ASYMPTOTIC EXPANSIONS FOR THE AVERAGE OF THE GENERALIZED OMEGA FUNCTION

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Abstract

We estimate the average of the difference $\Omega_s(k)-\Omega_0(k)$, where Ω_s is the generalized omega function, defined by $\Omega_s(k)=\sum_{p^\alpha\|k}\alpha^s$, and obtain a precise asymptotic expansion for $\sum_{k\leqslant n}\Omega_s(k)$ for each $s\geqslant 0$.

1. Introduction

Number theoretic omega functions $\omega(k)$ and $\Omega(k)$, which count the number of distinct prime divisors of the positive integer k, and the total number of prime divisors of k, respectively, are defined by

$$\omega(k)=\sum_{p|k}1, \quad \text{and} \quad \Omega(k)=\sum_{p^\alpha\|k}\alpha.$$

Concerning to the average order of these arithmetical functions, in 1917 Hardy and Ramanujan [5] proved that

$$\frac{1}{n}\sum_{k\leqslant n}\omega(k)=\log\log n+M+O\Big(\frac{1}{\log n}\Big),$$

and

$$\frac{1}{n}\sum_{k\leqslant n}\Omega(k)=\log\log n+M_1+O\Big(\frac{1}{\log n}\Big).$$

The constants M and M_1 are defined by

$$M=\gamma+\sum_p\left(\log\left(1-p^{-1}\right)+p^{-1}\right), \quad \text{and} \quad M_1=M+\sum_p\frac{1}{p(p-1)},$$

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where γ is Euler's constant, and $\sum_p$ denotes summation running over all primes.

For the fixed complex number s the generalized omega function $\Omega_s(k)$ is defined by

$$\Omega_s(k) = \sum_{p^\alpha \| k} \alpha^s.$$

Note that $\omega = \Omega_0$ and $\Omega = \Omega_1$. In 1962 Duncan [3] generalized the above result of Hardy and Ramanujan by showing

$$\frac{1}{n}\sum_{k\leqslant n} \Omega_s(k) = \log\log n + M_s + O\Big(\frac{1}{\log n}\Big), \tag{1}$$

for each nonnegative integer s. The constant M_s is an absolute constant depending only on the parameter s and defined by

$$M_s = M + M'_s \tag{2}$$

where

$$M'_s = \sum_p \sum_{\alpha\geqslant 2} \frac{\delta_s(\alpha)}{p^\alpha}, \quad \text{and} \quad \delta_s(\alpha) = \alpha^s - (\alpha-1)^s.$$

We will use this expression for $\delta_s(\alpha)$ throughout the paper.

In 1970 Saffari [8] used Dirichlet's hyperbola method to prove a more general result which implies that

$$\frac{1}{n}\sum_{k\leqslant n} \omega(k) = \log\log n + M + \sum_{j=1}^{m} \frac{a_j}{\log^j n} + O\Big(\frac{1}{\log^{m+1} n}\Big), \tag{3}$$

for each integer $m \geqslant 1$, with

$$a_j = -\int_1^\infty \frac{\{t\}}{t^2}(\log t)^{j-1}\mathrm{d}t. \tag{4}$$

More precisely, it is known that $a_1 = \gamma - 1$ (see [4] for more details). Later, in 1976 Diaconis [2] reproved (3) by applying Perron's formula on the Dirichlet series $\sum_{n=1}^{\infty} \omega(n)n^{-s}$ and using complex integration methods.

In this paper, we improve on Duncan's result (1) by proving the following precise asymptotic expansion.

Theorem 1. *For each $s \geqslant 0$ and for each fixed integer $m \geqslant 1$ we have*

$$\frac{1}{n}\sum_{k\leqslant n} \Omega_s(k) = \log\log n + M_s + \sum_{j=1}^{m} \frac{a_j}{\log^j n} + O\Big(\frac{1}{\log^{m+1} n}\Big),$$

where the constant M_s is defined by (2) *and the coefficients a_j are defined by* (4).

Our strategy to prove the above result is studying the average of the difference of the omega functions. To do this we define

$$\mathcal{J}_s(n) := \sum_{k=1}^{n} (\Omega_s(k) - \omega(k)),$$

and we prove the following result.

