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SQUARE CLASSES AND DIVISIBILITY PROPERTIES OF STERN POLYNOMIALS

Karl Dilcher[1]
Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, B3H 4R2, Canada
dilcher@mathstat.dal.ca

Hayley Tomkins[2]
Department of Mathematics and Statistics, University of Ottawa, Ottawa, Ontario, K1N 6N5, Canada
hayleytomkins@gmail.com

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Abstract

The classical Stern sequence was extended by Klavžar, Milutinović and Petr to the Stern polynomials $B_n(x)$ defined by $B_0(x) = 0$, $B_1(x) = 1$, $B_{2n}(x) = xB_n(x)$, and $B_{2n+1}(x) = B_n(x) + B_{n+1}(x)$. In this paper we prove several divisibility results for these polynomials. We also find several infinite classes of positive integers n such that the Stern polynomials with index n^2 are squares of polynomials which we give explicitly. We conjecture that, apart from two sporadic square Stern polynomials, we have characterized them all.

1. Introduction

The Stern polynomials $B_n(x)$, which were first introduced by Klavžar, Milutinović and Petr [10], have attracted some attention in recent years. They can be defined by $B_0(x) = 0$, $B_1(x) = 1$, and for $n \geq 1$ by

$$B_{2n}(x) = xB_n(x), \tag{1.1}$$

$$B_{2n+1}(x) = B_n(x) + B_{n+1}(x); \tag{1.2}$$

see Table 1 for the first 24 Stern polynomials. For $x = 1$, the Stern polynomials reduce to the classical Stern (diatomic) sequence $a(n)$ which is defined by $a(0) = 0$, $a(1) = 1$, and for $n \geq 1$ by

$$a(2n) = a(n), \qquad a(2n + 1) = a(n) + a(n + 1); \tag{1.3}$$

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see, e.g., [8] for some historical remarks and for some properties of this sequence. We have, therefore,

$$B_n(1) = a(n) \quad (n \geq 0). \tag{1.4}$$

Furthermore, it is easy to see by induction that

$$B_n(2) = n \quad (n \geq 0). \tag{1.5}$$

Numerous interesting properties of these polynomials were derived in [10], including connections with hyperbinary representations and the standard Gray code. Further properties were obtained in [9, 15, 16, 17].

At this point it should be mentioned that a different concept of Stern polynomials, also extending the Stern sequence (1.3), was independently introduced by the first author and K. B. Stolarsky [8]. Both concepts of Stern polynomials were subsequently generalized [4, 5], with further connections to hyperbinary representations.

Stern's diatomic sequence and both sequences of Stern polynomials display what can be called a binary structure. In fact, these sequences are 2-regular sequences in the sense of Allouche and Shallit, where the polynomial sequences are 2-regular over $\mathbb{Z}[x]$. See [1] for further details, or [2, Sect. 4] for a brief summary.

Returning to the polynomials $B_n(x)$, for ease of notation we will suppress the variable x and write B_n for $B_n(x)$ throughout this paper.

n	B_n	n	B_n	n	B_n
1	1	9	x^2+2x+1	17	x^3+x^2+2x+1
2	x	10	$2x^2+x$	18	x^3+2x^2+x
3	$x+1$	11	x^2+3x+1	19	$3x^2+3x+1$
4	x^2	12	x^3+x^2	20	$2x^3+x^2$
5	$2x+1$	13	$2x^2+2x+1$	21	$3x^2+4x+1$
6	x^2+x	14	x^3+x^2+x	22	x^3+3x^2+x
7	x^2+x+1	15	x^3+x^2+x+1	23	x^3+2x^2+3x+1
8	x^3	16	x^4	24	x^4+x^3

Table 1: The Stern polynomials B_n, $1 \leq n \leq 24$.

Questions of divisibility and irreducibility were major topics in the work of Schinzel [15] and Ulas [16, 17], with the conjecture that B_p is irreducible over $\mathbb{Q}$ whenever p is a prime. Based on studying the zero distribution of the Stern polynomials, this conjecture was proved in [7] for numerous classes of primes, and was computationally verified for all $p < 10^7$.

On the other hand, computations readily reveal numerous Stern polynomials that are reducible. Because of (1.1) we only need to consider odd-index polynomials.

The factorizations of the first 12 reducible odd-index Stern polynomials are shown in Table 2.

n	B_n	n	B_n
9	$(x+1)^2$	39	$(x+1)(2x^2+2x+1)$
15	$(x+1)(x^2+1)$	45	$(x+1)^2(2x+1)$
21	$(x+1)(3x+1)$	49	$(x^2+x+1)^2$
27	$(x+1)^3$	51	$(x+1)(3x^2+2x+1)$
33	$(x+1)(x^3+x+1)$	57	$(x+1)(x^3+2x^2+x+1)$
35	$(2x+1)(x^2+x+1)$	63	$(x+1)(x^2+x+1)(x^2-x+1)$

Table 2: The first 12 reducible odd-index Stern polynomials.

Among various emerging patterns in Table 2 we notice, in particular, that B_9 and B_{49} are squares. The fact that both 9 and 49 are squares is no surprise; by (1.5) this is a necessary condition for the polynomial to be a square.

It is the main purpose of this paper to give a characterization of those $n \in \mathbb{N}$ for which B_{n^2} is a square polynomial. To this end, we first present some preliminary results in Section 2, after which we prove some other divisibility results in Section 3. These results are of independent interest, but some of them are also used later. Our main results, on square classes of Stern polynomials, are stated and proved in Section 4, and we conclude this paper with some additional remarks in Section 5.

2. Some Properties of Stern Polynomials

Because of the binary structure mentioned in the Introduction, it is not surprising that various identities deal with powers of 2 in the subscripts. An example for this is the identity

$$B_{2^k-1} = \frac{x^k-1}{x-1}, \tag{2.1}$$

which will be used in the following section. It is easy to prove by induction, using (1.2) and the identity for B_{2^k}; see also [16, Corollary 2.2]. The most useful and important such identity, however, is due to Schinzel [15]; we will apply it in several of our results below.

Lemma 2.1 (Schinzel). *For all nonnegative integers a, m, and r with $0 \le r \le 2^a$ we have*

$$B_{m2^a+r} = B_{2^a-r}B_m + B_rB_{m+1}, \tag{2.2}$$

$$B_{m2^a-r} = B_{2^a-r}B_m + B_rB_{m-1}. \tag{2.3}$$

The identity (2.3) is not explicitly stated in [15], but it follows immediately if we replace r by $2^a - r$ in (2.2) and then replace m by $m-1$. Another identity, similar to (2.2) and (2.3), was proved in [7], but it will not be used here.

