A NOTE ON FINITE SUMS OF PRODUCTS OF BERNSTEIN BASIS POLYNOMIALS AND HYPERGEOMETRIC POLYNOMIALS

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Abstract

This note considers finite sums of products of Bernstein basis polynomials and Gauss hypergeometric polynomials for which all three parameters are non-positive integers. A simple formula is derived for such sums and an interesting binomial identity is obtained as a special case.

1. Preliminaries

The *kth Bernstein basis polynomial of degree* $n \in \mathbb{N}$ is defined by

$$B_{k,n}(z) = \binom{n}{k} z^k (1-z)^{n-k}, \quad z \in \mathbb{C}.$$

The set $\{B_{k,n}(z)\}_{k=0}^{n}$ is a basis for the space of polynomials of degree at most n with complex coefficients. Since

$$\sum_{k=0}^{n} B_{k,n}(z) = (z + (1-z))^n = 1,$$

the Bernstein basis polynomials of degree n form a partition of unity.

For $p, q \in \mathbb{N}$, the *generalized hypergeometric function* is defined by

$$_pF_q(\alpha_1, \ldots, \alpha_p; \gamma_1, \ldots, \gamma_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\gamma_1)_n \cdots (\gamma_q)_n} \frac{z^n}{n!}, \quad z \in \mathbb{C} \tag{1}$$

where $\alpha_1, \ldots, \alpha_p$, $\gamma_1, \ldots, \gamma_q \in \mathbb{C}$ and $(\rho)_n$ is the Pochhammer symbol defined by

$$(\rho)_n = \begin{cases} 1 & n = 0 \\ \rho(\rho+1)\cdots(\rho+n-1) & n > 0. \end{cases}$$

The $_2F_1$ case is called the Gauss hypergeometric function

$$_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n \, n!} z^n, \quad z \in \mathbb{C}. \tag{2}$$

Clearly ${}_2F_1$ is symmetric in α and β.

The series in (2) terminates if and only if either $-\alpha \in \mathbb{N}$ or $-\beta \in \mathbb{N}$. In this case, the series is called a hypergeometric polynomial. If $-\alpha \in \mathbb{N}$, then for $m = -\alpha$,

$$ {}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{m} \binom{m}{n} (-1)^n \frac{(\beta)_n}{(\gamma)_n} z^n. \tag{3} $$

If $-\beta \in \mathbb{N}$, first use symmetry of ${}_2F_1$ in α and β, then apply (3). In general, if $-\gamma \in \mathbb{N}$, the series in (2) does not converge, except in the case that the series terminates and $-\gamma > m$. (See [2] for a definition of ${}_2F_1$ explicitly including the case of negative integer values of γ.)

We note that in the special case $\beta = \gamma$, the hypergeometric polynomial simplifies as

$$ {}_2F_1(\alpha, \beta; \gamma; z) = {}_1F_0(\alpha; z) = (1-z)^{-\alpha}. \tag{4} $$

In the non-terminating case, Equation (4) only holds for $|z| < 1$, but in the polynomial case it is just a statement of the binomial theorem and thus holds for all $z \in \mathbb{C}$.

In this note, we prove some simple, but non-trivial results on sums of products of Bernstein basis polynomials and hypergeometric polynomials. More specifically, we will establish a formula for finite sums of the form

$$ \sum_{k=0}^{N} B_{k,N}(z)\, {}_2F_1(-k, -m; -N; \varphi(z)), $$

for $m, N \in \mathbb{N}$.

We will use the contiguous relationship

$$ \gamma\, {}_2F_1(\alpha, \beta; \gamma; z) - \alpha\, {}_2F_1(\alpha+1, \beta; \gamma+1; z) + (\alpha - \gamma)\, {}_2F_1(\alpha, \beta; \gamma+1; z) = 0 \tag{5} $$

proved in [1]. Also see [4] for an up-to-date list of known contiguous relations for the Gauss hypergeometric function. Since (5) holds for all $z \in \mathbb{C}$, it also holds if we replace z with $\varphi(z)$ for any $\mathbb{C}$-valued function φ, or branch of φ if it is multi-valued.