Theorem 2. *For each pair of fixed real numbers $s > 0$ and $\varepsilon > 0$, as $n \to \infty$ we have*

$$2^s \frac{\sqrt{n}}{\log n} \ll nM_s' - \mathcal{J}_s(n) \ll (2+\varepsilon)^s \frac{\sqrt{n}}{\log n}. \tag{5}$$

To show that Theorem 1 follows from Theorem 2, observe that the double-sided approximation (5) implies

$$\frac{1}{n}\mathcal{J}_s(n) = M_s' + O\Big(\frac{1}{\sqrt{n}\log n}\Big).$$

By considering the relation (2), using $M_0 = M$, and the approximation (3), we obtain

$$\begin{aligned}
\frac{1}{n}\sum_{k\leqslant n}\Omega_s(k) &= \frac{1}{n}\sum_{k\leqslant n}\omega(k) + M_s' + O\Big(\frac{1}{\sqrt{n}\log n}\Big)\\
&= \log\log n + M_0 + R_0(n) + M_s' + O\Big(\frac{1}{\sqrt{n}\log n}\Big)\\
&= \log\log n + M_s + \sum_{j=1}^{m}\frac{a_j}{\log^j n} + O\Big(\frac{1}{\log^{m+1} n}\Big).
\end{aligned}$$

Hence, Theorem 1 follows. The proof of Theorem 2, given in the last section, is based on some preliminary approximations formulated in Section 3 and proved in Section 4.

2. Some Remarks and Questions

The proof of the left-hand side of (5) ends in an explicit lower bound for $nM_s' - \mathcal{J}_s(n)$. Indeed, we show that for each real $s > 0$ and each integer $n \geqslant 1$,

$$(2^s - 1)\frac{\sqrt{n}}{\log n}\Big(2 - \frac{20}{\log n}\Big) < nM_s' - \mathcal{J}_s(n). \tag{6}$$

On the other hand, we conjecture that the true order of the right-hand side of (5) actually is $2^s \frac{\sqrt{n}}{\log n}$. To make this conjecture more clear, for each $s > 0$ and for each integer $n \geqslant 1$, let

$$\mathcal{K}_s(n) = 2^{-s}\,(nM_s' - \mathcal{J}_s(n))\,\frac{\log n}{\sqrt{n}}.$$

Question 1. Find the possible value of the following limit

$$\lim_{n\to\infty} \mathcal{K}_s(n).$$

Since $\delta_0(\alpha) = 0$ (note that $\alpha \geqslant 2$), thus we have $M_0' = 0$, and so $M_0 = M$. The constant M is known as the Meissel–Mertens constant. We recall (see [4], pp. 94–98) that to compute the constants M_0 and M_1 one may utilize the following rapidly converging series:

$$M_0 = \gamma + \sum_{k=2}^{\infty} \frac{\mu(k)\log\zeta(k)}{k}, \qquad \text{and} \qquad M_1 = \gamma + \sum_{k=2}^{\infty} \frac{\phi(k)\log\zeta(k)}{k},$$

where μ, ζ and ϕ denote the Möbius function, the Riemann zeta function, and the Euler function, respectively. By using the above series representations and careful computations run on Mathematica software, we get the precise values $M_0 \approxeq 0.26149721284764278376$ and $M_1 \approxeq 1.03465388189743791162$.

Question 2. For each real $s \geqslant 0$, does exist the arithmetic function f_s for which

$$M_s = \gamma + \sum_{k=2}^{\infty} \frac{f_s(k)\log\zeta(k)}{k}.$$

Related to the above question, note that $f_0 = \mu$ and $f_1 = \phi$.

Question 3. What is the behaviour of M_s as a function of the variable s defined over $[0,\infty)$? Is the series $\sum_{n\geqslant 0} M_n$ convergent?