The next two identities deal with sequences of subscripts that are related to the following interesting property of the Stern sequence (1.3). In each interval $2^{n-2} \le m \le 2^{n-1}$ the maximum value of $a(m)$ is the Fibonacci number F_n. It was apparently first shown by Lehmer [11] that this maximum occurs at

$$\alpha_n := \frac{1}{3}\left(2^n - (-1)^n\right) \quad \text{and} \quad \beta_n := \frac{1}{3}\left(5\cdot 2^{n-2} + (-1)^n\right) \quad (n \ge 2), \tag{2.4}$$

where α_n is also defined for $n = 0, 1$; see Table 3 for the first few values of both sequences. Numerous properties of these sequences can be found in [13] under A001045 and A048573, respectively. Here we mention only the homogeneous recurrence relations

$$\alpha_{n+1} = \alpha_n + 2\alpha_{n-1} \qquad \beta_{n+1} = \beta_n + 2\beta_{n-1}, \tag{2.5}$$

and the nonhomogeneous analogues

$$\alpha_{n+1} = 2\alpha_n + (-1)^n, \qquad \beta_{n+1} = 2\beta_n - (-1)^n. \tag{2.6}$$

These four identities can be obtained from (2.4) by easy manipulations. Also, by (1.4) and the remark preceding (2.4) we have

$$B_{\alpha_n}(1) = B_{\beta_n}(1) = F_n \qquad (n \ge 2).$$

The following recurrence relations for the subsequences B_{α_n} and B_{β_n} were obtained in [7]:

$$B_{\alpha_{n+1}} = B_{\alpha_n} + xB_{\alpha_{n-1}} \qquad (n \ge 1), \tag{2.7}$$

$$B_{\beta_{n+1}} = B_{\beta_n} + xB_{\beta_{n-1}} \qquad (n \ge 3). \tag{2.8}$$

We will now prove extensions of (2.7) and (2.8); the first one will be useful in Section 3 below.

Lemma 2.2. *For integers $n \ge 1$ and $0 \le j \le n-1$ we have*

$$B_{\alpha_n} = B_{\alpha_{j+1}}B_{\alpha_{n-j}} + xB_{\alpha_j}B_{\alpha_{n-j-1}}, \tag{2.9}$$

and for $n \ge 3$ and $0 \le j \le n-3$,

$$B_{\beta_n} = B_{\alpha_{j+1}}B_{\beta_{n-j}} + xB_{\alpha_j}B_{\beta_{n-j-1}}. \tag{2.10}$$

For $j = 0$, both identities are trivial, and for $j = 1$ we recover (2.7) and (2.8). The identity (2.9) is trivial also for $j = n-1$, and for $j = n-2$ it reduces to (2.7).

Proof of Lemma 2.2. We prove (2.9) by induction on n. By the remark above, the cases $n = 1$ and $n = 2$ are clear. We now let $n \geq 3$ and assume that (2.9) is true up to $n - 1$; we wish to show that it is true for n, i.e., as shown in (2.9). Again by the remark following the lemma, nothing remains to be done for $j = n - 1$ and $j = n - 2$. We now use, in sequence, the identity (2.7), then the induction hypothesis, and finally (2.7) again. We then obtain

$$\begin{aligned} B_{\alpha_n} &= B_{\alpha_{n-1}} + xB_{\alpha_{n-2}} \\ &= B_{\alpha_{j+1}} B_{\alpha_{n-1-j}} + xB_{\alpha_j} B_{\alpha_{n-2-j}} + x\left(B_{\alpha_{j+1}} B_{\alpha_{n-2-j}} + xB_{\alpha_j} B_{\alpha_{n-3-j}}\right) \\ &= B_{\alpha_{j+1}}\left(B_{\alpha_{n-1-j}} + xB_{\alpha_{n-2-j}}\right) + xB_{\alpha_j}\left(B_{\alpha_{n-2-j}} + xB_{\alpha_{n-3-j}}\right) \\ &= B_{\alpha_{j+1}} B_{\alpha_{n-j}} + xB_{\alpha_j} B_{\alpha_{n-j-1}}, \end{aligned}$$

which was to be shown.

To prove (2.10), we make use of the identity

$$B_{\beta_n} = xB_{\alpha_{n-1}} + B_{\alpha_{n-2}} \qquad (n \geq 2), \tag{2.11}$$

which can be proved by induction, using (2.7) and (2.8). Applying (2.11), then (2.9), and finally (2.11) again, we get for $0 \leq j \leq n - 3$,

$$\begin{aligned} B_{\beta_n} &= x\left(B_{\alpha_{j+1}} B_{\alpha_{n-1-j}} + xB_{\alpha_j} B_{\alpha_{n-2-j}}\right) + B_{\alpha_{j+1}} B_{\alpha_{n-2-j}} + xB_{\alpha_j} B_{\alpha_{n-3-j}} \\ &= B_{\alpha_{j+1}}\left(xB_{\alpha_{n-1-j}} + B_{\alpha_{n-2-j}}\right) + xB_{\alpha_j}\left(xB_{\alpha_{n-2-j}} + B_{\alpha_{n-3-j}}\right) \\ &= B_{\alpha_{j+1}} B_{\beta_{n-j}} + xB_{\alpha_j} B_{\beta_{n-j-1}}, \end{aligned}$$

which completes the proof. □

Next we state the following simple but useful identity which follows easily from (1.1) and (1.2).

Lemma 2.3. *For integers $n \geq 0$ we have*

$$B_{2n+2} = xB_{2n+1} - B_{2n}. \tag{2.12}$$

For much of Section 4 the following sequences, which supplement α_k and β_k, will be very useful. For each $k \geq 0$ we define

$$\gamma_k := \frac{2^{3k+3} + (-1)^k}{9}, \; \delta_k := \frac{5 \cdot 2^{3k+1} - (-1)^k}{9}, \; \varepsilon_k := \frac{25 \cdot 2^{3k-1} + (-1)^k}{9}. \tag{2.13}$$

With the exception of $\varepsilon_0 = 3/2$, all these terms are integers, which can be seen, for instance, with Euler's generalization of Fermat's little theorem. The first few terms are shown in Table 3. These three sequences satisfy the following nonhomogeneous recurrence relations.

Lemma 2.4. *For all $k \geq 1$ we have*

$$\gamma_k - (-1)^k = 8\gamma_{k-1}, \quad \delta_k + (-1)^k = 8\delta_{k-1}, \quad \varepsilon_k - (-1)^k = 8\varepsilon_{k-1}, \tag{2.14}$$

with initial conditions $\gamma_0 = \delta_0 = 1$, $\varepsilon_0 = 3/2$. We also have

$$\delta_k - (-1)^k = 10\gamma_{k-1}, \qquad \varepsilon_k + (-1)^k = 10\delta_{k-1}. \tag{2.15}$$

The proofs follow immediately from (2.13) and the fact that $9 - 1 = 2^3$. The identities in (2.14) now lead to the following three-term relations.

