 + 3v^2 + 7w^2,$$

where the integers $u, v$, and $w$ are given by

$$u = s - 2t, \quad v = r, \quad w = s + t.$$

Hence $2x^2 + 3y^2 + 7z^2$ is $(6, 3)$-universal. It is not $(2, 1)$-universal as it does not represent 1, and is not $(3, 3)$-universal as it does not represent 6. Thus $2x^2+3y^2+7z^2$ is properly $(6, 3)$-universal. □

**Proposition 13.** $x^2 + y^2 + 10z^2$ *is properly* $(6, 5)$*-universal.*

*Proof.* A proof that $x^2 + y^2 + 10z^2$ is $(6, 5)$-universal is given by Jagy [13]. The form $x^2 + y^2 + 10z^2$ does not represent 3 so it is not $(2, 1)$-universal, and it does not represent 38 so it is not $(3, 2)$-universal. Thus $x^2 + y^2 + 10z^2$ is properly $(6, 5)$-universal. □

**Proposition 14.** $x^2 + 4y^2 + 10z^2$ *is properly* $(6, 5)$*-universal.*

*Proof.* Let $n$ be a positive integer with $n \equiv 5 \pmod 6$. From Proposition 13 we know that $x^2 + y^2 + 10z^2$ is $(6, 5)$-universal. Hence there exist integers $u, v$, and $w$ such that $n = u^2 + v^2 + 10w^2$. Hence $1 \equiv u + v \pmod 2$ and thus one of $u$ and $v$ is even and one is odd. Interchanging $u$ and $v$ if necessary, we may suppose that $u$ is odd and $v$ is even. Set $v = 2v_1$, where $v_1 \in \mathbb{Z}$. Then $n = u^2 + 4v_1^2 + 10w^2$ so that $x^2 + 4y^2 + 10z^2$ is $(6, 5)$-universal. The form $x^2 + 4y^2 + 10z^2$ does not represent 3 so it is not $(2, 1)$-universal. Also it does not represent 2 so it is not $(3, 2)$-universal. Thus $x^2 + 4y^2 + 10z^2$ is properly $(6, 5)$-universal. □

**Proposition 15.** $x^2 + 2y^2 + 22z^2$ *is properly* $(8, 1)$*-universal.*

*Proof.* The genus of discriminant 44 containing the class of the form $x^2+2y^2+22z^2$ contains exactly one other class, namely the class of the form $x^2 + 6y^2 + 8z^2 + 4yz$ [4]. By Jones' theorem [15, Theorem, p. 99] the set of positive integers represented by either $x^2+2y^2+22z^2$ or $x^2+6y^2+8z^2+4yz$ or both is precisely the set of positive integers not of the form $4^k(8l + 5)$ for some $k, l \in \mathbb{N}_0$. Let $n \in \mathbb{N}$ be such that $n \equiv 1 \pmod 8$. As $n$ is not of the form $4^k(8l+5)$ for any $k, l \in \mathbb{N}_0$, the positive integer $n$ is represented by either $x^2+2y^2+22z^2$ or $x^2+6y^2+8z^2+4yz$. We show that $n$ is always represented by $x^2 + 2y^2 + 22z^2$. Suppose $n$ is represented by $x^2 + 6y^2 + 8z^2 + 4yz$. Then there are integers $u$, $v$, and $w$ such that $n = u^2 + 6v^2 + 8w^2 + 4vw$. Taking this equation modulo 2, we see that $u \equiv 1 \pmod 2$. Next, taking it modulo 4, we deduce that $1 \equiv 1 + 2v^2 \pmod 4$ so that $v \equiv 0 \pmod 2$. Define the integer $v_1$ by $v = 2v_1$. Then $n = u^2 + 2(v_1 + 2w)^2 + 22v_1^2$, so that $n$ is represented by $x^2 + 2y^2 + 22z^2$. Hence $x^2 + 2y^2 + 22z^2$ is $(8, 1)$-universal. But $x^2 + 2y^2 + 22z^2$ does not represent 5 so it is not $(4, 1)$-universal. Thus $x^2 + 2y^2 + 22z^2$ is properly $(8, 1)$-universal. □