3. Preliminary Approximations

During the proofs we will apply some explicit bounds concerning the function $\pi(x)$, which as usual counts the number of primes not exceeding x. Theorem 1 of [7] asserts that, for each real $x > 1$,

$$\pi(x) < \frac{x}{\log x}\Big(1 + \frac{3}{2\log x}\Big) =: u(x), \tag{7}$$

say, and also asserts the validity of $\frac{x}{\log x}(1 + \frac{1}{2\log x}) < \pi(x)$ for each real $x \geqslant 59$, from which by a simple computation we imply for each real $x > 1$ that

$$\frac{x}{\log x}\Big(1 - \frac{3}{4\log x}\Big) < \pi(x). \tag{8}$$

The above approximations imply the following useful results.

Proposition 1. *Assume that f is a positive, strictly decreasing, and continuously differentiable function on $[2,\infty)$, and $f(t)=o(\frac{1}{t})$ as $t\to\infty$. Then, for each $z>1$*

$$\sum_{p>z} f(p) < \frac{1}{\log z}\Big(1+\frac{3}{2\log z}\Big)\int_z^\infty f(t)\mathrm{d}t + \frac{9zf(z)}{4\log^2 z}, \tag{9}$$

provided the integral exists.

Proposition 2. *Assume that f is a positive, decreasing, and continuously differentiable function on $[2,\infty)$. Then, for each $z>1$*

$$\sum_{p\leqslant z} f(p) < C_f + \int_2^z \frac{f(t)}{\log t}\mathrm{d}t + \frac{1}{2}\int_2^z \frac{f(t)}{\log^2 t}\mathrm{d}t - \frac{1}{3}\int_2^z \frac{f(t)}{\log^3 t}\mathrm{d}t, \tag{10}$$

where $C_f = \frac{2}{\log 2}(1+\frac{3}{2\log 2})f(2)$.

Proposition 3. *For each real $z>1$*

$$\sum_{p>z}\frac{1}{p^2} > \frac{1}{z\log z} - \frac{5}{z\log^2 z}. \tag{11}$$

Finally, we recall (see [1], page 87) that for each arbitrary additive arithmetic function f one has

$$\sum_{k\leqslant n} f(k) = \sum_{k\leqslant n}\sum_{p^\alpha\| k} f(p^\alpha) = \sum_{k\leqslant n}\sum_{\substack{p^\alpha| k\\ \alpha\geqslant 1}}\left(f(p^\alpha)-f(p^{\alpha-1})\right),$$

and consequently

$$\sum_{k\leqslant n} f(k) = \sum_{\substack{p^\alpha\leqslant n\\ \alpha\geqslant 1}}\left(f(p^\alpha)-f(p^{\alpha-1})\right)\left[\frac{n}{p^\alpha}\right]. \tag{12}$$

4. Proofs of Propositions 1–3

Proof of Proposition 1. Let $\varpi(n)$ be 1 when n is prime and 0 otherwise. Thus $\sum_{n\leqslant x}\varpi(n)=\pi(x)$, and by Stieltjes integral and integrating by parts, for any function f which is continuously differentiable on the interval $[2,x]$, we get

$$\begin{aligned}\sum_{p\leqslant x} f(p) = \sum_{2\leqslant n\leqslant x}\varpi(n)f(n) &= \int_{2^-}^x f(t)\mathrm{d}\pi(t) \\ &= \pi(x)f(x) - \int_2^x \pi(t)\Big(\frac{\mathrm{d}}{\mathrm{d}t}f(t)\Big)\mathrm{d}t.\end{aligned} \tag{13}$$

Therefore

$$\sum_{p>z} f(p) = \lim_{b\to\infty} \sum_{z<p\leqslant b} f(p) = \lim_{b\to\infty} \Big(\sum_{p\leqslant b} f(p) - \sum_{p\leqslant z} f(p) \Big)$$
$$= -\pi(z)f(z) + \lim_{b\to\infty} \left(\pi(b)f(b) - \int_z^b \pi(t)\Big(\frac{\mathrm{d}}{\mathrm{d}t}f(t)\Big)\mathrm{d}t \right).$$

If further we assume that $f(b) = o(\frac{\log b}{b})$ as $b \to \infty$, then

$$\sum_{p>z} f(p) = -\pi(z)f(z) - \int_z^\infty \pi(t)\Big(\frac{\mathrm{d}}{\mathrm{d}t}f(t)\Big)\mathrm{d}t. \tag{14}$$