Lemma 2.5. *For any $k \geq 2$ we have*

$$B_{\gamma_{k+1}} = (x^2 + x + 1)B_{\gamma_k} + x^3 B_{\gamma_{k-1}}, \tag{2.16}$$
$$B_{\delta_{k+1}} = (x^2 + x + 1)B_{\delta_k} + x^3 B_{\delta_{k-1}}, \tag{2.17}$$
$$B_{\varepsilon_{k+1}} = (x^2 + x + 1)B_{\varepsilon_k} + x^3 B_{\varepsilon_{k-1}}, \tag{2.18}$$

where (2.16) *and* (2.17) *also hold for $k = 1$.*

Outline of proof. Using the first identity in (2.14) and (1.1), (1.2), we get

$$\begin{aligned} B_{\gamma_{k+1}} &= B_{4\gamma_k} + B_{4\gamma_k - (-1)^k} = x^2 B_{\gamma_k} + \left(B_{2\gamma_k} + B_{2\gamma_k - (-1)^k}\right) \\ &= (x^2 + x)B_{\gamma_k} + \left(B_{\gamma_k} + B_{\gamma_k - (-1)^k}\right) = (x^2 + x + 1)B_{\gamma_k} + B_{8\gamma_{k-1}} \\ &= (x^2 + x + 1)B_{\gamma_k} + x^3 B_{\gamma_{k-1}}, \end{aligned}$$

where in the second-last step we have used (2.14) again. The proofs of (2.17) and (2.18) are very similar. □

For ease of notation in Section 4, we introduce one more integer sequence which is similar in nature to $\gamma_k, \delta_k, \epsilon_k$ and supplements them. For $k \geq 0$ we define

$$\zeta_k := \frac{1}{9}\left(2^{3k+2} + 5(-1)^k\right) = \frac{1}{2}\left(\gamma_k + (-1)^k\right). \tag{2.19}$$

The first few values are listed in Table 3.

k	α_k	β_k	γ_k	δ_k	ε_k	ζ_k
0	0	–	1	1	3/2	1
1	1	–	7	9	11	3
2	1	2	57	71	89	29
3	3	3	455	569	711	227
4	5	7	3641	4551	5689	1821
5	11	13	29127	36409	45511	14563

Table 3: $\alpha_k, \ldots, \zeta_k$ for $0 \leq k \leq 5$.

Lemma 2.5 will now be used to prove the following identities.

Lemma 2.6. *For any $k \geq 1$ we have*

$$
\begin{aligned}
(2x+1)B_{\gamma_k} - x^2 B_{\varepsilon_k} &= (-x^2+x+1)B_{\delta_k}, && (2.20)\\
B_{\gamma_k} + xB_{\gamma_{k-1}} &= B_{\delta_k}, && (2.21)\\
B_{\delta_k} + xB_{\delta_{k-1}} &= B_{\varepsilon_k}, && (2.22)\\
B_{\delta_k} - x^2 B_{\delta_{k-1}} &= (2x+1)B_{\gamma_{k-1}}, && (2.23)\\
B_{\gamma_k} - x^2 B_{\gamma_{k-1}} &= B_{\zeta_k}, && (2.24)\\
B_{\delta_k} - x^2 B_{\varepsilon_{k-1}} &= (x+1)B_{\delta_{k-1}}. && (2.25)
\end{aligned}
$$

Outline of proof. (a) From Table 3 we know the values of $\gamma_k, \delta_k, \varepsilon_k$ for $k = 1, 2$. For $k = 1$, the corresponding Stern polynomials can be found in Table 1, and (2.20) is easy to verify in this case. Next we use (1.1), (1.2) and the entries in Table 1 to compute

$$
\begin{aligned}
&B_{\gamma_2} = x^4 + 3x^3 + 3x^2 + 2x + 1, \quad B_{\delta_2} = x^4 + 4x^3 + 4x^2 + 3x + 1,\\
&B_{\varepsilon_2} = x^4 + 5x^3 + 6x^2 + 4x + 1,
\end{aligned}
$$

and again (2.20) can be verified either by hand, or with a computer algebra system.

These two cases now serve as the beginning of an induction on k. The induction step follows immediately from (2.16)–(2.18). This completes the proof of (2.20).

(b) The identities (2.21)–(2.23) and (2.25) can be proved in a similar way: One can verify them for $k = 1$ and $k = 2$ by direct calculation, and then use induction via (2.16)–(2.18).

(c) To prove (2.24), we use the first identity in (2.14) and (1.1), (1.2) to get

$$B_{\gamma_k} = B_{4\gamma_{k-1}} + B_{4\gamma_{k-1}+(-1)^k} = x^2 B_{\gamma_{k-1}} + B_{(\gamma_k+(-1)^k)/2},$$

where we have used the fact that $4\gamma_{k-1} + (-1)^k = \frac{1}{2}(\gamma_k + (-1)^k)$ is equivalent to the first identity in (2.14). □

3. Divisibility Results

In this section we prove some divisibility results for Stern polynomials, some of which will be needed in the next section. Table 2 seems to indicate that for any odd index n that is divisible by 3, B_n is divisible by $B_3 = x + 1$. This is true in general:

Proposition 3.1. *For any $n \geq 1$ we have $B_3 \,|\, B_{3n}$.*

Proof. Towards a proof by induction, we note that the assertion is true for $n = 1$ and $n = 2$. Because of (1.1) there is no loss of generality if we choose an odd n and assume that the assertion is true for all subscripts less than n. We write $n = 4m \pm 1$ and use (2.2) and (2.3) to get

$$B_{3n} = B_{2^2(3m)\pm 3} = B_{2^2-3}B_{3m} + B_3 B_{3m\pm 1}. \tag{3.1}$$

Since $m < n$, by induction hypothesis B_{3m} is divisible by B_3, and so is the right-hand side of (3.1). This completes the proof by induction. □

We easily see from Table 2 that a similarly strong divisibility result does not hold for subscripts $5n$ or $7n$. However, we do have the following general result.

Proposition 3.2. *Let k, ℓ, m, and n be positive integers with $k \geq 2$, and assume that $B_k \,|B_{k\ell}$ and $B_k \,|B_{km}$. Then*

$$B_k \,\Big|B_{k(\ell\cdot 2^n \pm m)} \quad \textit{whenever} \quad km \leq 2^n. \tag{3.2}$$

Before proving this, we set $\ell = m = 1$ and note that the two divisibility conditions in the proposition are trivially true. We therefore have the following consequence.