**Proposition 16.** *$x^2 + y^2 + 10z^2$ is properly $(8, 2)$-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 2 \pmod 8$. Then $n/2 \equiv 1 \pmod 4$. By Table 4 $x^2 + y^2 + 5z^2$ is $(4, 1)$-universal. Hence there exist integers $u$, $v$, and $w$ such that $n/2 = u^2 + v^2 + 5w^2$. Thus $n = (u + v)^2 + (u - v)^2 + 10w^2$ so the form $x^2 + y^2 + 10z^2$ is $(8, 2)$-universal. It is not $(4, 2)$-universal as it does not represent 6. Hence $x^2 + y^2 + 10z^2$ is properly $(8, 2)$-universal. □

**Proposition 17.** *$x^2 + y^2 + 32z^2$ is properly $(8, 2)$-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 2 \pmod 8$. Then $n/2 \equiv 1 \pmod 4$. By Table 4 $x^2+y^2+16z^2$ is $(4, 1)$-universal. Hence there are integers $u$, $v$, and $w$ such that $n/2 = u^2+v^2+16w^2$. Thus $n = (u+v)^2+(u-v)^2+32w^2$ so the form $x^2+y^2+32z^2$ is $(8, 2)$-universal. It is not $(4, 2)$-universal as it does not represent 6. Hence $x^2 + y^2 + 32z^2$ is properly $(8, 2)$-universal. □

**Proposition 18.** *$x^2 + 2y^2 + 9z^2$ is properly $(8, 2)$-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 2 \pmod 8$. Then $n/2 \equiv 1 \pmod 4$. By Table 4 $x^2 + y^2 + z^2$ is $(4, 1)$-universal, so there are integers $u$, $v$, and $w$ such that $\frac{n}{2} = u^2 + v^2 + w^2$. If $u \equiv 2 \pmod 3$ we replace $u$ by $-u$ so that $u \equiv 1 \pmod 3$. We do the same for $v$ and $w$. Hence each of $u, v$, and $w$ is $\equiv 0$ or $1 \pmod 3$. By Dirichlet's box principle at least two of $u, v$, and $w$ are congruent modulo 3. We interchange two of $u, v$, and $w$ so that $u \equiv v \pmod 3$. Then we define integers $r, s$, and $t$ by $r = u + v$, $s = w$, $t = (u - v)/3$ so that

$$n = 2u^2 + 2v^2 + 2w^2 = (u + v)^2 + (u - v)^2 + 2w^2 = r^2 + 2s^2 + 9t^2.$$

Hence the form $x^2 + 2y^2 + 9z^2$ is $(8, 2)$-universal. It is properly $(8, 2)$-universal as it does not represent 14. □

**Proposition 19.** *$x^2 + 2y^2 + 18z^2$ is properly $(8, 3)$-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 3 \pmod 8$. The genus of discriminant 36 containing the class of the form $x^2 + 2y^2 + 18z^2$ contains only one other class, namely the class of $3x^2 + 3y^2 + 5z^2 + 2yz + 2zx + 2xy$ [4]. Every positive integer not of the form $4^k(8l + 7)$ $(k, l \in \mathbb{N}_0)$ is represented by the genus, that is by $x^2 + 2y^2 + 18z^2$ or $3x^2+3y^2+5z^2+2yz+2zx+2xy$ (or both). As $n$ is not of the form $4^k(8l+7)$ for any $k, l \in \mathbb{N}_0$, $n$ is represented by either $x^2+2y^2+18z^2$ or $3x^2+3y^2+5z^2+2yz+2zx+2xy$ (or both). If $n$ is not represented by $3x^2 + 3y^2 + 5z^2 + 2yz + 2zx + 2xy$ then $n$ is represented by $x^2+2y^2+18z^2$. If $n$ is represented by $3x^2+3y^2+5z^2+2yz+2zx+2xy$

we must show that $n$ is represented by $x^2+2y^2+18z^2$. Suppose $n = 3u^2+3v^2+5w^2+2uv+2uw+2vw$. Taking this equation modulo 8, we obtain

$$\begin{aligned} 3 \equiv n &\equiv 3u^2+3v^2+5w^2+(u+v+w)^2-u^2-v^2-w^2 \\ &\equiv (u+v+w)^2+2u^2+2v^2+4w^2 \\ &\equiv 1+2u^2+2v^2+4w^2 \pmod 8; \end{aligned}$$