Moreover, if f is positive and strictly decreasing and $f(b) = o(\frac{1}{b})$ as $b \to \infty$, then (7) gives

$$\int_z^\infty \pi(t)\Big(-\frac{\mathrm{d}}{\mathrm{d}t}f(t)\Big)\mathrm{d}t < \int_z^\infty \frac{t}{\log t}\Big(1+\frac{3}{2\log t}\Big)\Big(-\frac{\mathrm{d}}{\mathrm{d}t}f(t)\Big)\mathrm{d}t$$
$$\leqslant \frac{1}{\log z}\Big(1+\frac{3}{2\log z}\Big)\int_z^\infty t\Big(-\frac{\mathrm{d}}{\mathrm{d}t}f(t)\Big)\mathrm{d}t$$
$$= \frac{1}{\log z}\Big(1+\frac{3}{2\log z}\Big)\left(zf(z) + \int_z^\infty f(t)\mathrm{d}t\right).$$

Also, by using (8) for each $z > 1$ we get

$$-\pi(z)f(z) < -\frac{z}{\log z}\Big(1-\frac{3}{4\log z}\Big)f(z).$$

Combining the above bounds completes the proof of (9) for each $z > 1$. □

Proof of Proposition 2. Since $\frac{\mathrm{d}}{\mathrm{d}t}f(t) < 0$, the identity (13) and the bound (7) give

$$\sum_{p\leqslant z} f(p) < u(z)f(z) + \int_2^z u(t)\Big(-\frac{\mathrm{d}}{\mathrm{d}t}f(t)\Big)\mathrm{d}t.$$

Integrating by part implies that

$$u(z)f(z) + \int_2^z u(t)\Big(-\frac{\mathrm{d}}{\mathrm{d}t}f(t)\Big)\mathrm{d}t = u(2)f(2) + \int_2^z f(t)\Big(\frac{\mathrm{d}}{\mathrm{d}t}u(t)\Big)\mathrm{d}t.$$

We note that $C_f = u(2)f(2)$ and $\frac{\mathrm{d}}{\mathrm{d}t}u(t) = \frac{1}{\log t} + \frac{1}{2\log^2 t} - \frac{1}{3\log^3 t}$. This completes the proof. □

Proof of Proposition 3. We utilize the relation (14), and then the bounds (7) and (8), to write the following for each $z > 1$:

$$S_4(z) = -\frac{\pi(z)}{z^2} + 2\int_z^\infty \frac{\pi(t)}{t^3}\mathrm{d}t$$
$$> -\frac{u(z)}{z^2} + 2\int_z^\infty \frac{1}{t^2\log t}\Big(1-\frac{3}{4\log t}\Big)\mathrm{d}t = -\frac{u(z)}{z^2} + 2h_1(z) - \frac{3}{2}h_3(z),$$

where

$$h_1(z) = \int_z^\infty \frac{\mathrm{d}t}{t^2 \log t}, \qquad \text{and} \qquad h_3(z) = \int_z^\infty \frac{\mathrm{d}t}{t^2 \log^2 t}.$$

Note that $h_1(z) \sim \frac{1}{z \log z}$ as $z \to \infty$. Let $h_2(z) = h_1(z) - \frac{1}{z \log z} + \frac{1}{z \log^2 z}$. Since $\frac{\mathrm{d}}{\mathrm{d}z} h_2(z) = -\frac{2}{z^2 \log^3 z} < 0$ and $\lim_{z\to\infty} h_2(z) = 0$, we obtain $h_2(z) > 0$ for each $z > 1$, and hence $h_1(z) > \frac{1}{z \log z} - \frac{1}{z \log^2 z}$. Similarly, $h_3(z) \sim \frac{1}{z \log^2 z}$ as $z \to \infty$. Let $h_4(z) = h_3(z) - \frac{1}{z \log^2 z}$, for which $\frac{\mathrm{d}}{\mathrm{d}z} h_4(z) = \frac{2}{z^2 \log^3 z} > 0$ and $\lim_{z\to\infty} h_4(z) = 0$. Hence $h_4(z) < 0$ for each $z > 1$, and consequently $h_3(z) < \frac{1}{z \log^2 z}$. We use the obtained bounds for $h_1(z)$ and $h_3(z)$ to conclude the proof. □