2. Main Results

Theorem 1. *Let $N \in \mathbb{N}^+$, $m \in \mathbb{N}$ with $m \leq N$. For φ a $\mathbb{C}$-valued function (or a branch of a $\mathbb{C}$-valued function) and z in its domain, if*

$$ S_m(n) = \sum_{k=0}^{n} B_{k,n}(z)\, {}_2F_1(-k, -m; -n; \varphi(z)) $$

then $S_m(N) = S_m(m)$. Thus, if $N \in \mathbb{N}^+$, $m \in \mathbb{N}$ with $m \leq N$, the sum $S_m(N)$ depends only on m.

Proof. Note that $S_m(N) = S_m(m) + \sum_{n=m+1}^{N} S_m(n) - S_m(n-1)$. We prove that $\sum_{n=m+1}^{N} S_m(n) - S_m(n-1) = 0$.

For $n > m$,

$$\begin{aligned}
& S_m(n) - (1-z)S_m(n-1) \\
= & \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} \, {}_2F_1(-k,-m;-n;\varphi(z)) \\
& \quad -(1-z) \sum_{k=0}^{n-1} \binom{n-1}{k} z^k (1-z)^{n-1-k} \, {}_2F_1(-k,-m;-(n-1);\varphi(z)) \\
= & \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} \, {}_2F_1(-k,-m;-n;\varphi(z)) \\
& \quad - \sum_{k=0}^{n-1} \binom{n-1}{k} z^k (1-z)^{n-k} \, {}_2F_1(-k,-m;-(n-1);\varphi(z)) \\
= & (1-z)^n + z^n \, {}_2F_1(-n,-m;-n;\varphi(z)) \\
& \quad + \sum_{k=1}^{n-1} \binom{n}{k} z^k (1-z)^{n-k} \, {}_2F_1(-k,-m;-n;\varphi(z)) \\
& \quad -(1-z)^n - \sum_{k=1}^{n-1} \binom{n-1}{k} z^k (1-z)^{n-k} \, {}_2F_1(-k,-m;-(n-1);\varphi(z)) \\
= & z^n \, {}_2F_1(-n,-m;-n;\varphi(z)) + \sum_{k=1}^{n-1} z^k (1-z)^{n-k} \Big[\binom{n}{k} \, {}_2F_1(-k,-m;-n;\varphi(z)) \\
& \qquad - \binom{n-1}{k} \, {}_2F_1(-k,-m;-(n-1);\varphi(z)) \Big] \\
= & z^n \, {}_2F_1(-n,-m;-n;\varphi(z)) \\
& \quad + \sum_{k=1}^{n-1} z^k (1-z)^{n-k} \Big[\frac{n}{k} \binom{n-1}{k-1} \, {}_2F_1(-k,-m;-n;\varphi(z)) \\
& \qquad - \frac{n-k}{k} \binom{n-1}{k-1} \, {}_2F_1(-k,-m;-(n-1);\varphi(z)) \Big] \qquad (6) \\
= & z^n \, {}_2F_1(-n,-m;-n;\varphi(z)) \\
& \quad + \sum_{k=1}^{n-1} \binom{n-1}{k-1} z^k (1-z)^{n-k} \Big[\frac{1}{k} \Big(n \, {}_2F_1(-k,-m;-n;\varphi(z)) \\
& \qquad -(n-k) \, {}_2F_1(-k,-m;-(n-1);\varphi(z)) \Big) \Big] \\
= & z^n \, {}_2F_1(-n,-m;-n;\varphi(z)) \\
& \quad + \sum_{k=1}^{n-1} \binom{n-1}{k-1} z^k (1-z)^{n-k} \Big[\frac{1}{k} \Big(k \, {}_2F_1(-(k-1),-m;-(n-1);\varphi(z)) \Big) \Big] \qquad (7)
\end{aligned}$$