Corollary 3.3. *For any integers $k \geq 2$ and $n \geq 1$ with $k \leq 2^n$ we have*

$$B_k \,\Big|B_{k(2^n\pm 1)}\ . \tag{3.3}$$

This explains, for instance, why B_{35} and B_{45} are both divisible by B_5, and B_{49} and B_{63} are both divisible by B_7; see Table 2.

Proof of Proposition 3.2. This is an easy application of Lemma 2.1. Indeed, if we rewrite $k(\ell \cdot 2^n \pm m) = (kl)2^n \pm km$, then for $km \leq 2^n$, (2.2) and (2.3) together give

$$B_{k(\ell\cdot 2^n\pm m)} = B_{2^n-km}B_{k\ell} + B_{km}B_{k\ell\pm 1}. \tag{3.4}$$

If both divisibility conditions hold, then (3.4) immediately implies (3.2). □

In some cases we can actually say more. As noted above, Corollary 3.3 shows that B_5 divides both B_{35} and B_{45}. Now, using this first divisibility, Proposition 3.2 shows that B_5 divides $B_{5\cdot 57}$ and $B_{5\cdot 71}$. Table 2 shows that, in fact,

$$B_{5\cdot 7} = B_5 B_7 \qquad \text{and} \qquad B_{5\cdot 9} = B_5 B_9,$$

and an easy computation shows that the same is true for $B_{5\cdot 57}$ and $B_{5\cdot 71}$. From Table 3 we see that the subscripts 7, 9, 57, and 71 are $\gamma_1, \delta_1, \gamma_2$, and δ_2, respectively. In fact, we have the following result.

Proposition 3.4. *For all $k \geq 0$ we have*

$$B_{5\gamma_k} = B_5 B_{\gamma_k} \qquad \textit{and} \qquad B_{5\delta_k} = B_5 B_{\delta_k}. \tag{3.5}$$

Proof. We prove these identities by induction on k. For $k = 0$ they are trivial since $\gamma_0 = \delta_0 = 1$. Now assume that they hold for some $k \geq 0$. Using the first identity in (2.14), followed by (1.1) and (1.2), we get

$$\begin{aligned} B_{5\gamma_{k+1}} &= B_{40\gamma_k - 5(-1)^k} = B_{20\gamma_k - 3(-1)^k} + B_{20\gamma_k - 2(-1)^k} \\ &= \cdots = (x+1)B_{5\gamma_k} + (2x+1)B_{5\gamma_k - (-1)^k}, \end{aligned} \tag{3.6}$$

where the dots indicate a repeated application of (1.1) and (1.2). Similarly,

$$B_{5\delta_{k+1}} = (x+1)B_{5\delta_k} + (2x+1)B_{5\delta_k + (-1)^k}. \tag{3.7}$$

Next we note that

$$5\gamma_k - (-1)^k = \frac{5 \cdot 2^{3k+3} + 5(-1)^k - 9(-1)^k}{9} = 4\frac{5 \cdot 2^{3k+1} - (-1)^k}{9} = 4\delta_k, \tag{3.8}$$

$$5\delta_k + (-1)^k = \frac{25 \cdot 2^{3k+1} - 5(-1)^k + 9(-1)^k}{9} = 4\frac{25 \cdot 2^{3k-1} + (-1)^k}{9} = 4\varepsilon_k, \tag{3.9}$$

and with (1.1) and (3.6), (3.7) we get

$$B_{5\gamma_{k+1}} = (x+1)B_{5\gamma_k} + (2x+1)x^2 B_{\delta_k}, \qquad B_{5\delta_{k+1}} = (x+1)B_{5\delta_k} + (2x+1)x^2 B_{\varepsilon_k}.$$

Combining these with (2.16), (2.17), we see that both identities in (3.5), with $k+1$ in place of k, follow from the induction hypothesis (3.5) if we can show that

$$\begin{aligned} (x+1)B_5 B_{\gamma_k} + (2x+1)x^2 B_{\delta_k} &= B_5 \cdot \left((x^2+x+1)B_{\gamma_k} + x^3 B_{\gamma_{k-1}}\right), \\ (x+1)B_5 B_{\delta_k} + (2x+1)x^2 B_{\varepsilon_k} &= B_5 \cdot \left((x^2+x+1)B_{\delta_k} + x^3 B_{\delta_{k-1}}\right). \end{aligned}$$

Noting that $B_5 = 2x+1$, these last identities are easily transformed to (2.21), (2.22) which completes the proof. □

The method of proof of Propositions 3.1 and 3.2 can also be used to prove that an infinite class of subsequences of Stern polynomials are divisibility sequences. For a fixed integer $r \geq 1$ we define

$$\alpha(r;k) := \frac{2^{rk} - (-1)^k}{2^r + 1}, \qquad k = 0, 1, 2, \ldots \tag{3.10}$$

Clearly $\alpha(r;0) = 0, \alpha(r;1) = 1$, and $\alpha(r;2) = 2^r - 1$. Also, comparing (3.10) with (2.4) and (2.13), we see that

$$\alpha_k = \alpha(1;k) \qquad \text{and} \qquad \gamma_k = \alpha(3;k+1). \tag{3.11}$$

We are now ready to state and prove the following result.

Proposition 3.5. *Let $r, k \geq 1$ be integers. Then for all integers $\ell \geq 1$,*

$$B_{\alpha(r;k)} \,\big|\, B_{\alpha(r;\ell k)}. \tag{3.12}$$

Proof. By rearranging the right-hand side of (3.10), we get

$$\begin{aligned}\alpha(r;\ell k) &= \frac{2^{r((\ell-1)k+k)}-(-1)^{\ell k}}{2^r+1}\\ &= \frac{2^{r(\ell-1)k}2^{rk}-(-1)^k 2^{r(\ell-1)k}+(-1)^k 2^{r(\ell-1)k}-(-1)^{\ell k}}{2^r+1}\\ &= 2^{r(\ell-1)k}\frac{2^{rk}-(-1)^k}{2^r+1}+(-1)^k\frac{2^{r(\ell-1)k}-(-1)^{(\ell-1)k}}{2^r+1}\\ &= 2^{r(\ell-1)k}\alpha(r;k)+(-1)^k\alpha(r;(\ell-1)k).\end{aligned}$$

With (2.2) and (2.3) we now get

$$B_{\alpha(r;\ell k)} = B_R B_{\alpha(r;k)} + B_{\alpha(r;(\ell-1)k)} B_{\alpha(r;k)+(-1)^k}, \tag{3.13}$$

where for simplicity we have set $R := 2^{r(\ell-1)k} - \alpha(r;(\ell-1)k)$. We now prove (3.12) by induction on ℓ. For $\ell = 1$ this is obvious. Suppose now that it is true for some $\ell - 1$. Then $B_{\alpha(r;k)}$ divides the right-hand side of (3.13), so the statement is also true for ℓ; this completes the proof. □

With (3.11) we immediately get the following consequence.