so that

$$u^2+v^2+2w^2 \equiv 1 \pmod 4.$$

If $w \equiv 1 \pmod 2$ then $u^2+v^2 \equiv 3 \pmod 4$, which is impossible. Hence $w \equiv 0 \pmod 2$ so that $w = 2w_1$ for some $w_1 \in \mathbb{Z}$. Then

$$n = (u-v)^2 + 2(u+v+w_1)^2 + 18w_1^2.$$

Hence $n$ is represented by $x^2+2y^2+18z^2$. Thus $x^2+2y^2+18z^2$ is $(8,3)$-universal. As $x^2+2y^2+18z^2$ does not represent 7, it is not $(4,3)$-universal. Thus $x^2+2y^2+18z^2$ is properly $(8,3)$-universal. $\square$

**Proposition 20.** $x^2+2y^2+24z^2$ *is properly* $(8,3)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 3 \pmod 8$. The genus of discriminant 48 containing the class of the form $x^2+2y^2+24z^2$ contains exactly one other class, namely the class of $3x^2+3y^2+6z^2+2xy$ [4] . The set of positive integers represented by either of these two forms is precisely the set of positive integers not of the forms $\{8k+7 \mid k \in \mathbb{N}_0\}$ and $\left\{4^k(8l+5) \mid k,l \in \mathbb{N}_0\right\}$. As $n$ is not in these arithmetic progressions, it is represented by $x^2+2y^2+24z^2$ or $3x^2+3y^2+6z^2+2xy$ (or both). If $n$ is not represented by $3x^2+3y^2+6z^2+2xy$ then it is represented by $x^2+2y^2+24z^2$. If $n$ is represented by $3x^2+3y^2+6z^2+2xy$, we wish to show that $n$ is represented by $x^2+2y^2+24z^2$. In this case there are integers $u$, $v$, and $w$ such that

$$n = 3u^2+3v^2+6w^2+2uv.$$

Taking this equation modulo 2, we obtain $1 \equiv u+v \pmod 2$. Interchanging $u$ and $v$ if necessary, we may suppose that $u \equiv 1 \pmod 2$ and $v \equiv 0 \pmod 2$. Next, taking the equation modulo 4, we obtain $3 \equiv 3+0+2w^2+0 \pmod 4$ so that $w \equiv 0 \pmod 2$. Define $w_1 \in \mathbb{Z}$ by $w = 2w_1$. Then

$$n = (u-v)^2 + 2(u+v)^2 + 24w_1^2.$$

Hence $n$ is represented by $x^2+2y^2+24z^2$. This proves that $x^2+2y^2+24z^2$ is $(8,3)$-universal. Now $x^2+2y^2+24z^2$ does not represent 7 so it is not $(4,3)$-universal. Thus $x^2+2y^2+24z^2$ is properly $(8,3)$-universal. $\square$

**Proposition 21.** $x^2+y^2+10z^2$ *is properly* $(8,4)$*-universal.*

*Proof.* Dickson [8, Corollary, p. 341] has shown that $x^2 + y^2 + 10z^2$ represents all even positive integers except those of the form $4^k(16l + 6)$ for some $k, l \in \mathbb{N}_0$. Let $n \in \mathbb{N}$ satisfy $n \equiv 4 \pmod 8$. Clearly $n$ is not of the form $4^k(16l + 6)$ $(k, l \in \mathbb{N}_0)$. Hence $n$ is represented by $x^2 + y^2 + 10z^2$, so that $x^2 + y^2 + 10z^2$ is $(8, 4)$-universal. It is not $(4, 4)$-universal as it does not represent 24. Hence $x^2+y^2+10z^2$ is properly $(8, 4)$-universal. □

**Proposition 22.** *$x^2 + 2y^2 + 9z^2$ is properly $(8, 4)$-universal.*

*Proof.* The fact that $x^2+2y^2+9z^2$ is $(8, 4)$-universal is proved in [26, Lemma 2.2.]. It does not represent 56, so it is not $(4, 4)$-universal. Hence $x^2+2y^2+9z^2$ is properly $(8, 4)$-universal. □