5. Proof of Theorem 2

The function Ω_s is additive. Hence by using (12) with $f(k) = \Omega_s(k)$, and also considering $\delta_s(1) = 1$, we obtain

$$\sum_{k \leqslant n} \Omega_s(k) = \sum_{\substack{p^\alpha \leqslant n \\ \alpha \geqslant 1}} \delta_s(\alpha) \left[\frac{n}{p^\alpha}\right] = \sum_{p \leqslant n} \left[\frac{n}{p}\right] + \sum_{\substack{p^\alpha \leqslant n \\ \alpha \geqslant 2}} \delta_s(\alpha) \left[\frac{n}{p^\alpha}\right].$$

By using (12) with $f(k) = \omega(k)$ one has $\sum_{k \leqslant n} \omega(k) = \sum_{p \leqslant n} [\frac{n}{p}]$. Therefore

$$\mathcal{J}_s(n) = \sum_{\substack{p^\alpha \leqslant n \\ \alpha \geqslant 2}} \delta_s(\alpha) \left[\frac{n}{p^\alpha}\right]. \tag{15}$$

5.1. Proofs of (6) and the Left-hand Side of (5)

We observe that $\delta_s(\alpha) > 0$ for $s > 0$. Thus, for each integer $n \geqslant 1$,

$$nM_s' - \mathcal{J}_s(n) = \sum_p \sum_{\alpha \geqslant 2} \delta_s(\alpha) \left\{\frac{n}{p^\alpha}\right\} \geqslant \sum_p \delta_s(2) \left\{\frac{n}{p^2}\right\} \geqslant \delta_s(2) \sum_{p > \sqrt{n}} \left\{\frac{n}{p^2}\right\}.$$

The condition $p > \sqrt{n}$ is equivalent by $0 < \frac{n}{p^2} < 1$, for which $\{\frac{n}{p^2}\} = \frac{n}{p^2}$, and from this, by applying the bound (11) we get

$$\sum_{p > \sqrt{n}} \left\{\frac{n}{p^2}\right\} = n \sum_{p > \sqrt{n}} \frac{1}{p^2} > \frac{\sqrt{n}}{\log n}\Big(2 - \frac{20}{\log n}\Big).$$

We have $\delta_s(2) = 2^s - 1$. Thus we obtain (6) for $s > 0$ and each integer $n \geqslant 1$, and we have established the validity of the left-hand side of (5).

5.2. Proof of the Right-hand Side of (5)

Let $N = [\frac{\log n}{\log p}]$. By considering $\sum_{\alpha=2}^{N} \delta_s(\alpha) = N^s - 1$,

$$\begin{aligned}\mathcal{J}_s(n) &\geqslant \sum_{p\leqslant\sqrt{n}} \sum_{\alpha=2}^{N} \delta_s(\alpha) \left[\frac{n}{p^\alpha}\right] \\ &> \sum_{p\leqslant\sqrt{n}} \sum_{\alpha=2}^{N} \delta_s(\alpha)\Big(\frac{n}{p^\alpha} - 1\Big) = nM'_s - n\Sigma_1(n,s) - n\Sigma_2(n,s) - \Sigma_3(n,s),\end{aligned}$$

where

$$\Sigma_1(n,s) = \sum_{p>\sqrt{n}} \sum_{\alpha\geqslant 2} \frac{\delta_s(\alpha)}{p^\alpha}, \qquad \Sigma_2(n,s) = \sum_{p\leqslant\sqrt{n}} \sum_{\alpha>N} \frac{\delta_s(\alpha)}{p^\alpha},$$

and

$$\Sigma_3(n,s) = \sum_{p\leqslant\sqrt{n}} (N^s - 1).$$

Approximations of the above sums in the next sections yield

$$nM'_s - \mathcal{J}_s(n) < n\Sigma_1(n,s) + n\Sigma_2(n,s) + \Sigma_3(n,s) \ll (2+\varepsilon)^s \frac{\sqrt{n}}{\log n},$$

for $s > 0$ and $\varepsilon > 0$, thus implying the right-hand side of (5). To complete the proof, it remains to approximate the sums $\Sigma_1(n,s)$, $\Sigma_2(n,s)$ and $\Sigma_3(n,s)$.