Corollary 3.6. *Let $k \geq 1$ be an integer and α_k, γ_k be as defined in Section 2. Then for all integers $\ell \geq 1$,*

$$B_{\alpha_k} \,|\, B_{\alpha_{\ell k}} \qquad \textit{and} \qquad B_{\gamma_{k-1}} \,\big|\, B_{\gamma_{\ell k-1}}.$$

Remarks. (1) One consequence of Proposition 3.5 is the fact that $B_{\alpha(r;n)}$ has numerous factors if n does. For instance,

$$B_{\alpha_{12}} = B_{1365} = (x+1)(2x+1)(3x+1)(x^2+4x+1).$$

(2) Given the second-order recurrence relations (2.7) and (2.16) and an existing general theory of polynomial divisibility sequences (see, e.g., [12]), Corollary 3.6 is not too surprising. In fact, Proposition 3.5 would also fit into this more general framework. Furthermore, it can be shown that there is a close relationship between the polynomials $B_{\alpha(r;n)}$ and the Chebyshev polynomials of the second kind, $U_k(x)$. While a further investigation of this would be beyond the scope of this paper, we mention that a first instance of this connection was recently obtained in [7, Equation (3.2)].

(3) The sequences $\alpha(r;k)$, in a slightly different notation, recently occurred in connection with certain classes of continued fractions related to generalized Stern polynomials; see [6, Sections 5, 6].

4. Square Classes of Stern Polynomials

From Tables 2 and 1 we see that

$$B_{3^2} = (x+1)^2 = (B_3)^2, \qquad B_{7^2} = (x^2+x+1)^2 = (B_7)^2.$$

These, together with the trivial identity $B_{1^2} = (B_1)^2$, are the first instances of an infinite class of Stern polynomials that are squares. We obtain this class and several others in this section.

Theorem 4.1. *For every integer $k \geq 1$ we have*

$$B_{(2^k-1)^2} = (B_{2^k-1})^2. \tag{4.1}$$

Proof. Noting that $(2^k-1)^2 = 2^{k+1}(2^{k-1}-1)+1$, we use (2.2) with $a = k+1$, $m = 2^{k-1}-1$, and $r = 1$. With (2.1) we then obtain

$$\begin{aligned} B_{(2^k-1)^2} &= B_{2^{k+1}-1}B_{2^{k-1}-1} + B_1B_{2^{k-1}} \\ &= \frac{x^{k+1}-1}{x-1}\cdot\frac{x^{k-1}-1}{x-1} + x^{k-1} \\ &= \frac{x^{2k}-x^{k+1}-x^{k-1}+1+(x^2-2x+1)x^{k-1}}{(x-1)^2} \\ &= \frac{x^{2k}-2x^k+1}{(x-1)^2} = \left(\frac{x^k-1}{x-1}\right)^2 = \left(B_{2^k-1}\right)^2, \end{aligned}$$

which was to be shown. ∎

The next result deals with two related, but sparser, classes of Stern polynomials that are squares.

Theorem 4.2. *Let $k \geq 0$ be an integer, and let*

$$n = \frac{1}{3}(2^{3k}-(-1)^k) \quad \textit{or} \quad n = \frac{1}{3}(5\cdot 2^{3k+1}-(-1)^k). \tag{4.2}$$

Then

$$B_{n^2} = \left(B_3B_{n/3}\right)^2. \tag{4.3}$$

We prove this result in two parts, according to the first, resp. second form of n in (4.2).

Proof. (1) We assume that n has the form of the left identity in (4.2). However, regardless of the particular form of n, as long as it is odd and divisible by 3, we have

$$\begin{aligned} \left(B_3B_{n/3}\right)^2 &= \left(B_{n/3}\right)^2(x^2+2x+1) = B_{n/3}\cdot\left((x^2+x+1)B_{n/3}+xB_{n/3}\right) \\ &= B_{n/3}\cdot\left((x^2+x+1)B_{n/3}+B_{n/3-1}+B_{n/3+1}\right), \end{aligned} \tag{4.4}$$

where we have used (2.12) in the last line. Now we use (2.2) and (2.3) with $a = 3$, $m = n/3$, and $r = 1$, obtaining

$$B_{8n/3+(-1)^k} = (x^2 + x + 1)B_{n/3} + B_{n/3+(-1)^k},$$

and so (4.4) becomes

$$\begin{aligned}\left(B_3 B_{n/3}\right)^2 &= B_{n/3} \cdot \left(B_{8n/3+(-1)^k} + B_{n/3-(-1)^k}\right)\\ &= B_{n/3}B_{n/3-(-1)^k} + B_{8n/3+(-1)^k}B_{n/3}.\end{aligned} \tag{4.5}$$

On the other hand, we have

$$\begin{aligned}n^2 &= \frac{1}{9}\left(2^{6k} - 2(-1)^k 2^{3k} + 1\right)\\ &= 2^{3k}\left(\frac{2^{3k} - (-1)^k}{9} - (-1)^k\right) + (-1)^k\left(8 \cdot \frac{2^{3k} - (-1)^k}{9} + (-1)^k\right),\end{aligned}$$

which is easy to verify by simply expanding the right-hand side. Using the definition of n, we therefore have

$$n^2 = 2^{3k}\left(\frac{n}{3} - (-1)^k\right) + (-1)^k\left(8\frac{n}{3} + (-1)^k\right). \tag{4.6}$$

We now use (2.2) when k is even and (2.3) when k is odd, with $a = 3k$, $m = \frac{n}{3} - (-1)^k$, and $r = 8\frac{n}{3} + (-1)^k$ in both cases. This gives, with (4.6),

$$B_{n^2} = B_{2^{3k}-8n/3-(-1)^k}B_{n/3-(-1)^k} + B_{8n/3-(-1)^k}B_{n/3}, \tag{4.7}$$

where the last term on the right comes from $m + (-1)^k$. Finally, we note that

$$2^{3k} - \frac{8n}{3} - (-1)^k = 2^{3k} - 8\frac{2^{3k} - (-1)^k}{9} - (-1)^k = \frac{2^{3k} - (-1)^k}{9} = \frac{n}{3},$$

which means that the right-hand side of (4.7) is the same as the right-hand side of (4.6). This proves the first part of Theorem 4.2.