**Proposition 23.** *$x^2 + 2y^2 + 11z^2$ is properly $(8, 4)$-universal.*

*Proof.* The genus of discriminant 22 containing the class of the form $x^2+2y^2+11z^2$ contains exactly one other class, namely the class of the form $2x^2 + 3y^2 + 4z^2 + 2yz$ [4]. Every positive integer not of the form $2^{2k+1}(8l + 5)$ for some $k, l \in \mathbb{N}_0$, is represented by this genus. Let $n \in \mathbb{N}$ satisfy $n \equiv 4 \pmod 8$. As $n$ is not of the form $2^{2k+1}(8l + 5)$ for any $k, l \in \mathbb{N}_0$, every positive integer $n \equiv 4 \pmod 8$ is represented by $x^2 + 2y^2 + 11z^2$ or $2x^2 + 3y^2 + 4z^2 + 2yz$ (or both). If $n$ is not represented by $2x^2 + 3y^2 + 4z^2 + 2yz$, then $n$ must be represented by $x^2 + 2y^2 + 11z^2$. If $n$ is represented by $2x^2 + 3y^2 + 4z^2 + 2yz$, then there are integers $u$, $v$, and $w$ such that

$$n = 2u^2 + 3v^2 + 4w^2 + 2vw.$$

As $n \equiv 0 \pmod 2$ we see that $v \equiv 0 \pmod 2$. Define an integer $v_1$ by $v = 2v_1$. Then

$$n = (v_1 + 2w)^2 + 2u^2 + 11v_1^2.$$

This proves that $x^2 + 2y^2 + 11z^2$ is $(8, 4)$-universal. As $x^2 + 2y^2 + 11z^2$ does not represent 40 it is not $(4, 4)$-universal. Hence $x^2 + 2y^2 + 11z^2$ is properly $(8, 4)$-universal. □

**Proposition 24.** *$x^2 + 2y^2 + 12z^2$ is properly $(8, 4)$-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 4 \pmod 8$ so $n/4 \equiv 1 \pmod 2$. From Table 4 we see that $x^2 + 2y^2 + 3z^2$ is $(2, 1)$-universal. Hence there exist integers $u$, $v$, and $w$ such that

$$\frac{n}{4} = u^2 + 2v^2 + 3w^2.$$

Thus

$$n = (2u)^2 + 2(2v)^2 + 12w^2$$

so that $x^2 + 2y^2 + 12z^2$ is $(8, 4)$-universal. It is not $(4, 4)$-universal as it does not represent 40. Thus $x^2 + 2y^2 + 12z^2$ is properly $(8, 4)$-universal. □

**Proposition 25.** *$x^2 + 3y^2 + 8z^2$ is properly $(8, 4)$-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 4 \pmod 8$ so $n/4 \equiv 1 \pmod 2$. From Table 4 we see that $x^2+2y^2+3z^2$ is $(2,1)$-universal. Thus there exist integers $u$, $v$ and $w$ such that $\frac{n}{4} = u^2+2v^2+3w^2$. Hence $n = (2u)^2+3(2w)^2+8v^2$ so that $x^2+3y^2+8z^2$ is $(8,4)$-universal. It is not $(4,4)$-universal as it does not represent 40. Thus $x^2+3y^2+8z^2$ is properly $(8,4)$-universal. □

**Proposition 26.** *$x^2+5y^2+6z^2$ is properly $(8,4)$-universal.*

*Proof.* The genus of discriminant 30 containing the class of the form $x^2+5y^2+6z^2$ contains exactly one other class, namely that of the form $3x^2+3y^2+4z^2+2xz+2yz$ [4]. Every positive integer not of the form $2^{2k+1}(8l+1)$ for some $k, l \in \mathbb{N}_0$ is represented by this genus, that is by $x^2+5y^2+6z^2$ or $3x^2+3y^2+4z^2+2xz+2yz$ (or both). Let $n \in \mathbb{N}$ satisfy $n \equiv 4 \pmod 8$. As $n$ is not of the form $2^{2k+1}(8l+1)$ it is represented by either $x^2+5y^2+6z^2$ or $3x^2+3y^2+4z^2+2xz+2yz$ (or both). If $n$ is not represented by $3x^2+3y^2+4z^2+2xz+2yz$, then $n$ is represented by $x^2+5y^2+6z^2$. If $n$ is represented by $3x^2+3y^2+4z^2+2xz+2yz$, then we wish to show that $n$ is represented by $x^2+5y^2+6z^2$. In this case there are integers $u$, $v$, and $w$ such that

$$n = 3u^2+3v^2+4w^2+2uw+2vw.$$

As $n \equiv 0 \pmod 2$ this equation modulo 2 gives $u \equiv v \pmod 2$. Hence we can define integers $u_1$ and $v_1$ by

$$u_1 = \frac{u+v}{2},\ v_1 = \frac{u-v}{2}.$$

Then

$$n = (u_1+2w)^2+5u_1^2+6v_1^2.$$

Hence $x^2+5y^2+6z^2$ is $(8,4)$-universal. It is not $(4,4)$-universal, as it does not represent 8. Thus $x^2+5y^2+6z^2$ is properly $(8,4)$-universal. □