5.3. Approximation of $\Sigma_1(n,s)$

For each $\alpha \geqslant 1$ we have $\delta_s(\alpha) \leqslant \alpha^s$, and hence

$$\sum_{\alpha\geqslant 2} \frac{\delta_s(\alpha)}{p^\alpha} \leqslant \sum_{\alpha\geqslant 2} \frac{\alpha^s}{p^\alpha} = \mathrm{Li}_{-s}\Big(\frac{1}{p}\Big) - \frac{1}{p},$$

where $\mathrm{Li}_s(z)$ denotes the polylogarithm function (see [6], pp. 610–612), defined for each real or complex s and z by $\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$. Since for each s,

$$\lim_{p\to\infty} \left(\mathrm{Li}_{-s}\Big(\frac{1}{p}\Big) - \frac{1}{p}\right) p^2 = 2^s,$$

hence

$$\mathrm{Li}_{-s}\Big(\frac{1}{p}\Big) - \frac{1}{p} \leqslant \frac{2^s + 1}{p^2},$$

for each s, and for p sufficiently large. If we assume that $s > 0$ then $2^s + 1 \leqslant 2^{s+1}$, and hence

$$\sum_{\alpha\geqslant 2} \frac{\alpha^s}{p^\alpha} \leqslant \frac{2^{s+1}}{p^2},$$

for $s > 0$ and p sufficiently large. Proposition 1 with $f(t) = \frac{1}{t^2}$ gives for each $n \geqslant 149$,

$$\sum_{p>\sqrt{n}} \frac{1}{p^2} < \frac{1}{\log\sqrt{n}}\Big(1+\frac{3}{2\log\sqrt{n}}\Big)\int_{\sqrt{n}}^{\infty}\frac{\mathrm{d}t}{t^2} + \frac{9}{4\sqrt{n}\log^2\sqrt{n}}$$
$$= \frac{2}{\sqrt{n}\log n} + \frac{15}{\sqrt{n}\log^2 n} \leqslant \frac{5}{\sqrt{n}\log n}.$$

Thus, for each $s > 0$ and for each $n \geqslant 149$ we obtain

$$\Sigma_1(n,s) \leqslant 10\frac{2^s}{\sqrt{n}\log n}.$$

5.4. Approximation of $\Sigma_2(n,s)$

For $\alpha \geqslant 1$, we have $\delta_s(\alpha) \leqslant \alpha^s$. Thus

$$\sum_{\alpha>N}\frac{\delta_s(\alpha)}{p^\alpha} \leqslant \sum_{\alpha\geqslant\frac{\log n}{\log p}}\frac{\alpha^s}{p^\alpha} = \frac{1}{n}\Phi\Big(\frac{1}{p}, -s, \frac{\log n}{\log p}\Big),$$

where $\Phi(z,s,a) = \sum_{k=0}^{\infty} z^k (k+a)^{-s}$ denotes the Hurwitz Lerch transcendent (see [6], page 612). Hence

$$n\Sigma_2(n,s) \leqslant \sum_{p\leqslant\sqrt{n}}\Phi\Big(\frac{1}{p}, -s, \frac{\log n}{\log p}\Big).$$