(2) We now assume that n has the form of the second identity in (4.2). We note again that n is divisible by 3, and in analogy with (4.4) we write

$$\begin{aligned}\left(B_3 B_{n/3}\right)^2 &= B_{n/3} \cdot \left((2x^2 + x)B_{n/3} - (x^2 - x - 1)B_{n/3}\right)\\ &= B_{\delta_k} \cdot \left((2x^2 + x)B_{\delta_k} - (x^2 - x - 1)B_{\delta_k}\right).\end{aligned} \tag{4.8}$$

On the other hand, we obtain

$$\begin{aligned}n^2 &= \frac{1}{9}\left(25 \cdot 2^{6k+2} - 10(-1)^k 2^{3k+1} + 1\right)\\ &= 2^{3k+1} \cdot 5 \cdot \frac{5 \cdot 2^{3k+1} - (-1)^k}{9} - (-1)^k \cdot \frac{5 \cdot 2^{3k+1} - (-1)^k}{9}\\ &= 2^{3k+1} \cdot 5 \cdot \delta_k - (-1)^k \delta_k,\end{aligned}$$

where we have used (2.13). We now use (2.2) and (2.3) with $a = 3k+1$, $m = 5\delta_k$, and $r = \delta_k$. Then

$$B_{n^2} = B_{2^{3k+1}-\delta_k} B_{5\delta_k} + B_{\delta_k} B_{5\delta_k-(-1)^k}. \tag{4.9}$$

Now a simple calculation using (2.13) shows that

$$2^{3k+1} - \delta_k = \gamma_k,$$

and with Lemma 2.3 we get

$$B_{5\delta_k-(-1)^k} = xB_{5\delta_k} - B_{5\delta_k+(-1)^k} = xB_{5\delta_k} - x^2 B_{\varepsilon_k},$$

where in the last equation we have used (3.9) and (1.1). Combining these last two identities with (4.9) and using (3.5), we obtain

$$\begin{aligned} B_{n^2} &= B_{\delta_k} \cdot \left(B_5 B_{\gamma_k} + B_5 B_{\delta_k} - x^2 B_{\varepsilon_k} \right) \\ &= B_{\delta_k} \cdot \left((2x^2 + x) B_{\delta_k} + (2x+1) B_{\gamma_k} - x^2 B_{\varepsilon_k} \right). \end{aligned}$$

Comparing this with (4.8), we see that we are done if

$$(2x+1)B_{\gamma_k} - x^2 B_{\varepsilon_k} = -(x^2 - x - 1) B_{\delta_k}$$

holds. But this last identity is the same as (2.20). The proof of Theorem 4.2 is now complete. □

The final two sequences of square cases are much sparser still than those in (4.2). Although similar in nature to each other, the next two results are sufficiently different to require separate statements and proofs.

Theorem 4.3. *Let $k \geq 0$ be an integer, and let*

$$n = \frac{1}{3}\left(2^{3k} + (-1)^k\right)\left(2^{3k+2} - (-1)^k\right). \tag{4.10}$$

Then

$$B_{n^2} = \Big(B_{\zeta_k} B_{2\varepsilon_k-(-1)^k} + x^3 B_{\gamma_{k-1}}^2 \Big)^2. \tag{4.11}$$

Proof. We prove this result in several steps.

1. With n as in (4.10), it is easy to verify that

$$n^2 = 2^{6k}\frac{1}{9}\left(2^{6k+4} + 3(-1)^k 2^{3k+3} + 5\right) - \frac{1}{9}\left(2^{6k+2} + 3(-1)^k 2^{3k+1} - 1\right), \tag{4.12}$$

and if we set

$$m := \frac{1}{9}\left(2^{6k+4} + 3(-1)^k 2^{3k+3} + 5\right), \quad r := \frac{1}{9}\left(2^{6k+2} + 3(-1)^k 2^{3k+1} - 1\right), \tag{4.13}$$

$$R := 2^{6k} - r = \frac{1}{9}\left(5 \cdot 2^{6k} - 3(-1)^k 2^{3k+1} + 1\right), \tag{4.14}$$

then (2.3) gives $B_{n^2} = B_R B_m + B_r B_{m-1}$. The two identities in (4.13) show that $m-1 = 4r$, so that by (1.1) we have $B_{m-1} = x^2 B_r$, and thus

$$B_{n^2} = B_R B_m + (xB_r)^2 . \tag{4.15}$$

The idea now is to find polynomials S_k, T_k with

$$B_R B_m = S_k T_k \qquad \text{such that} \qquad S_k - T_k = 2xB_r. \tag{4.16}$$

Then by (4.15) we would have

$$B_{n^2} = S_k T_k + \left(\frac{S_k - T_k}{2}\right)^2 = \left(\frac{S_k + T_k}{2}\right)^2 , \tag{4.17}$$

which would complete the proof.

2. With (4.14) and (2.13) we can verify that

$$R = 2^{3k} 5\gamma_{k-1} + (-1)^{k-1}\gamma_{k-1},$$

and (2.2), (2.3) together give

$$B_R = B_{2^{3k}-\gamma_{k-1}} B_{5\gamma_{k-1}} + B_{\gamma_{k-1}} B_{5\gamma_{k-1}+(-1)^{k-1}}. \tag{4.18}$$

Now it is easy to verify with (2.13) that

$$2^{3k} - \gamma_{k-1} = \gamma_k, \qquad 5\gamma_{k-1} - (-1)^{k-1} = 4\delta_{k-1}, \tag{4.19}$$

and (2.12), and then the second identity in (4.19), give

$$B_{5\gamma_{k-1}+(-1)^{k-1}} = xB_{5\gamma_{k-1}} - B_{5\gamma_{k-1}-(-1)^{k-1}} = xB_{5\gamma_{k-1}} - x^2 B_{\delta_{k-1}}.$$

This and the first identity in (4.19) substituted into (4.18), and using the first identity in (3.5), give

$$\begin{aligned} B_R &= B_{\gamma_{k-1}} \left((2x+1)(B_{\gamma_k} + xB_{\gamma_{k-1}}) - x^2 B_{\delta_{k-1}}\right) \\ &= B_{\gamma_{k-1}} \left((2x+1)B_{\delta_k} - x^2 B_{\delta_{k-1}}\right), \end{aligned} \tag{4.20}$$

where in the second equation we have used (2.21).

Next, with (4.13) and (2.19) we verify that $m = 2^{3k+2}\zeta_k + (-1)^k\zeta_k$, and (2.2), (2.3) give

$$B_m = B_{2^{3k+2}-\zeta_k} B_{\zeta_k} + B_{\zeta_k} B_{\zeta_k+(-1)^k}. \tag{4.21}$$

Using (2.19) and (2.13) we verify that $2^{3k+2} - \zeta_k = 4\gamma_k - (-1)^k$, and we rewrite (4.21) as

$$B_m = B_{\zeta_k} \left(B_{4\gamma_k-(-1)^k} + B_{\zeta_k+(-1)^k}\right). \tag{4.22}$$

Next we will show that

$$S_k := B_{\zeta_k}\left((2x+1)B_{\delta_k} - x^2 B_{\delta_{k-1}}\right), \tag{4.23}$$
$$T_k := B_{\gamma_{k-1}}\left(B_{4\gamma_k-(-1)^k} + B_{\zeta_k+(-1)^k}\right) \tag{4.24}$$

satisfy both identities in (4.16). The first one is clear by (4.20) and (4.22).