**Proposition 27.** *$x^2+8y^2+11z^2$ is properly $(8,4)$-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 4 \pmod 8$. By Proposition 23 we see that $x^2+2y^2+11z^2$ is $(8,4)$-universal, so there exist integers $u$, $v$, and $w$ such that $n = u^2+2v^2+11w^2$. Taking this equation modulo 2, we obtain $u \equiv w \pmod 2$. Thus $u^2 \equiv w^2 \pmod 4$, and so $u^2+11w^2 \equiv 12w^2 \equiv 0 \pmod 4$. Hence

$$0 \equiv n = u^2+2v^2+11w^2 \equiv 2v^2 \pmod 4;$$

so that $v \equiv 0 \pmod 2$. Define $v_1 \in \mathbb{Z}$ by $v = 2v_1$. Then $n = u^2+8v_1^2+11w^2$. Hence $x^2+8y^2+11z^2$ is $(8,4)$-universal. It is not $(4,4)$-universal as it does not represent 40. Thus $x^2+8y^2+11z^2$ is properly $(8,4)$-universal. □

**Proposition 28.** *$x^2+8y^2+12z^2$ is properly $(8,4)$-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 4 \pmod 8$. Then $\frac{n}{4} \equiv 1 \pmod 2$. From Table 4 we see that $x^2+2y^2+3z^2$ is $(2,1)$-universal. Hence there exist integers $u$, $v$, and $w$ such that $\frac{n}{4} = u^2+2v^2+3w^2$. Thus $n = (2u)^2+8v^2+12w^2$ so that $x^2+8y^2+12z^2$ is $(8,4)$-universal. It is properly $(8,4)$-universal, as it does not represent 40. □

**Proposition 29.** $2x^2 + 3y^2 + 4z^2$ *is properly* $(8, 4)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 4 \pmod 8$. Then $n/4 \equiv 1 \pmod 2$. By Table 4 $x^2 + 2y^2 + 3z^2$ is $(2, 1)$-universal, so there are integers $u$, $v$, and $w$ such that $n/4 = u^2 + 2v^2 + 3w^2$. Hence $n = 2(2v)^2 + 3(2w)^2 + 4u^2$. Thus $2x^2 + 3y^2 + 4z^2$ is $(8, 4)$-universal. It is properly $(8, 4)$-universal, as it does not represent 40. □

**Proposition 30.** $3x^2 + 4y^2 + 8z^2$ *is properly* $(8, 4)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 4 \pmod 8$. Then $n/4 \equiv 1 \pmod 2$. By Table 4 $x^2 + 2y^2 + 3z^2$ is $(2, 1)$-universal, so there are integers $u$, $v$, and $w$ such that $n/4 = u^2+2v^2+3w^2$. Hence $n = 3(2w)^2+4u^2+8v^2$. Thus $3x^2+4y^2+8z^2$ is $(8, 4)$-universal. It is properly $(8, 4)$-universal, as it does not represent 40. □

**Proposition 31.** $x^2 + 2y^2 + 11z^2$ *is properly* $(8, 6)$*-universal.*

*Proof.* The class of the positive ternary quadratic form $x^2 + 2y^2 + 6z^2 + 2yz$ of discriminant 11 is alone in its genus [4]. By Jones' theorem [15, p. 99] the integers represented by this class (and so by the form $x^2+2y^2+6z^2+2yz$) are those integers not of the form $2^{2k}(8l+5)$ for any $k, l \in \mathbb{N}_0$. Let $n \in \mathbb{N}$ be such that $n \equiv 6 \pmod 8$ so that $n/2 \equiv 3 \pmod 4$. Thus, as $n/2$ is not of the form $2^{2k}(8l + 5)$ for any $k, l \in \mathbb{N}_0$, $n/2$ is represented by $x^2 + 2y^2 + 6z^2 + 2yz$. Hence there are integers $r, s$ and $t$ such that