Let $b > 2$. We observe that for fixed numbers $p \geqslant 2$ and $s > 0$, the function $g(t) = \Phi(\frac{1}{p}, -s, t)$ is strictly increasing for $t > 0$. Thus

$$\sum_{p\leqslant\sqrt{n}}\Phi\Big(\frac{1}{p}, -s, \frac{\log n}{\log p}\Big) = \sum_{p\leqslant n^{\frac{1}{b}}}\Phi\Big(\frac{1}{p}, -s, \frac{\log n}{\log p}\Big) + \sum_{n^{\frac{1}{b}}<p\leqslant\sqrt{n}}\Phi\Big(\frac{1}{p}, -s, \frac{\log n}{\log p}\Big)$$
$$\leqslant \sum_{p\leqslant n^{\frac{1}{b}}}\Phi\Big(\frac{1}{p}, -s, \frac{\log n}{\log 2}\Big) + \sum_{n^{\frac{1}{b}}<p\leqslant\sqrt{n}}\Phi\Big(\frac{1}{p}, -s, \frac{\log n}{\log(n^{\frac{1}{b}})}\Big)$$
$$=: T_1(s,n) + T_2(s,n),$$

say, respectively. For fixed $s > 0$ and $a > 0$, the function $h(t) = \Phi(t, -s, a)$ is strictly increasing and convex for $0 < t < 1$, and satisfies $\lim_{t\to 0^+} h(t) = a^s$ and $\lim_{t\to 1^-} h(t) = +\infty$. Thus, for $0 < t \leqslant \frac{1}{2}$ we have $h(t) \leqslant a^s + 2u(s,a)t$, where

$$u(s,a) = \Phi\Big(\frac{1}{2}, -s, a\Big) - a^s = \sum_{k=1}^{\infty}\frac{(k+a)^s}{2^k}$$
$$= \sum_{1\leqslant k\leqslant a}\frac{k^s}{2^k}\Big(1+\frac{a}{k}\Big)^s + \sum_{k>a}\frac{k^s}{2^k}\Big(1+\frac{a}{k}\Big)^s =: u_1(s,a) + u_2(s,a),$$

say, respectively. Since $\frac{a}{k} \leqslant a$ for each $k \geqslant 1$, we obtain

$$u_1(s,a) \leqslant (a+1)^s \sum_{1 \leqslant k < a} \frac{k^s}{2^k} \leqslant (a(a+1))^s \sum_{1 \leqslant k < a} \frac{1}{2^k} < (a(a+1))^s.$$

Also, we have

$$u_2(s,a) < 2^s \sum_{k>a} \frac{k^s}{2^k} = 2^s \sum_{j+a>a} \frac{(j+a)^s}{2^{j+a}} = 2^{s-a} \Phi\Big(\frac{1}{2}, -s, a\Big).$$

Thus, for fixed $s > 0$ and $a > 0$,

$$\Phi\Big(\frac{1}{2}, -s, a\Big) - a^s < (a(a+1))^s + 2^{s-a} \Phi\Big(\frac{1}{2}, -s, a\Big).$$

If we further assume that $s < a$, then

$$\Phi\Big(\frac{1}{2}, -s, a\Big) < \frac{a^s + (a(a+1))^s}{1 - 2^{s-a}}.$$

Hence, for all real numbers s, a, t with $a > s > 0$ and $0 < t \leqslant \frac{1}{2}$ we obtain

$$\Phi(t, -s, a) < a^s + 2a^s \Big(\frac{2^{s-a} + (a+1)^s}{1 - 2^{s-a}}\Big) t. \tag{16}$$

We assume that $n > 2^s$ and we apply (16) to get

$$\begin{aligned} T_1(s,n) &\leqslant \Big(\frac{\log n}{\log 2}\Big)^s \pi(n^{\frac{1}{b}}) + 2\Big(\frac{\log n}{\log 2}\Big)^s \left(\frac{2^s + n(1 + \frac{\log n}{\log 2})^s}{n - 2^s}\right) \mathcal{A}(n^{\frac{1}{b}}) \\ &\leqslant \Big(\frac{\log n}{\log 2}\Big)^s \pi(n^{\frac{1}{b}}) + \frac{2n}{n - 2^s}\Big(\frac{\log n}{\log 2}\Big)^s \left(1 + \Big(1 + \frac{\log n}{\log 2}\Big)^s\right) \mathcal{A}(n^{\frac{1}{b}}), \end{aligned}$$

where $\mathcal{A}(x) = \sum_{p \leqslant x} \frac{1}{p}$. We utilize the bounds $\pi(x) \ll \frac{x}{\log x}$ and $\mathcal{A}(x) \ll \log\log x$ to obtain $T_1(s,n) \ll v(s,n)$ for numbers $s > 0$ and $b > 2$, with