$$
\begin{aligned}
&= z^n \,{}_2F_1(-n,-m;-n;\varphi(z)) \\
&\quad + \sum_{k'=0}^{n-2} \binom{n-1}{k'} z^{k'+1}(1-z)^{n-k'-1} \,{}_2F_1(-k',-m;-(n-1);\varphi(z)) \qquad (8)\\
&= z^n \,{}_2F_1(-n,-m;-n;\varphi(z)) - z^n \,{}_2F_1(-(n-1),-m;-(n-1);\varphi(z)) \\
&\quad + \sum_{k'=0}^{n-1} \binom{n-1}{k'} z^{k'+1}(1-z)^{n-k'-1} \,{}_2F_1(-k',-m;-(n-1);\varphi(z)) \\
&= z^n(1-z)^m - z^n(1-z)^m \\
&\quad + z\sum_{k'=0}^{n-1} \binom{n-1}{k'} z^{k'}(1-z)^{n-1-k'} \,{}_2F_1(-k',-m;-(n-1);\varphi(z)) \qquad (9)\\
&= zS_m(n-1),
\end{aligned}
$$

where (6) follows from the binomial coefficient identities

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \quad \text{and} \quad \binom{n-1}{k} = \frac{n-k}{k}\binom{n-1}{k-1},$$

(7) uses the contiguous relationship in (5), (8) follows from the substitution $k' = k-1$, and (9) follows from the symmetry of ${}_2F_1$ in α and β, and an application of (4).

This argument implies $S_m(n) = S_m(n-1)$, and the result follows. □

Corollary 1. *Let $N \in \mathbb{N}^+$, $m \in \mathbb{N}$ with $m \leq N$. If φ is a $\mathbb{C}$-valued function (or a branch of a $\mathbb{C}$-valued function) and z is in its domain, then*

$$\sum_{k=0}^{N} B_{k,N}(z) \,{}_2F_1(-k,-m;-N;\varphi(z)) = (1-z\varphi(z))^m. \qquad (10)$$

Proof. The result follows from Theorem 1 and the binomial theorem:

$$
\begin{aligned}
\sum_{k=0}^{N} B_{k,N}(z) \,{}_2F_1(-k,-m;-N;\varphi(z)) &= \sum_{k=0}^{m} B_{k,m}(z) \,{}_2F_1(-k,-m;-m;\varphi(z)) \\
&= \sum_{k=0}^{m} B_{k,m}(z) \,{}_1F_0(-k;\varphi(z)) \\
&= \sum_{k=0}^{m} \binom{m}{k} z^k(1-z)^{m-k}(1-\varphi(z))^k \\
&= \sum_{k=0}^{m} \binom{m}{k} (z-z\varphi(z))^k(1-z)^{m-k} \\
&= (1-z\varphi(z))^m.
\end{aligned}
$$

□

Specific identities that are special cases of Corollary 1 include,

(i) $\displaystyle\sum_{k=0}^{N} B_{k,N}(z)\,{}_2F_1(-k,-m;-N;z) = (1-z^2)^m$; and in particular
$\displaystyle\sum_{k=0}^{N} B_{k,N}(i)\,{}_2F_1(-k,-m;-N;i) = 2^m$,

(ii) $\displaystyle\sum_{k=0}^{N} B_{k,N}(z)\,{}_2F_1(-k,-m;-N;1-z) = (z^2-z+1)^m$,

(iii) $\displaystyle\sum_{k=0}^{N} B_{k,N}(z)\,{}_2F_1\Big(-k,-m;-N;\frac{1}{z}\Big) = \begin{cases} 1 & m=0 \\ 0 & m>0, \end{cases}$

(iv) $\displaystyle\sum_{k=0}^{N} B_{k,N}(z)\,{}_2F_1\Big(-k,-m;-N;1+\frac{1}{z}\Big) = z^m$.

The following corollary establishes a binomial identity that is also an immediate consequence of Corollary 1.