$$\frac{n}{2} = r^2 + 2s^2 + 6t^2 + 2st.$$

Therefore

$$n = u^2 + 2v^2 + 11w^2,$$

where $u, v$, and $w$ are the integers given by

$$u = 2s + t,\ v = r,\ w = t.$$

Hence $x^2 + 2y^2 + 11z^2$ is $(8, 6)$-universal. As $x^2 + 2y^2 + 11z^2$ does not represent 10, it is not $(4, 2)$-universal. Thus $x^2 + 2y^2 + 11z^2$ is properly $(8, 6)$-universal. □

**Proposition 32.** $x^2 + 2y^2 + 12z^2$ *is properly* $(8, 6)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 6 \pmod 8$. Then $n/2 \equiv 3 \pmod 4$. By Table 4 $x^2 + 2y^2 + 6z^2$ is $(4, 3)$-universal. Hence there exist integers $u$, $v$, and $w$ such that $n/2 = u^2 + 2v^2 + 6w^2$. Thus $n = (2v)^2 + 2u^2 + 12w^2$. Hence $x^2 + 2y^2 + 12z^2$ is $(8, 6)$-universal. As $x^2 + 2y^2 + 12z^2$ does not represent 10, it is properly $(8, 6)$-universal. □

**Proposition 33.** $x^2 + 5y^2 + 6z^2$ *is properly* $(8, 6)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ be such that $n \equiv 6 \pmod 8$ so that $n/2 \equiv 3 \pmod 4$. The class of the ternary quadratic form $2x^2 + 3y^2 + 3z^2 + 2xy$ of discriminant 15 is alone in its genus [4]. By Jones' theorem [15, p. 99] all positive integers except those of the form $2^{2k}(8l + 5)$ for some $k, l \in \mathbb{N}_0$ are represented by this class and so by the form $2x^2 + 3y^2 + 3z^2 + 2xy$. As $n/2$ is not of the form $2^{2k}(8l + 5)$ $(k, l \in \mathbb{N}_0)$ there are integers $u, v$, and $w$ such that

$$\frac{n}{2} = 2u^2 + 3v^2 + 3w^2 + 2uv.$$

Hence $n = r^2 + 5s^2 + 6t^2$, where the integers $r, s$, and $t$ are given by $r = 2u + v$, $s = v$, $t = w$. Thus the ternary form $x^2 + 5y^2 + 6w^2$ is $(8, 6)$-universal. It is not $(4, 2)$-universal, as it does not represent 2. Hence $x^2 + 5y^2 + 6w^2$ is properly $(8, 6)$-universal. □

**Proposition 34.** $2x^2 + 3y^2 + 4z^2$ *is properly* $(8, 6)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 6 \pmod 8$. Then $n/2 \equiv 3 \pmod 4$. By Table 4 $x^2 + 2y^2 + 6z^2$ is $(4, 3)$-universal. Hence there exist integers $u$, $v$, and $w$ such that $n/2 = u^2 + 2v^2 + 6w^2$. Thus $n = 2u^2 + 3(2w)^2 + 4v^2$. Hence $2x^2 + 3y^2 + 4z^2$ is $(8, 6)$-universal. As $2x^2 + 3y^2 + 4z^2$ does not represent 10, it is properly $(8, 6)$-universal. □

**Proposition 35.** $x^2 + 6y^2 + 8z^2$ *is properly* $(8, 7)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 7 \pmod 8$. From Table 4 we see that $x^2 + 2y^2 + 6z^2$ is $(4, 3)$-universal. Hence there are integers $u$, $v$, and $w$ such that $n = u^2 + 2v^2 + 6w^2$. Taking this equation modulo 8, we obtain $7 \equiv u^2 + 2v^2 + 6w^2 \pmod 8$. Clearly $u$ is odd, so $u^2 \equiv 1 \pmod 8$, and thus $3 \equiv v^2 + 3w^2 \pmod 4$. If $w \equiv 0 \pmod 2$ then $w^2 \equiv 0 \pmod 4$ so $v^2 \equiv 3 \pmod 4$, which is impossible. Hence $w \equiv 1 \pmod 2$. Thus $w^2 \equiv 1 \pmod 4$ and $v^2 \equiv 0 \pmod 4$, so $v \equiv 0 \pmod 2$, say $v = 2v_1$, $v_1 \in \mathbb{Z}$. Then $n = u^2 + 6w^2 + 8v_1^2$ and the form $x^2 + 6y^2 + 8z^2$ is $(8, 7)$-universal. It is properly $(8, 7)$-universal as it does not represent 3. □