$$v(s,n) = b\Big(\frac{\log n}{\log 2}\Big)^s \frac{n^{\frac{1}{b}}}{\log n} + \frac{n}{n - 2^s}\Big(\frac{\log n}{\log 2}\Big)^s \left(1 + \Big(1 + \frac{\log n}{\log 2}\Big)^s\right) \log\log n,$$

as $n \to \infty$. We have $v(s,n) = o(\sqrt{n})$ for numbers $s > 0$ and $b > 2$, as $n \to \infty$. Thus, as $n \to \infty$ we imply

$$T_1(s,n) \ll b^s \frac{\sqrt{n}}{\log n}.$$

On the other hand, $\lim_{t \to 0^+} h(t) = a^s$ implies $h(t) \leqslant 1 + a^s$ for each sufficiently small $t > 0$. Hence, as $n \to \infty$ we get

$$T_2(s,n) \leqslant \sum_{n^{\frac{1}{b}} < p \leqslant \sqrt{n}} (1 + b^s) = (1 + b^s)\left(\pi(\sqrt{n}) - \pi(n^{\frac{1}{b}})\right) \ll b^s \frac{\sqrt{n}}{\log n}.$$

We let $b = 2 + \varepsilon$ with $\varepsilon > 0$ being a fixed small real. Thus we obtain

$$\Sigma_2(n, s) \ll \frac{(2+\varepsilon)^s}{\sqrt{n}\log n},$$

as $n \to \infty$.

5.5. Approximation of $\Sigma_3(n, s)$

We have

$$\Sigma_3(n, s) \leqslant \sum_{p \leqslant \sqrt{n}} \left(\left(\frac{\log n}{\log p} \right)^s - 1 \right) = (\log n)^s \, S_3(\sqrt{n}, s) - \pi(\sqrt{n}),$$

where

$$S_3(z, s) = \sum_{p \leqslant z} \frac{1}{(\log p)^s}.$$

For $s > 0$, by using Proposition 2 with $f(t) = \frac{1}{\log^s t}$, the inequality

$$S_3(z, s) \leqslant 2 \int_2^z \frac{\mathrm{d}t}{(\log t)^{s+1}},$$

is valid for sufficiently large z. Since $\int_2^z \frac{\mathrm{d}t}{(\log t)^{s+1}} \sim \frac{z}{(\log z)^{s+1}}$ as $z \to \infty$, there exists a constant $c_1 > 0$ such that

$$S_3(\sqrt{n}, s) \leqslant c_1 \frac{2^s \sqrt{n}}{(\log n)^{s+1}},$$

for sufficiently large n. The prime number theorem implies validity of the inequality $\pi(\sqrt{n}) \leqslant 3\frac{\sqrt{n}}{\log n}$ for sufficiently large n, too. Therefore, as $n \to \infty$ we get the approximation

$$\Sigma_3(n, s) \ll \frac{2^s \sqrt{n}}{\log n}.$$

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References

[1] J.-M. De Koninck and F. Luca, *Analytic Number Theory: Exploring the Anatomy of Integers*, American Mathematical Society, Providence, 2012.

[2] P. Diaconis, *Asymptotic expansions for the mean and variance of the number of prime factors of a number n*, Technical Report No. 96, Department of Statistics, Stanford University, 1976.

[3] R.L. Duncan, A class of additive arithmetical functions, *Amer. Math. Monthly* **69** (1962), 34-36.

[4] S.R. Finch, *Mathematical constants*, Cambridge University Press, Cambridge, 2003.

[5] G. Hardy and S. Ramanujan, The normal number of prime factors of a number n, *Quart. J. Math.* **48** (1917), 76-92.

[6] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and Ch. W. Clark (editors), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.

[7] J. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois J. Math.* **6** (1962), 64-94.

[8] B. Saffari, Sur quelques applications de la “méthode de l'hyperbole” de Dirichlet a la théorie des nombres premiers, *Enseignement Math.* **14** (1970), 205-224.