3. To show that the second identity in (4.16) holds, we begin by determining B_r. Using (4.13), (2.19), and (2.13), we verify that $r = 2^{3k}\zeta_k + (-1)^k\gamma_{k-1}$, and once again using (2.2) and (2.3), we get

$$B_r = B_{2^{3k}-\gamma_{k-1}}B_{\zeta_k} + B_{\gamma_{k-1}}B_{\zeta_k+(-1)^k} = B_{\gamma_k}B_{\zeta_k} + B_{\gamma_{k-1}}B_{\zeta_k+(-1)^k}, \tag{4.25}$$

where in the second equation we have used the first identity in (4.19). Using (4.23)–(4.25) and collecting terms, we see that the right identity in (4.16) holds if and only if

$$B_{\zeta_k}\left((2x+1)B_{\delta_k} - x^2 B_{\delta_{k-1}} - 2xB_{\gamma_k}\right) = B_{\gamma_{k-1}}\left(B_{4\gamma_k-(-1)^k} + (2x+1)B_{\zeta_k+(-1)^k}\right). \tag{4.26}$$

The expression in large parentheses on the left can be rewritten as

$$2x\left(B_{\delta_k} - B_{\gamma_k}\right) + \left(B_{\delta_k} - x^2 B_{\delta_{k-1}}\right) = (2x^2+2x+1)B_{\gamma_{k-1}},$$

where we have used (2.21) and (2.23). Hence (4.26) is equivalent to

$$(2x^2+2x+1)B_{\zeta_k} = B_{4\gamma_k-(-1)^k} + (2x+1)B_{\zeta_k+(-1)^k}. \tag{4.27}$$

To verify this, we first use (1.2) twice, followed by (2.14) and (1.1), to obtain

$$\begin{aligned} B_{4\gamma_k-(-1)^k} &= (x+1)B_{\gamma_k} + x^3 B_{\gamma_{k-1}} \\ &= x\left(B_{\gamma_k} - x^2 B_{\gamma_{k-1}}\right) + B_{\gamma_k} + 2x^3 B_{\gamma_{k-1}} \\ &= xB_{\zeta_k} + B_{\gamma_k} + 2x^3 B_{\gamma_{k-1}}, \end{aligned} \tag{4.28}$$

where we have used (2.24) in the second equality. Next we use (2.13) and (2.19) to verify that $\zeta_k - (-1)^k = 4\gamma_{k-1}$; then we get with (2.12) and (1.1),

$$B_{\zeta_k+(-1)^k} = xB_{\zeta_k} - B_{\zeta_k-(-1)^k} = xB_{\zeta_k} - x^2 B_{\gamma_{k-1}}, \tag{4.29}$$

and with this and (4.28) we rewrite the right-hand side of (4.27) as

$$(2x^2+2x)B_{\zeta_k} + \left(B_{\gamma_k} - x^2 B_{\gamma_{k-1}}\right) = (2x^2+2x+1)B_{\zeta_k},$$

where we have used (2.24). We have thus verified (4.16), and by (4.17) we know that B_{n^2} is the square of the polynomial $\frac{1}{2}(S_k + T_k)$.

4. It only remains to simplify this last expression. With (4.28) and (4.29) and using (2.24) again, we get

$$\begin{aligned} B_{4\gamma_k-(-1)^k} + B_{\zeta_k+(-1)^k} &= 2xB_{\zeta_k} + \left(B_{\gamma_k} - x^2 B_{\gamma_{k-1}}\right) + 2x^3 B_{\gamma_{k-1}} \\ &= (2x+1)B_{\zeta_k} + 2x^3 B_{\gamma_{k-1}}, \end{aligned}$$

and then (4.23), (4.24) give

$$\begin{aligned} S_k + T_k &= B_{\zeta_k}\left((2x+1)B_{\delta_k} - x^2 B_{\delta_{k-1}} + (2x+1)B_{\gamma_{k-1}}\right) + 2x^3 B_{\gamma_{k-1}}^2, \\ &= 2B_{\zeta_k}\left((x+1)B_{\delta_k} - x^2 B_{\delta_{k-1}}\right) + 2x^3 B_{\gamma_{k-1}}^2, \end{aligned} \tag{4.30}$$

where in the second equality we have used (2.23). On the other hand, using (1.2), followed by (2.22) and (2.14), we get

$$\begin{aligned} B_{2\varepsilon_k-(-1)^k} &= B_{\varepsilon_k} + B_{\varepsilon_k-(-1)^k} = \left(B_{\delta_k} + xB_{\delta_{k-1}}\right) + x^3 B_{\varepsilon_{k-1}} \\ &= B_{\delta_k} + xB_{\delta_{k-1}} + x\left(B_{\delta_k} - (x+1)B_{\delta_{k-1}}\right) \\ &= (x+1)B_{\delta_k} - x^2 B_{\delta_{k-1}}, \end{aligned}$$

where in the second-last step we have used (2.25). This, with (4.30), gives (4.11), and the proof is complete. □

Our next and final result is similar to Theorem 4.3.

Theorem 4.4. *Let $k \geq 0$ be an integer, and let*

$$n = \frac{1}{3}\left(2^{3k+1} + (-1)^k\right)\left(2^{3k+3} - (-1)^k\right). \tag{4.31}$$

Then

$$B_{n^2} = \left(B_{\gamma_k}\left((x^2+x+1)B_{\delta_k} + x^2 B_{\gamma_{k-1}}\right) - x^4 B_{\gamma_{k-1}} B_{\delta_k}\right)^2. \tag{4.32}$$

Proof. We follow the outline of the proof of Theorem 4.3.

1. We verify that

$$n^2 = 2^{6k+2}\frac{1}{9}\left(2^{6k+6} + 3(-1)^k 2^{3k+4} + 5\right) - \frac{1}{9}\left(2^{6k+4} + 3(-1)^k 2^{3k+2} - 1\right) \tag{4.33}$$

and set

$$m := \frac{1}{9}\left(2^{6k+6} + 3(-1)^k 2^{3k+4} + 5\right), \quad r := \frac{1}{9}\left(2^{6k+4} + 3(-1)^k 2^{3k+2} - 1\right), \tag{4.34}$$

$$R := 2^{6k+2} - r = \frac{1}{9}\left(5 \cdot 2^{6k+2} - 3(-1)^k 2^{3k+2} + 1\right). \tag{4.35}$$

It is again clear from both identities in (4.34) that $m-1 = 4r$, and then the identities (4.15)–(4.17) also hold in this case.