**Proposition 36.** $2x^2 + 2y^2 + 5z^2$ *is properly* $(8, 7)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 7 \pmod 8$. From Table 4 we see that $x^2 + y^2 + 5z^2$ is $(8, 7)$-universal. Hence there exist integers $u$, $v$, and $w$ such that $n = u^2 + v^2 + 5w^2$. Suppose $u \not\equiv v \pmod 2$. Without loss of generality we may suppose that $u \equiv 1 \pmod 2$ and $v \equiv 0 \pmod 2$. Define $v_1 \in \mathbb{Z}$ by $v = 2v_1$. Then, taking $n = u^2 + v^2 + 5w^2$ modulo 8, we obtain $7 \equiv 1 + 4v_1^2 + 5w^2 \pmod 8$, so that $w \equiv 0 \pmod 2$, say $w = 2w_1$, $w_1 \in \mathbb{Z}$. Thus $3 \equiv 2v_1^2 + 10w_1^2 \pmod 4$, which is clearly impossible. Hence $u \equiv v \pmod 2$. We define integers $t$ and $k$ by $t = \frac{u+v}{2}$ and $k = \frac{u-v}{2}$. Thus $u^2 + v^2 = 2t^2 + 2k^2$ and $n = 2t^2 + 2k^2 + 5w^2$, proving that the form $2x^2 + 2y^2 + 5z^2$ is $(8, 7)$-universal. It is properly $(8, 7)$-universal, as it does not represent 3. □

**Proposition 37.** $2x^2 + 5y^2 + 8z^2$ *is properly* $(8, 7)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 7 \pmod 8$. By Proposition 36 we see that $2x^2+2y^2+5z^2$ is $(8,7)$-universal, so there are integers $u$, $v$, and $w$ such that $n = 2u^2+2v^2+5w^2$. Suppose $u \equiv v \equiv 1 \pmod 2$. Then $7 \equiv 2+2+5w^2 \pmod 8$, so that $w^2 \equiv 7 \pmod 8$, which is impossible. Hence $u$ or $v$ is even. Interchanging $u$ and $v$ if necessary, we may suppose that $v$ is even. Set $v = 2v_1$, where $v_1 \in \mathbb{Z}$. Then $n = 2u^2+5w^2+8v_1^2$ so that $2x^2+5y^2+8z^2$ is $(8,7)$-universal. It is properly $(8,7)$-universal, as it does not represent 3. □

**Proposition 38.** *$x^2+3y^2+27z^2$ is properly $(9,3)$-universal.*

*Proof.* From Table 4 we see that $x^2+3y^2+9z^2$ is $(3,1)$-universal. Let $n \in \mathbb{N}$ satisfy $n \equiv 3 \pmod 9$. Then $n/3 \equiv 1 \pmod 3$ and there exist integers $u$, $v$, and $w$ such that $n/3 = u^2+3v^2+9w^2$. Thus $n = (3v)^2+3u^2+27w^2$. Hence $x^2+3y^2+27z^2$ is $(9,3)$-universal. It is properly $(9,3)$-universal, as it does not represent 6. □

**Proposition 39.** *$2x^2+3y^2+27z^2$ is properly $(9,3)$-universal.*

*Proof.* From Table 4 we see that $x^2+6y^2+9z^2$ is $(3,1)$-universal. Let $n \in \mathbb{N}$ satisfy $n \equiv 3 \pmod 9$. Then $n/3 \equiv 1 \pmod 3$ and there exist integers $u$, $v$, and $w$ such that $n/3 = u^2+6v^2+9w^2$. Thus $n = 2(3v)^2+3u^2+27w^2$. Hence $2x^2+3y^2+27z^2$ is $(9,3)$-universal. It is properly $(9,3)$-universal, as it does not represent 6. □