Corollary 2. *Let $N \in \mathbb{N}^+$, $m \in \mathbb{N}$ with $m \leq N$. Then*

$$\sum_{k=0}^{N} B_{k,N}(x)\,{}_2F_1\Big(-k,-m;-N;\frac{1+x-y}{x}\Big) = (y-x)^m, \quad y \in \mathbb{C},\ x \in \mathbb{C}\setminus\{0\}. \tag{11}$$

A remarkable aspect of this identity is the way that monomial terms of degree other than m in the summation in (11) combine and vanish, leaving only terms of degree m involved in the binomial expansion. For example, in the case with $m=3$ and $N=4$,

$$\begin{aligned}
&\sum_{k=0}^{4} B_{k,4}(x)\,{}_2F_1\Big(-k,-3;-4;\frac{1+x-y}{x}\Big)\\
&= x^4 - 4\,x^3 + 6\,x^2 - 4\,x + 1\\
&\quad - x^4 - 3\,x^3y + 6\,x^3 + 9\,x^2y - 12\,x^2 - 9\,xy + 10\,x + 3\,y - 3\\
&\quad + 3\,x^3y + 3\,x^2y^2 - 3\,x^3 - 12\,x^2y - 6\,xy^2 + 9\,x^2 + 15\,xy + 3\,y^2 - 9\,x - 6\,y + 3\\
&\quad - 3\,x^2y^2 - xy^3 + 6\,x^2y + 6\,xy^2 + y^3 - 3\,x^2 - 9\,xy - 3\,y^2 + 4\,x + 3\,y - 1\\
&\quad + xy^3 - 3\,xy^2 + 3\,xy - x\\
&= y^3 - 3xy^2 + 3x^2y - x^3\\
&= (y-x)^3.
\end{aligned}$$

As another application, Corollary 1 can be used to derive the generating function for Krawtchouk polynomials. For $0 < p < 1$, $N \in \mathbb{N}$, and $x \in \mathbb{N}$ with $x \leq N$, the

Krawtchouk polynomials are defined for $\nu \in \mathbb{N}$ with $\nu \leq x$ by

$$K_\nu(x, p, N) = {}_2F_1\Big(-\nu, -x; -N; \frac{1}{p}\Big),$$

(see [3], for instance). In (10), let $z = \frac{t}{1+t}$ and $\varphi(z) = \frac{1}{p}$. Then

$$\begin{aligned}(1+t)^{-N} \sum_{\nu=0}^{N} \binom{N}{\nu} {}_2F_1\left(-\nu, -x; -N; \varphi\left(\frac{t}{1+t}\right)\right) t^\nu \\ = \sum_{n=0}^{N} B_{\nu,N}\left(\frac{t}{1+t}\right) {}_2F_1\left(-\nu, -x; -N; \varphi\left(\frac{t}{1+t}\right)\right) \\ = \left(1 - \left(\frac{t}{1+t}\right)\frac{1}{p}\right)^x = \left(1 - \left(\frac{1-p}{p}\right)t\right)^x (1+t)^{-x}.\end{aligned}$$

Thus

$$\sum_{\nu=0}^{N} \binom{N}{\nu} {}_2F_1\Big(-\nu, -x; -N; \frac{1}{p}\Big) t^\nu = \left(1 - \left(\frac{1-p}{p}\right)t\right)^x (1+t)^{N-x},$$

which is the generating function for Krawtchouk polynomials as given in [3]

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References

[1] Y.-J. Cho, T.-Y. Seo, and J.-S. Choi, A note on contiguous function relations, *East Asian Math. J.* **15** (1999), 29-38.

[2] A. Erdelyi (ed.), *Higher Transcendental Functions: Vol. I: Bateman manuscript project*, McGraw-Hill Book Company, New York, 1953.

[3] S. Karlin and J. McGregor, Ehrenfest Urn Models, *J. Appl. Probab.* **2** (1965), 352-376.

[4] M. A. Rakha, A. K. Rathie, and P. Chopra, On some new contiguous relations for the Gauss hypergeometric function with applications, *Comput. Math. Appl.* **61** (2011), 620-629.