2. With (4.35) and (2.13) it is easy to verify that

$$R = 10 \cdot 2^{3k+1}\gamma_{k-1} + (-1)^k\gamma_k,$$

and (2.2), (2.3) together give

$$B_R = B_{2^{3k+1}-\gamma_k} B_{10\gamma_{k-1}} + B_{\gamma_k} B_{10\gamma_{k-1}+(-1)^k}. \tag{4.36}$$

With (2.13) we verify that $2^{3k+1} - \gamma_k = \delta_k$, and using (3.5) and (2.15), we can rewrite (4.36) as

$$B_R = B_{\delta_k}\left((2x^2 + x)B_{\gamma_{k-1}} + B_{\gamma_k}\right). \tag{4.37}$$

Next, using (4.34) and (2.13), we verify $m = 2^{3k+3}\gamma_k + (-1)^k 5\gamma_k$, and with (2.2), (2.3) we get

$$\begin{aligned} B_m &= B_{2^{3k+3}-5\gamma_k} B_{\gamma_k} + B_{5\gamma_k} B_{\gamma_k+(-1)^k} \\ &= B_{\gamma_k}\left(B_{4\gamma_k-(-1)^k} + (2x+1)B_{\gamma_k+(-1)^k}\right), \end{aligned} \tag{4.38}$$

where we have used (3.5) and the identity $2^{3k+3} - 5\gamma_k = 4\gamma_k - (-1)^k$, which is easy to verify using (2.13). In analogy to (4.23) and (4.24) we now set

$$S_k := B_{\delta_k}\left(B_{4\gamma_k-(-1)^k} + (2x+1)B_{\gamma_k+(-1)^k}\right), \tag{4.39}$$

$$T_k := B_{\gamma_k}\left((2x^2 + x)B_{\gamma_{k-1}} + B_{\gamma_k}\right). \tag{4.40}$$

By (4.37) and (4.38), the first identity in (4.16) is clearly satisfied.

3. To show that the second identity in (4.16) holds as well, we once again begin by determining B_r. With (4.34) and (2.13), we verify that $r = 2^{3k+1}\gamma_k + (-1)^k\delta_k$, and (2.2), (2.3) give

$$B_r = B_{2^{3k+1}-\delta_k} B_{\gamma_k} + B_{\delta_k} B_{\gamma_k+(-1)^k} = B_{\gamma_k}^2 + B_{\delta_k} B_{\gamma_k+(-1)^k}, \tag{4.41}$$

where we have used $2^{3k+1} - \delta_k = \gamma_k$, an identity already applied in the line following (4.36). Now we use (2.21) and (4.39)–(4.41) to obtain

$$\begin{aligned} S_k - T_k - 2xB_r &= B_{\delta_k}\left(B_{4\gamma_k-(-1)^k} + B_{\gamma_k+(-1)^k}\right)) \\ &\qquad - (2x+1)B_{\gamma_k}\left(xB_{\gamma_{k-1}} + B_{\gamma_k}\right) \\ &= B_{\delta_k}\left(B_{4\gamma_k-(-1)^k} + B_{\gamma_k+(-1)^k} - (2x+1)B_{\gamma_k}\right). \end{aligned} \tag{4.42}$$

Finally, the first line of (4.28) gives

$$B_{4\gamma_k-(-1)^k} = (x+1)B_{\gamma_k} + x^3 B_{\gamma_{k-1}}, \tag{4.43}$$

and (2.12), followed by (2.14) and (1.1), give

$$B_{\gamma_k+(-1)^k} = xB_{\gamma_k} - B_{\gamma_k-(-1)^k} = xB_{\gamma_k} - x^3 B_{\gamma_{k-1}}. \tag{4.44}$$

If we substitute these last two identities into the right-hand side of (4.42), we see that this expression vanishes, and therefore the second identity of (4.16) holds. We have thus shown that B_{n^2} is the square of the polynomial $\frac{1}{2}(S_k+T_k)$.

4. As in the previous proof, it remains to simplify this last expression. We use (4.43) and (4.44) to rewrite S_k, as defined in (4.39), as

$$S_k = B_{\delta_k}\left((2x^2+2x+1)B_{\gamma_k} - 2x^4 B_{\gamma_{k-1}}\right).$$

Using (2.21), we rewrite T_k as $T_k = 2x^2 B_{\gamma_k} B_{\gamma_{k-1}} + B_{\gamma_k} B_{\delta_k}$. It is now clear that $\frac{1}{2}(S_k+T_k)$ is the right-hand side of (4.32), which completes the proof. □

5. Further Remarks

Interestingly, there are two "sporadic" cases that are not covered by the four theorems in Section 4. Computations show that

$$B_{227^2} = \left(1+5x+10x^2+10x^3+4x^4+x^5\right)^2, \tag{5.1}$$

$$B_{349^2} = \left(1+4x+7x^2+7x^3+6x^4+3x^5+x^6\right)^2. \tag{5.2}$$

We note that 227 and 349 are both primes, and that the square root polynomials on the right are irreducible; however, they are not themselves Stern polynomials.

With these cases and the results in Section 4 in mind, we propose the following conjecture which we confirmed by computation for all $n < 10^{10}$.

Conjecture 5.1. *Apart from the sporadic cases* (5.1)*,* (5.2)*, all square Stern polynomials are given by Theorems* 4.1–4.4*.*

For a final remark we recall that Theorem 4.1 shows

$$B_{n^2} = (B_n)^2 \quad \text{when} \quad n = 2^k - 1. \tag{5.3}$$

The only other such index we were able to find is $n = 27$. This case is covered by Theorem 4.2, more exactly by the right-hand term in (4.2) for $k = 1$. However, it is interesting to note that

$$B_{3^6} = (x+1)^6 = (B_{3^3})^2. \tag{5.4}$$

In this connection it is worth mentioning that $B_{3^k} = (x+1)^k$ appears to hold only for $k = 1, 2, 3$ and 6; we verified this by computation for all $k \leq 200$. Known results on the degrees of Stern polynomials (see [16], Corollary 4.4) or on the sizes of the Stern numbers $B_n(1)$ (see [14, p. 472] or [3]) would not preclude this from happening for higher k. Finally, slightly rewriting (5.4) gives the cubic relation

$$B_{9^3} = (B_9)^3.$$

Apart from $n = 1, 3, 9$, we have not found any other odd index with this property.

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