**Proposition 40.** *$x^2+y^2+10z^2$ is properly $(10,5)$-universal.*

*Proof.* The form $x^2+y^2+2z^2$ is $(2,1)$-universal. Let $n \in \mathbb{N}$ satisfy $n \equiv 5 \pmod{10}$ so that $n/5 \equiv 1 \pmod 2$. Hence there exist integers $u$, $v$, and $w$ such that $n/5 = u^2+v^2+2w^2$. Thus $n = (u+2v)^2+(2u-v)^2+10w^2$. Hence $x^2+y^2+10z^2$ is $(10,5)$-universal. As $x^2+y^2+10z^2$ does not represent 70, it is properly $(10,5)$-universal. □

**Proposition 41.** *$x^2+4y^2+10z^2$ is properly $(10,5)$-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 5 \pmod{10}$. Now $x^2+y^2+10z^2$ is $(10,5)$-universal by Proposition 40, so there exist integers $u$, $v$, and $w$ such that $n = u^2+v^2+10w^2$. As $n \equiv 1 \pmod 2$, we have $1 \equiv u+v \pmod 2$, so that $u$ or $v$ is even. Interchanging $u$ and $v$ if necessary, we may suppose that $v$ is even, say $v = 2v_1$ ($v_1 \in \mathbb{Z}$). Then $n = u^2+4v_1^2+10w^2$, so that the form $x^2+4y^2+10z^2$ is $(10,5)$-universal. It is properly $(10,5)$-universal, as it does not represent 70. □

**Proposition 42.** *$x^2+5y^2+6z^2$ is properly $(10,5)$-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 5 \pmod{10}$ so that $n/5 \equiv 1 \pmod 2$. As $x^2+2y^2+3z^2$ is $(2,1)$-universal (Table 4) there are integers $r, s$, and $t$ such that

$$\frac{n}{5} = r^2+2s^2+3t^2.$$

Then

$$n = u^2+5v^2+6w^2,$$

where the integers $u, v$, and $w$ are given by

$$u = 2s + 3t, \quad v = r, \quad w = s - t,$$

so that $x^2 + 5y^2 + 6z^2$ is $(10, 5)$-universal. It is not $(2, 1)$-universal, as it does not represent 3, and it is not $(5, 5)$-universal, as it does not represent 50. Hence $x^2 + 5y^2 + 6z^2$ is properly $(10, 5)$-universal. $\square$

**Proposition 43.** $x^2 + 5y^2 + 14z^2$ *is properly* $(10, 5)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 5 \pmod{10}$ so that $n/5 \equiv 1 \pmod 2$. As $x^2 + 3y^2 + 5z^2 + 2yz$ is $(2, 1)$-universal [18, p. 213] there exist integers $r$, $s$, and $t$ such that

$$\frac{n}{5} = r^2 + 3s^2 + 5t^2 + 2st.$$

Hence

$$n = u^2 + 5v^2 + 14w^2,$$

where the integers $u, v$, and $w$ are given by

$$u = s + 5t, \quad v = r, \quad w = s.$$

Thus $x^2 + 5y^2 + 14z^2$ is $(10, 5)$-universal. It is not $(2, 1)$-universal, as it does not represent 3, nor $(5, 5)$-universal, as it does not represent 10. Hence $x^2 + 5y^2 + 14z^2$ is properly $(10, 5)$-universal. $\square$

**Proposition 44.** $2x^2 + 5y^2 + 7z^2$ *is properly* $(10, 5)$*-universal.*

*Proof.* Let $n \in \mathbb{N}$ satisfy $n \equiv 5 \pmod{10}$ so that $n/5 \equiv 1 \pmod 2$. By [18, p. 213] $x^2 + 3y^2 + 5z^2 + 2yz$ is $(2, 1)$-universal. Hence there exist integers $r$, $s$, and $t$ such that

$$\frac{n}{5} = r^2 + 3s^2 + 5t^2 + 2st.$$

Thus

$$n = 2u^2 + 5v^2 + 7w^2,$$

where the integers $u, v$, and $w$ are given by

$$u = 2s + 3t, \quad v = r, \quad w = s - t.$$

Hence $2x^2 + 5y^2 + 7z^2$ is $(10, 5)$-universal. It is not $(2, 1)$-universal, as it does not represent 1, and it is not $(5, 5)$-universal, as it does not represent 10. Thus it is properly $(10, 5)$-universal. $\square$